PHYSICAL REVIEW VOLUME 92, NUMBER 5 DECEMBER 1, 1953

Liquid Helium II: Bose-Einstein Condensation and Two-Fluid Model

P. R. ZILSEL
University of Connecticut, Storrs, Connecticut
(Received June 19, 1953)

The Bose-Einstein condensation mechanism is shown to be capable of accounting for the existence of separate hydrodynamic velocity fields for the normal and the superfluid, provided suitable assumptions are made with regard to the single-particle energy spectrum. We consider the effect on the microcanonical gas distribution of imposing a nonzero value of the total momentum **P**. For an ordinary gas the effect is trivial, the whole distribution being merely shifted in momentum space. However, in the case of a degenerate Bose gas whose single-particle energy spectrum has a sharp minimum (gap or cusp), only the excited part of the gas (normal fluid) participates in the imposed motion, the condensate (superfluid) remaining "frozen" in momentum space. This rigidity of the condensate in momentum space plays the same role as the rigidity of the superelectrons on imposition of a magnetic field in the London theory of superconductivity. **P** acts as an additional thermodynamic variable, states with $\mathbf{P}\neq 0$ being macroscopically metastable and corresponding to the existence of a relative velocity between normal and superfluid. The basic hydrodynamic assumption of the two-fluid model is thus reduced to an assumption concerning the form of an effective single-particle energy spectrum, and the parallelism between the theories of superconductivity and superfluidity is clearly exhibited. The present theory permits, in particular, the introduction of the mathematical form of Landau's phonon and "roton" spectra within the framework of the Bose-Einstein condensation picture. The statistical-thermodynamic formulas are derived and are shown to lead to characteristic two-fluid equations derived previously from a variational principle.

I. INTRODUCTION

THE characteristic properties of liquid helium, which distinguish it from all other fluids except the "electron gas" in superconductors, may be classified phenomenologically into two types.

(1) purely thermostatic equilibrium properties. The most significant are the stability of the liquid phase apparently down to absolute zero, and the λ transition within the liquid phase.

(2) the dynamic "superfluid" properties of the liquid below the λ transition (liquid helium II). These may be summarized by the statement that reversible—or, at least, almost reversible—transport processes exist within the liquid.

In contrast to the case of superconductivity, where the existence of strictly reversible transport processes (persistent currents) is definitely established, the stricly reversible character of the "superfluid" processes in He II is still to some extent a matter of dispute, since the persistence or decay of a current of neutral helium atoms is much more difficult to establish by direct observation than the persistence or decay of a current of charged electrons. Nevertheless, such experimental evidence, direct and indirect, as exists, as well as the very suggestive and probably quite fundamental parallelism with superconductivity[1]—about which more later—favors the assumption of strict reversibility under suitable conditions. We shall, moreover, attempt to show that, just as in the case of superconductivity, strict reversibility in the case of He II is less difficult to understand theoretically than the alternative assumption, which involves the necessity of explaining why dissipative interactions which are very strong in other fluids, including liquid helium I, should be extremely weak, though qualitatively present, in liquid helium II.

A successful molecular theory of liquid helium has to account simultaneously for both the thermostatic properties (1) and the dynamic properties (2), presumably on the basis of quantum statistical mechanics. Such a theory, again as in the case of superconductivity, does not yet exist. However, a basic point of view from which attempts at building such theories may proceed, has been expressed by London.[1] According to this point of view both superconductivity and superfluidity result from quantum statistical mechanisms producing condensation into a more or less rigid lattice in momentum space.

Attempts at developing this point of view into complete theories have led to a peculiar contrast between the status of the theory of superconductivity, on the one hand, and of liquid helium, on the other. In the case of superconductivity, the concept of a rigid lattice in momentum space leads almost directly to a generally satisfactory phenomenological theory of the dynamical properties of superconductors (the London theory). The question, however, of what is the mechanism producing this condensation of the electron fluid into a rigid lattice in momentum space, and thus, ultimately, the question of the statistical thermodynamics of the superconductive state and the phase transition into this state, is a very difficult one. Only quite recently has any real progress been made toward its solution.[2]

The situation with respect to liquid helium is almost exactly opposite. Here a mechanism which can account at least qualitatively for the condensation in momentum space, and thus for the existence of the λ transition, is provided by the Bose-Einstein condensation,[3] which predicts for an ideal Bose gas the existence of a transition temperature T_0 below which a finite fraction of all the molecules condenses progressively into the lowest

[1] See F. London, *Superfluids* (John Wiley and Sons, Inc., New York, 1950), Vol. 1.

[2] H. Froehlich, Phys. Rev. **79**, 845 (1950); Proc. Phys. Soc. (London) **A64**, 129 (1951). J. Bardeen, Phys. Rev. **80**, 567 (1950); Revs. Modern Phys. **23**, 261 (1951).

[3] F. London, Nature **141**, 643 (1938); Phys. Rev. **54**, 947 (1938).

energy level. While liquid helium is not, of course, an ideal gas, there is good reason for believing that the qualitative features of the Bose-Einstein condensation remain valid for a liquid with such a large specific volume and such small intermolecular forces as liquid helium.[4] In particular, Guggenheim's smoothed potential model of a liquid[5] exhibits a Bose-Einstein condensation very similar to that of an ideal gas, except that the properties of the phase transition approach more closely to those actually observed in liquid helium.

While, however, the Bose-Einstein condensation has provided at least a qualitative understanding of the *thermostatic* properties of liquid helium, it has not, until now, served as a convincing basis for an understanding of the *two-fluid dynamic* properties. It is true that, historically, Tisza's development of the "two-fluid model",[6] which furnishes a generally successful phenomenological theory of the dynamic properties, sprang from ideas suggested by the Bose-Einstein condensation. With its division of the gas into two parts—the entropiless condensate in momentum space, and the remainder of the gas distributed statistically among the excited energy levels—the Bose-Einstein condensation indeed furnished a basis for *some* features of the two-fluid model. But the crucial hydrodynamic feature of the model, that is the existence of separate hydrodynamic velocity fields for the condensate (superfluid) and the excited part (normal fluid), appeared as a very questionable assumption which could be justified only *a posteriori* by the success of the model. It seemed to be necessary to assume a completely unintelligible absence of collisions between the atoms constituting the superfluid and those constituting the normal fluid, in order to account for the apparently frictionless flow of the two fluids relative to each other. At the same time, collisions are, of course, necessary to establish local thermal equilibrium within the liquid, without which the application of thermodynamic reasoning, a very essential feature of the two-fluid theory, would be quite without justification.

The key to the only way out of this dilemma would seem to be the hypothesis put forth by London, that, under given macroscopic boundary conditions, a state of relative motion between the normal and the superfluid is macroscopically metastable, in the same way as a supercurrent is macroscopically metastable in the presence of a given external magnetic field. This means that the superflow is truly reversible; it is maintained not by the absence of collisions, but—under the given conditions—is the thermodynamic equilibrium state established by the collisions.

It is the purpose of the present paper to show that, under suitable assumptions with regard to the single-particle energy spectrum, a degenerate Bose "gas" below its condensation temperature behaves in accordance with London's hypothesis. We shall see that the imposition of nonzero *total momentum* on the degenerate Bose gas plays here a role analogous to that of the imposition of a magnetic field in the London theory of superconductivity. If one calculates the microcanonical gas distribution under the condition of fixed (nonzero) total momentum as well as fixed total energy and number of particles, the equilibrium distribution turns out to involve relative motion between the condensate and the center of mass of the excited part of the gas.

In this way we find that the Bose-Einstein condensation mechanism can furnish at least a qualitative kinetic insight into the existence of two distinct hydrodynamic velocity fields, as well as into the thermostatic properties of liquid helium II. The general features of the two-fluid hydrodynamics can thus be put on a statistical basis within the framework of the Bose-Einstein condensation picture. This is significant especially in view of the apparent absence of superfluidity in pure $_2He^3$, which obeys Fermi-Dirac statistics. The present treatment, further, exhibits in a new light the parallelism between superfluidity and superconductivity.

The theory is still phenomenological in that it considers the liquid in a single-particle approximation, neglecting correlations between the molecules and considering intermolecular forces only to the extent that they are assumed to modify the effective single-particle energy spectrum. To obtain statistical results corresponding to the two-fluid behavior, it is further necessary to assume that this effective single-particle energy spectrum, as a function of the single-particle momentum, has a sharp minimum—either a cusp or a gap. In particular one may use, for example, the mathematical forms of Landau's phonon or "roton" spectrum.[7] The present treatment, further, is confined to a discussion of the *statistical* aspects of two-fluid theory. No attempt is made to solve the fundamental quantum mechanical problem concerning the dynamic behavior of the pure ground state of an ideal quantum liquid.

Some of the statistical-thermodynamic formulas of the present theory will be seen to be similar to some of those given by Landau,[7] and by Dingle[8] and Temperley[9] in further elaboration of Landau's approach. However the significance and interpretation of these equations will be very different.[10]

[4] F. London, *Superfluids*, Vol. 2 (to be published). See also R. P. Feynman, Phys. Rev. **90**, 1116 (1953).

[5] E. A. Guggenheim, Proc. Roy. Soc. (London) **A135**, 181 (1932).

[6] L. Tisza, Compt. rend. **207**, 1035 and 1186 (1938); Phys. Rev. **72**, 838 (1947).

[7] L. Landau, J. Phys. U.S.S.R. **5**, 71 (1941).

[8] R. B. Dingle, Phil. Mag. Supplement **1**, 111 (1952).

[9] H. N. V. Temperley, Proc. Phys. Soc. (London) **A65**, 490 (1952).

[10] It should perhaps be pointed out that, in adopting here the approach from the gas-like Bose-Einstein approximation, I do not wish to maintain that this is the best available approximation at all temperatures. It seems very likely from the experimental evidence that at the lowest temperatures (below about 0.6°K) the excitations in He II are better described by Landau's Debye-type phonon picture. There seems to be little doubt, however, that in the region of the λ transition and down to about 1°K the gas-like picture is more nearly adequate, and is the only one of the two which can account for the existence of the λ transition

II. PRELIMINARY THERMODYNAMIC CONSIDERATIONS

The thermodynamic system under discussion will be taken to be a *unit volume* of fluid. For a simple one-component system such as liquid helium (neglecting the irrelevant complication of a small admixture of the isotope $_2He^3$ in natural helium) the specification of a thermostatic equilibrium state requires two variables of state, for which we shall take the entropy S and the number of particles N in the unit volume. However, we wish to consider the effect on the system of imposing and adiabatically varying certain external conditions—which we may call, somewhat loosely, "boundary conditions." In the language of statistical mechanics, these boundary conditions correspond to constraints or inhibitions imposed on the assembly representing our thermodynamic system. The thermodynamic description of the system then requires at least one additional variable or parameter suitably specifying the effect of the imposed constraint. Without enquiring at the moment what the physical significance of this variable might be for the case we are interested in, we shall denote it by Y.

The First Law of Thermodynamics for reversible processes, including adiabatic changes in the imposed constraints, then reads

$$dE'=TdS+\mu dN+\varphi dY, \tag{2.1}$$

where $E'(S, N, Y)$ is the total energy. The temperature is given by

$$T=(\partial E'/\partial S)_{N,Y}, \tag{2.2}$$

the chemical potential by

$$\mu=(\partial E'/\partial N)_{S,Y}, \tag{2.3}$$

and the quantity φ is defined by

$$\varphi=(\partial E'/\partial Y)_{S,N}. \tag{2.4}$$

It is clear that the quantity φ must be such as to represent a generalized force and dY the corresponding infinitesimal displacement, so that $-\varphi dY$ is the infinitesimal work done by the system against the yielding constraint.

If, then, in the two-fluid model, the states of relative motion between the normal and the superfluid are metastable equilibrium states resulting from the application of suitable macroscopic constraints, it follows that the thermodynamic functions occurring in the two-fluid model must depend on at least one additional variable beside the usual two. In an earlier paper[11] it was found from a detailed analysis of the two-fluid hydrodynamics that such must indeed be the case, if both the requirements of reversibility of superflow and of conservation of total energy and momentum are to be met. The variable which appeared in that discussion was the *relative density of the normal fluid*, $x=\rho_n/\rho$, and it was shown that the *internal energy per gram* U depends on x through the relation

$$(\partial U/\partial x)_{s,\rho}=v^2/2, \tag{2.5}$$

so that

$$dU=Tds+(\mathfrak{P}/\rho^2)d\rho+(v^2/2)dx. \tag{2.6}$$

Here $s=S/\rho$ is the entropy per gram, $\mathfrak{P}$ is the pressure, and $\mathbf{v}$ is the relative velocity of the two fluids.[12]

The problem now before us is to determine whether it is possible to express the conditions responsible for the appearance of a relative velocity in liquid He II in terms of a physically meaningful statistical constraint on the distribution of a degenerate Bose gas, and what the nature of such a constraint might be. For this purpose it will be useful to consider the analogous situation in the London theory of superconductivity.

The physical constraint responsible for the appearance of a supercurrent is the application of a magnetic field, $\mathbf{H}$, and the term corresponding to our φdY in Eq. (2.1) is $(1/4\pi)\mathbf{H}\cdot d\mathbf{B}$, where $\mathbf{B}$ is the magnetic induction per unit volume. The electric current per superelectron is $-(e^2/mc)\mathbf{A}$. The vector potential $\mathbf{A}$ is determined in the present case by $\mathrm{curl}\mathbf{A}=\mathbf{h}$, $\mathrm{div}\mathbf{A}=0$, where $\mathbf{h}$ is the local value of the magnetic field. These relations are not very useful for our purpose in their present form. They can, however, be re-expressed in such a way as to provide a clue to the corresponding situation in liquid helium.

The momentum of a charged particle in the presence of a magnetic field is given by

$$\mathbf{p}_i=m\mathbf{v}_i+(e/c)\mathbf{A}(\mathbf{r}_i),$$

and the corresponding electric current by

$$\mathbf{j}_i=e\mathbf{v}_i,$$

where $\mathbf{v}_i$ is the velocity of the particle. Consider now a gas of free electrons in statistical equilibrium. In the absence of a magnetic field both the total momentum

$$\mathbf{P}=\sum_i \mathbf{p}_i,$$

and the total current

$$\mathbf{J}=\sum_i \mathbf{j}_i,$$

will be zero. If a magnetic field is applied, however, the total current will classically[13] still be zero, but there will now be a total momentum given by

$$\mathbf{P}=\sum_i(e/c)\mathbf{A}(\mathbf{r}_i).$$

Thus, for a free electron gas, the application of a mag-

itself (see F. London, reference 4, and also H. N. V. Temperley, reference 9).

[11] P. R. Zilsel, Phys. Rev. 79, 309 (1950).

[12] The treatment of x as a thermodynamic variable and, in particular, Eq. (2.5) has been criticized as inconsistent by H. N. V. Temperley, Proc. Phys. Soc. (London) A64, 105 (1951), and by Dingle (reference 8). We shall see, however, that both the treatment and the equation follow directly from the statistical considerations of the present paper. Moreover, Eq. (2.5) can easily be shown to follow from Dingle's own equation (116) when the latter is interpreted correctly.

[13] There is a small quantum effect, the Landau-Peierls diamagnetism, which is irrelevant to our present purpose.

netic field is equivalent to the imposition of a total momentum.

Now the basic assumption of the London theory is that in the superconducting state a finite fraction of the electron gas is "frozen" into a rigid lattice in momentum space, so that the momentum of this fraction is unaffected by the application of a weak magnetic field. Only the "normal" electrons then contribute to the net total momentum of the gas in the presence of a field; the average momentum of the superelectrons remains zero, and there results a supercurrent

$$\mathbf{J}_s = -\sum{}'(e^2/mc)\mathbf{A},$$

(the summation extending over the "super"-fraction of the electron gas).

Guided by the analogy between superconductivity and superfluidity we will, then, consider the effect of imposing a nonzero value of the total momentum on our system. The variable we have called Y will now be the total momentum $\mathbf{P}$ in the unit volume, and the quantity φ will be a velocity defined by the relation

$$\mathbf{v} = (\partial E'/\partial \mathbf{P})_{S,N}. \qquad (2.7)$$

In this case Eq. (2.1) becomes

$$dE' = TdS + \mu dN + \mathbf{v}\cdot d\mathbf{P}, \qquad (2.8)$$

the infinitesimal work done in increasing the momentum of the system being $\mathbf{v}\cdot d\mathbf{P}$. The Helmholtz free energy, defined as

$$F' = E' - ST, \qquad (2.9)$$

satisfies the equation

$$d(F'/T) = -(E'/T^2)dT + (\mu/T)dN + \mathbf{v}\cdot d\mathbf{P}/T. \qquad (2.10)$$

For a closed system the total momentum is a constant of the motion. Thus the state resulting from the imposition of a total momentum different from zero is not subject to internal dissipation into the absolute equilibrium state characterized by $\mathbf{P}=0$, but is indeed macroscopically metastable.

Note that, since the total momentum is an extensive variable, the chemical potential is no longer equal to the Gibbs potential per particle G'/N, where

$$G' = E' - ST + \mathfrak{P}\mathfrak{V}, \qquad (2.11)$$

and the volume $\mathfrak{V}$ is unity in the present case. Instead we have

$$\mu = (G' - \mathbf{P}\cdot\mathbf{v})/N. \qquad (2.12)$$

Of course, for an ordinary thermodynamic system the imposition of a total momentum is trivial. It results simply in the total system being put into motion, without significant effect on its statistical properties, and is equivalent to a Galilean coordinate transformation. The velocity $\mathbf{v}$, defined by Eq. (2.7), is then the resulting center-of-mass velocity of the system as a whole and satisfies the relation

$$\mathbf{P} = \rho\mathbf{v}. \qquad (2.13)$$

We shall see in the next section, however, that in the case of a degenerate Bose gas with a suitably generalized energy spectrum, an effect can occur which is quite analogous to the "freezing" of the superelectrons in momentum space which is assumed by the London theory of superconductivity.

III. THE DEGENERATE BOSE GAS WITH FIXED TOTAL MOMENTUM

We proceed to derive the most probable distribution in a Bose gas with a generalized energy spectrum $\epsilon_j = \epsilon(\mathbf{p}_j)$, under the conditions of fixed total number of particles N, total energy E', and total momentum $\mathbf{P}$, that is,

$$\sum_j n_j = N, \quad \sum_j n_j\epsilon_j = E', \quad \sum_j n_j\mathbf{p}_j = \mathbf{P}. \qquad (3.1)$$

The entropy corresponding to a given distribution (occupation numbers n_j) is[14]

$$S_D = \sum_j S_j = \sum_j k[g_j \ln(1+n_j/g_j) + n_j \ln(1+g_j/n_j)], \qquad (3.2)$$

where g_j is the degeneracy of the j'th energy level.

The most probable distribution is the one for which S_D is a maximum subject to the restrictions (3.1). It is, then, determined by the condition that for all i

$$(\partial/\partial n_i)[S/k - \alpha\sum n_j - \beta\sum n_j\epsilon_j - \boldsymbol{\gamma}\cdot\sum n_j\mathbf{p}_j] = 0, \qquad (3.3)$$

that is,

$$\ln(1+g_j/n_j) - \alpha - \beta\epsilon_j - \boldsymbol{\gamma}\cdot\mathbf{p}_j = 0. \qquad (3.4)$$

Thus we have for the most probable occupation numbers

$$n_j = g_j/[\exp(\alpha + \beta\epsilon_j + \boldsymbol{\gamma}\cdot\mathbf{p}_j) - 1]. \qquad (3.5)$$

This distribution differs from the usual one, determined without specifying the total momentum, only through the presence of the extra term $\boldsymbol{\gamma}\cdot\mathbf{p}_j$ in the exponent.[15]

The Lagrange multipliers α, β, $\boldsymbol{\gamma}$, are determined by the conditions (3.1). α and β have their usual significance,

$$k\alpha = -\mu/T, \qquad (3.6)$$

$$k\beta = 1/T, \qquad (3.7)$$

and $\boldsymbol{\gamma}$ is given by

$$k\boldsymbol{\gamma} = -\mathbf{v}/T, \qquad (3.8)$$

where $\mathbf{v}$ is the velocity defined by the thermodynamic relation (2.7).

Equation (3.8) plus, incidentally, Eqs. (3.6) and (3.7) may be obtained quickly, though somewhat nonrigorously, by a slight extension of a method used by Mayer and Mayer:[14] we consider the gas as consisting of two parts, a particular energy level j and the remainder, denoted by r. For the most probable distribution the total entropy, $S = S_j + S_r$, is stationary under the transfer of

[14] See J. E. Mayer and M. G. Mayer, *Statistical Mechanics* (John Wiley and Sons, Inc., New York, 1940), p. 124.

[15] A similar result was obtained by Temperley (reference 9) for an assembly of Debye modes in connection with Landau's phonon model.

a small number of particles,

$$\delta n_j = -\delta n_r, \qquad \text{(i)}$$

into the state j from the remainder of the gas. This transfer has to be such as to conserve not only the total number of particles, but also the total energy and momentum, so that we have, in addition to (i),

$$\delta E_r' = -\delta E_j' = -\epsilon_j \delta n_j; \quad \delta \mathbf{P}_r = -\delta \mathbf{P}_j = -\mathbf{p}_j \delta n_j. \qquad \text{(ii)}$$

The condition that the entropy is stationary gives

$$0 = \delta S = (dS_j/dn_j - \delta S_r/\delta n_r)\delta n_j. \qquad \text{(iii)}$$

From (3.4)

$$dS_j/dn_j = k \ln(1+g_j/n_j) = k\alpha + k\beta\epsilon_j + k\boldsymbol{\gamma}\cdot\mathbf{p}_j. \qquad \text{(iv)}$$

To evaluate $\delta S_r/\delta n_r$ we assume that the part r is so nearly the whole system that we can apply to it the thermodynamic formulas valid for the gas as a whole. Thus

$$\delta S_r/\delta n_r = \partial S/\partial N + (\partial S/\partial E')\delta E_r'/\delta n_r + (\partial S/\partial \mathbf{P})\cdot\delta \mathbf{P}_r/\delta n_r. \quad \text{(v)}$$

Equating (iv) to (v), making use of the relations (ii) and the identity

$$(\partial S/\partial \mathbf{P})_{E',N} = -(\partial S/\partial E')_{\mathbf{P},N}(\partial E'/\partial \mathbf{P})_{S,N} = -\mathbf{v}/T, \qquad \text{(vi)}$$

we obtain the desired result.

Equation (3.5) for the most probable distribution thus takes the form

$$n_j = g_j/\{\exp[(\epsilon_j - \mathbf{v}\cdot\mathbf{p}_j - \mu)/kT] - 1\}. \qquad (3.9)$$

To consider the properties of this distribution function it is convenient to introduce a quantity ϵ_j', defined as

$$\epsilon_j' = \epsilon_j - \mathbf{v}\cdot\mathbf{p}_j. \qquad (3.10)$$

The distribution (3.9) is the same function of ϵ' as the usual Bose distribution without imposed momentum is of ϵ. Condensation will set in at the temperature at which μ becomes virtually equal to the minimum value of ϵ', and the condensation will be into the energy level ϵ_0 determined by the condition that

$$\epsilon_0' = \epsilon_0 - \mathbf{v}\cdot\mathbf{p}_0 = \min(\epsilon'). \qquad (3.11)$$

In general the identity of this level will, of course, depend on the value of $\mathbf{v}$, which means, ultimately, on the value of the imposed momentum.

Ideal-Gas Energy Spectrum

Consider in particular the case where the energy spectrum is that of an ordinary ideal gas,

$$\epsilon_j = p_j^2/2m.$$

In this case

$$\epsilon_j' = p_j^2/2m - \mathbf{v}\cdot\mathbf{p}_j = (1/2m)|\mathbf{p}_j - m\mathbf{v}|^2 - mv^2/2,$$

so that, by Eq. (3.11), the condensation occurs into the level

$$\epsilon_0 = mv^2/2$$

with momentum

$$\mathbf{p}_0 = m\mathbf{v}.$$

The effect of imposing a total momentum on the gas is, then, what we should expect for an ordinary system: the whole distribution in momentum space is merely shifted by the amount $m\mathbf{v}$. The total momentum is

$$\mathbf{P} = Nm\mathbf{v} = \rho\mathbf{v},$$

in agreement with Eq. (2.13), and the velocity $\mathbf{v}$ has the significance of the center-of-mass velocity of the gas as a whole.

Energy Spectrum with Cusp or Gap

If, on the other hand, the energy spectrum $\epsilon(\mathbf{p})$ is a monotonically increasing function of the magnitude of $\mathbf{p}$ with a finite slope at the origin, such as

$$\epsilon_j = c|\mathbf{p}_j| + c'p_j^2 + \cdots, \qquad \text{(a)}$$

or, *a fortiori*, if there is a finite energy gap separating the state of zero momentum from the other levels, e.g.,

$$\epsilon_j = 0, \quad \mathbf{p}_j = 0; \quad \epsilon_j = p_j^2/2m + \Delta, \quad \mathbf{p}_j \neq 0, \qquad \text{(b)}$$

the situation is quite different. For sufficiently small $\mathbf{v}$ the minimum of ϵ' remains at the level with zero momentum.

$$\epsilon_0 = \epsilon(\mathbf{p} = 0),$$

so that *with this type of energy spectrum, only the excited part of the gas contributes to the total imposed momentum.* We thus have, in this case, an actual model of a macroscopically metastable state involving relative motion between the excited part (normal fluid) and the condensed part (superfluid) of the gas. As long as a finite fraction of the gas is condensed into the level with zero momentum, that is as long as we remain below the condensation temperature, the ratio of the total momentum $\mathbf{P}$ to the velocity $\mathbf{v}$ is less than the total mass ρ in the unit volume of gas, since only the excited part of the gas is accelerated when the momentum is increased. If, following Landau,[7] we define the effective density of the normal fluid as this ratio of the total momentum to what is in his treatment the "drift velocity" of the excitations,

$$\mathbf{P} = \rho_n \mathbf{v} = \rho x \mathbf{v}, \qquad (3.12)$$

the velocity $\mathbf{v}$ takes on the significance of the relative velocity between the normal and the superfluid.[16]

The distortion of the equilibrium distribution by the imposition of a total momentum is shown schematically in Fig. 1 for the case of the energy spectrum (b). In this particular case

$$\mathbf{P} = mN_e\mathbf{v},$$

so that, by (3.12),

$$\rho_n = mN_e,$$

where

$$N_e = N - n_0$$

is the number of excited particles. Note that N_e is an increasing

[16] The statistical formulas of Landau (reference 7) and Dingle (reference 8) refer to assemblies of "excitations" moving with a drift velocity $\mathbf{v}$ relative to an unspecified rest system. These authors assume that this rest system is the one in which the "ground state" of the liquid (superfluid) is at rest. But since the ground state is not part of their statistical system, it is not clear in their treatment whether $\mathbf{v}$ is in fact a relative velocity between the assembly of excitations and the superfluid.

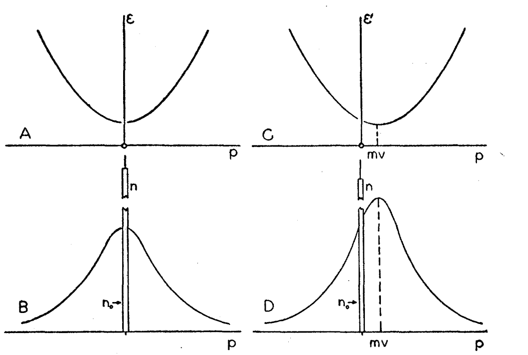

FIG. 1. Effect on the equilibrium distribution of imposing a total momentum on a degenerate ($T<T_0$) Bose gas, with energy spectrum $\epsilon_j=0$, for $\mathbf{p}_j=0$; $\epsilon_j=p_j^2/2m+\Delta$, for $\mathbf{p}_j\neq 0$ (schematic). A: energy spectrum as a function of $\mathbf{p}$; the circle at the origin indicates the lowest level. B: equilibrium distribution, n as a function of $\mathbf{p}$, for zero total momentum; the occupation number n_0 of the lowest level (condensate) is indicated schematically by the broken bar. C: quantity $\epsilon_j'=\epsilon_j-\mathbf{v}\cdot\mathbf{p}_j$, which governs the distribution for the case of nonzero total momentum. D: equilibrium distribution for nonzero total momentum. The average momentum of the excited particles is $m\mathbf{v}$, whereas the condensate (n_0) has zero momentum. The relative velocity between normal and superfluid is $\mathbf{v}$.

function of v^2. For the energy spectrum (b) the dependence is

$$N_e(v^2)=N_e(0)\exp[mv^2/2kT].$$

It should be noted that the conclusions of the preceding paragraphs are valid only for sufficiently small $\mathbf{v}$, since for any energy spectrum with finite slope or finite gap, a sufficiently large value of $\mathbf{v}$ will shift the minimum of ϵ' away from the state of zero momentum. In this way the notion of a critical velocity arises in the present considerations. A quantitative investigation of this and other detailed features which depend on the particular choice of the energy spectrum, may be left to a later time. It may be pointed out, however, that in particular the mathematical form of Landau's phonon or roton spectrum—the latter corresponding to our spectrum (b)—may be used in the present model, and that, in that case, our results are formally similar to those given by Landau,[7] even though the point of view is quite different.

We conclude this section by constructing the statistical equation for the Helmholtz Free Energy. For this purpose we introduce the function Q, which is the logarithm of the Grand Partition Function:

$$Q=\sum_j g_j \ln[1-\exp(-\alpha-\beta\epsilon_j-\boldsymbol{\gamma}\cdot\mathbf{p}_j)]. \quad (3.13)$$

Considered as a function of α, β, and $\boldsymbol{\gamma}$, Q has the following properties, as is easily verified:

$$\partial Q/\partial\alpha=N,\quad \partial Q/\partial\beta=E',\quad \partial Q/\partial\boldsymbol{\gamma}=\mathbf{P}. \quad (3.14)$$

The free energy F' is a function of T (or β), N, and $\mathbf{P}$, and can be obtained from Q essentially by a Legendre transformation. We shall verify that

$$\beta F'=Q-\alpha N-\boldsymbol{\gamma}\cdot\mathbf{P}. \quad (3.15)$$

Taking the differential of Eq. (3.15) and using (3.14), we have

$$d(\beta F')=E'd\beta-\alpha dN-\boldsymbol{\gamma}\cdot d\mathbf{P},$$

or, using Eqs. (3.6)–(3.8),

$$d(F'/T)=E'd(1/T)+(\mu/T)dN+(\mathbf{v}/T)\cdot d\mathbf{P},$$

which agrees with the thermodynamic equation (2.10), thus verifying (3.15) and, incidentally, the identifications (3.6)–(3.8) of the Lagrange multipliers α, β, and $\boldsymbol{\gamma}$.

We thus have explicitly for the Helmholtz Free Energy of a unit volume of the gas, as a function of the temperature, the number of particles, and the momentum, the following expression:[17]

$$F'=kT\sum_j g_j \ln\{1-\exp[(\mu+\mathbf{v}\cdot\mathbf{p}_j-\epsilon_j)/kT]\} +N\mu+\mathbf{v}\cdot\mathbf{P}. \quad (3.16)$$

IV. CONNECTION WITH THE TWO-FLUID MODEL. DISCUSSION

In discussing the thermo-hydrodynamics of the two-fluid model it is customary[18] to consider the total energy per unit volume, E, to consist of two parts: the hydrodynamic kinetic energy of the two fluids, and the "internal energy." Thus

$$E=\rho_n v_n^2/2+\rho_s v_s^2/2+\rho U, \quad (4.1)$$

where

$$\rho_n=x\rho,\quad \rho_s=(1-x)\rho, \quad (4.2)$$

are the densities of the normal and the superfluid, respectively, and $\mathbf{v}_n$ and $\mathbf{v}_s$ are their respective hydrodynamic velocities.

The energy E' considered in the preceding sections clearly corresponds neither to E nor to ρU: it is the total energy per unit volume in a coordinate system in which the condensate (superfluid) is at rest. To obtain the thermodynamic properties of E or of ρU, we have to transform to a coordinate system in which the superfluid has the velocity $\mathbf{v}_s$, that is a system moving with the velocity $-\mathbf{v}_s$ relative to the one for which the statistical formulas of the preceding section hold. The relation between E and E' is then found to be

$$E=E'+\rho v_s^2/2+\rho_n\mathbf{v}\cdot\mathbf{v}_s, \quad (4.3)$$

so that

$$\rho U=E'-\rho_n v^2/2. \quad (4.4)$$

Remembering the relation (3.12), whereby, in Eq. (2.8) for dE'

$$\mathbf{v}\cdot d\mathbf{P}=\mathbf{v}\cdot d(\rho x\mathbf{v})=xv^2d\rho+\rho v^2dx+\rho x dv^2/2,$$

we obtain from Eqs. (4.4) and (2.8)

$$dU=dE'/\rho-E'd\rho/\rho^2-d(xv^2)/2$$
$$=Tds+(ST+\rho xv^2-E'+N\mu)d\rho/\rho^2+(1/2)v^2dx. \quad (4.5)$$

[17] The corresponding formula given by Dingle [reference 8, Eq. (116)] for an assembly of excitations is obtained by a Galilean coordinate transformation involving a *fixed* velocity $\mathbf{v}$, and therefore omits the term $-\boldsymbol{\gamma}\cdot\mathbf{P}$ in the Legendre transformation (3.15). Hence Dingle's expression really stands for the function $F'-\mathbf{v}\cdot\mathbf{P}$, which accounts for its being a decreasing function of v^2.

[18] See, e.g., reference 11.

Substitution of the expression (2.12) for μ into (4.5) yields the characteristic equation (2.6) of the two-fluid model.

It will be well, in conclusion, to discuss briefly some questions relating to the physical significance of the formal statistical results we have obtained. We have seen that a degenerate Bose gas with a suitable energy spectrum exhibits the characteristic hydrodynamic property of the two-fluid model of liquid He II, that is the property of being able to sustain, under appropriate macroscopic conditions, a stable relative velocity between the condensed (superfluid) and the excited (normal) part of the gas. It is significant, however, that this result does not obtain for an *ideal gas* of helium atoms (energy spectrum $\epsilon = p^2/2m$), but requires the existence of an energy gap or, at least, a finite slope at the origin in the energy spectrum of an atom as a function of the magnitude of its momentum.

Now it is quite reasonable to expect that a single-particle approximation to the description of liquid helium would involve a spectrum of this type.[19] Such a single-particle approximation, or the smoothed potential model of a liquid,[5] for which the essential parts of the treatment in Sec. III are still valid, takes account of the interactions between the atoms in an approximate way by considering each atom to be moving in a potential obtained by averaging over the wave functions of all the other atoms. The energy spectrum of an atom in such a model therefore actually depends on the states of motion of all the atoms; it is a function of the statistical distribution. In our discussion we have assumed, however, that the single-particle energy spectrum is unaffected by the change in the distribution resulting from the imposition of a total momentum.

Quite apart, therefore, from the limitations of any model neglecting the correlations between the atoms in the description of a liquid, our results can be expected to have only limited qualitative validity as a basis for understanding the actual hydrodynamic properties of liquid He II. They will be valid only for situations in which the effective single-particle energy spectrum is not altered appreciably by the imposition of a total momentum on the fluid. We may expect that this will be the case if two conditions are met. First, the resultant relative velocity $\mathbf{v}$ must be sufficiently small, so that the distribution (3.9) is not greatly distorted by the presence of the $\mathbf{v}\cdot\mathbf{p}_j$ term. Second, the momentum must be imposed in such a way that not all the atoms are accelerated simultaneously, because otherwise one would expect the minimum of the single-particle energy spectrum to shift away from the state of zero momentum.

[19] A spectrum with a gap has been proposed by Bijl, deBoer, and Michels, Physica **8**, 655 (1941); and by Toda, Prog. Theoret. Phys. **6**, 458 (1951). See also Landau, reference 7.

The first of these conditions suggests that the reversibility of superflow will probably break down at considerably lower relative velocities than those one would deduce from the statistical considerations which assume a fixed energy spectrum. The second condition will bear a closer look at this time. In Sec. III we have assumed that the single-particle energy spectrum $\epsilon(\mathbf{p})$ has its sharp minimum at $\mathbf{p}=0$. However, the interactions between the atoms, which would be the cause of this sharp minimum, can depend only on *relative* momenta, so that in general we should expect the energy spectrum to be a function

$$\epsilon = \epsilon(|\mathbf{p}-\mathbf{p}_s|), \tag{4.6}$$

the particles of the condensate moving with a momentum $\mathbf{p}_s$. In this case the total momentum in the absence of a relative velocity $\mathbf{v}$ would be not zero, but $\mathbf{P}_0 = N\mathbf{p}_s$. Clearly the treatment of Sec. III still holds for this case, a relative velocity resulting now from the imposition of a total momentum differing from $\mathbf{P}_0$. The application of forces which affect all the particles simultaneously, however, such as for example the force of gravity or body forces in general, would now be expected to result not in a relative velocity $\mathbf{v}$, but rather in a change of $\mathbf{p}_s$, that is, a shift in the minimum of the energy spectrum (4.6). On the other hand, "forces" which act statistically by collisions, such as viscous friction between layers of the fluid or between the fluid and a wall, will not affect the energy spectrum appreciably, but will instead lead to the emergence of a relative velocity between the condensate and the excited part of the fluid. This, then, is the kind of physical situation corresponding to the imposition of the statistical constraint of a total momentum (different from $\mathbf{P}_0$).

The behavior of $\mathbf{p}_s$ is presumably described by the equation of motion of the superfluid velocity $\mathbf{v}_s$ in the two-fluid model. The problem of a rigorous microscopic theory of this behavior is essentially the problem of a rigorous theory of the ground state of an ideal quantum liquid. It is, therefore, in the main a problem not of statistical mechanics but of pure quantum mechanics, and cannot be treated by the statistical methods of the present paper. It's complete solution must await the development of a theory of the pure quantum states of a liquid. We have seen, on the other hand, that the *statistical aspects* of the two-fluid hydrodynamics can be elucidated at least qualitatively within the framework of the Bose-Einstein condensation picture.

I am greatly indebted to Professor F. London for letting me see the manuscript of his forthcoming book before publication and for several valuable discussions. I am also indebted to Mr. H. M. Fried for assistance with some of the calculations, and to the Research Corporation for a financial grant.