Theory of Polarized Particles and Gamma Rays in Nuclear Reactions*

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(Received August 6, 1953)

The complete specification of the spin states of a particle of spin i resulting from a nuclear reaction requires a knowledge of all irreducible spin tensor moments $T_\kappa^q(|\kappa| \leq q \leq 2i)$. A general calculation is made of the spin tensor moments arising from a nuclear reaction initiated by an arbitrarily polarized initial beam. All sums over magnetic quantum numbers are performed by the use of the S matrix and Racah formalisms. The results are expressed in terms of the G function which is related to the Fano X function. All selection rules follow from the properties of the G function. In particular, a generalization of the Eisner-Sachs selection rules are given. The problem of the detection of polarized particles is considered.

The S-matrix formalism is extended to include the possibility of gamma-rays in nuclear reactions. Analogous formulas to those of Blatt and Biedenharn are given for the angular distribution of gamma-rays. A simple recipe yields a general result for the polarization and angular distribution of radiation from aligned nuclei. Finally, the theory is extended to include the possibility of an arbitrarily polarized target nucleus as well.

I. INTRODUCTION

THE spin state of a beam of particles of spin i may be specified by giving the elements of its density matrix. The density matrix, however, can always be written as a linear sum of the irreducible spin tensor moments T_κ^q, where q is the rank of the tensor and κ is its component ($|\kappa| \leq q \leq 2i$). In a previous paper[1] an expression was given for the spin tensor moments arising from a nuclear reaction initiated by an unpolarized beam; i.e., one in which all initial tensor moments other than $q=\kappa=0$ vanish. It is the purpose of this paper to generalize the result to the case of an arbitrarily polarized initial beam.

In Sec. II an expression is obtained for the expectation value of a spin tensor operator resulting from a nuclear reaction. The complete dependence of the expression on the magnetic quantum numbers of the initial and final states is contained in Clebsch-Gordan coefficients. All sums over these magnetic quantum numbers are performed in Sec. III, and the final result is expressed in terms of the S matrix and a geometrical function for the initial and final states. Several general selection rules, including the generalization of the Eisner-Sachs selection rules, are listed in Sec. IV. Applications of the general result to the problems of the angular distribution and polarization of particles in nuclear reactions, detection of polarized particles, and radiations from polarized nuclei are made in Sec. V. Section VI discusses the extension of the formalism to include gamma rays in nuclear reactions. Section VII indicates the modifications which must be made if the target nuclei, as well as the incident particles, are polarized.

This paper can properly be regarded as the generalization of the results of Blatt and Biedenharn[2] to include the observation of the spin polarizations of the initial and final states of a nuclear reaction. It should be pointed out to the reader that there is a complete correspondence between this theory and the theory of angular correlation.[3] In this regard it should be noted that, although the X coefficient of Fano is the natural function to use in angular correlation problems, the geometrical G function introduced below seems to be more convenient for the problem of nuclear reactions in which the initial state is a plane wave. This same effect is apparent in the results of BB where the Z coefficient is a more natural function for the angular distribution than the Racah coefficient. The G function is the generalization of the Z coefficient and bears the same relation to the Fano X function as the Z coefficient does to the Racah function.

II. NOTATION AND GENERAL EXPRESSION FOR THE POLARIZATION

As in Sec. I, we consider the reaction

$$a+X\rightarrow Y+b, \tag{2.1}$$

in which particle a collides with nucleus X. Particle b emerges at an angle θ to the direction of the incident beam, and Y is the residual nucleus. All quantities are measured in the center-of-gravity system. It is assumed that the spin polarization measurements in the final state are made upon particle b.

The system before collision is described by the channel spin s, the relative orbital angular momentum l, and the channel index α which defines the type of incoming particle (neutron, proton, etc.) as well as its energy and the state of the target nucleus. The channel spin s is the total spin angular momentum in the entrance channel and is formed by the vector addition of the intrinsic spin i of the incoming particle and the

* This paper is based on work performed for the U. S. Atomic Energy Commission at the Oak Ridge National Laboratory.

[1] A. Simon and T. A. Welton, Phys. Rev. 90, 1036 (1953); references to this paper will be designated by I.

[2] J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. 24, 258 (1952); references to this paper will be designated by BB.

[3] U. Fano, National Bureau of Standards Report 1214 (unpublished). The author wishes to take this opportunity to thank Dr. Fano for permission to see a manuscript by Fano and Racah in advance of publication.

spin I of the target nucleus. The state of the system after the reaction will be described by primed quantities.

A convenient expression for the asymptotic form of the outgoing wave is provided by the use of the S-matrix formalism. If the colliding system, which is taken to be a plane wave along the z axis, has the initial quantum numbers α, s, and m_s, the corresponding final wave function in the state $\alpha' s' m_s'$ has the form [see BB (3.12)]

$$\psi(\alpha s m_s, \alpha' s' m_s', \theta\phi) \sim i\lambda_\alpha \left(\frac{v_\alpha}{v_{\alpha'}}\right)^{\frac{1}{2}} \frac{\exp(ik_{\alpha'} r_{\alpha'})}{r_{\alpha'}} \times \phi_{\alpha'} q(\alpha s m_s, \alpha' s' m_s'; \theta\phi) \chi(s' m_s'),$$

where v_α and k_α are the relative velocity and wave number, respectively, in channel α: $\phi_{\alpha'}$ is the product of internal wave functions of nucleus Y, and particle b corresponding to the specification α'; $\chi(s'm_s')$ is the final channel spin wave function and

$$q(\alpha s m_s, \alpha' s' m_s'; \theta\phi) = \pi^{\frac{1}{2}} \sum_{J=0}^{\infty} \sum_{l=|J-s|}^{J+s} \sum_{l'=|J-s'|}^{J+s'} \sum_{\pi=\pm1} i^{l-l'}(2l+1)^{\frac{1}{2}} \times (l\, s\, 0\, m_s | l\, s\, J\, m_s)(l'\, s'\, \mu'\, m_s' | l'\, s'\, J\, m_s) \times R(\alpha s l, \alpha' s' l'; J\pi) Y_{l'\mu'}(\theta\phi). \quad (2.3)$$

The quantity $(l\, s\, 0\, m_s | l\, s\, J\, m_s)$ is the Clebsch-Gordan coefficient defined as in Condon and Shortley.[4] The R matrix, which is related to the scattering matrix S by $R \equiv 1-S$, is evaluated in the representation specified by the quantum numbers $\alpha s l J m_j$ and π. Here J is the total angular momentum of the colliding system, m_j is its orientation, and π is the parity. It has been assumed that no external forces are acting on the system. As a result, then, S must be diagonal in J and π and must be independent of m_j as has been made clear in the notation. In this representation the S matrix is symmetric and unitary. The summations over l and l' in Eq. (2.3) are to be extended only over those values which satisfy the parity condition. For pure elastic scattering S is related to the phase shift δ by $S = \exp(2i\delta)$.

The initial spin state of the colliding system in channel α is most conveniently specified by the use of the density matrix. For our purposes this may be defined as follows. The initial system may be described as a statistical mixture of channel spin states χ^γ, and the results of any *physical* measurement are obtained by performing a suitable average over the elements γ of the statistical ensemble. If we now expand the statistical spin states in terms of the basic channel spin functions $\chi(sm_s)$, we have

$$\chi^\gamma = \sum_{s, m_s} a^\gamma(sm_s)\chi(sm_s).$$

An element of the density matrix is then defined as

$$\rho(s_1 m_1; s_2 m_2) = \langle a^\gamma(s_1 m_1)^* a^\gamma(s_2 m_2)\rangle_\gamma, \quad (2.4)$$

where the brackets denote an average over γ and where for simplicity m_{s1} is replaced by m_1, etc. It is clear that the measured value of any operator O in the final channel α' is now given by the average over the elements γ of the initial ensemble of the expression

$$\sum a^\gamma(s_1 m_1)^* a^\gamma(s_2 m_2)(\psi^*(\alpha s_1 m_1, \alpha' s_1' m_1'; \theta\phi) \times |O| \psi(\alpha s_2 m_2, \alpha' s_2' m_2'; \theta\phi)),$$

where the sum is over $s_1 m_1 s_2 m_2 s_1' m_1' s_2'$ and m_2'. Hence, by Eq. (2.4) the resultant expression, after averaging, becomes

$$\sum \rho(s_1 m_1; s_2 m_2)(\psi^*(\alpha s_1 m_1, \alpha' s_1' m_1'; \theta\phi) \times |O| \psi(\alpha s_2 m_2, \alpha' s_2' m_2'; \theta\phi)).$$

If it is assumed that the operator O corresponds to a measurement upon the spin $\mathbf{i}'$ of particle b at an angle θ, we have

$$O \equiv O(\mathbf{i}') r_{\alpha'}^2 d\Omega$$

where $d\Omega$ is the solid angle. This expression, combined with Eq. (2.2), yields the following result for the measured intensity (note that normalization is to unit incident flux)

$$\langle O(\mathbf{i}')\rangle = \lambda_\alpha^2 \sum \rho(s_1 m_1; s_2 m_2) q^*(\alpha s_1 m_1, \alpha' s_1' m_1'; \theta\phi) \times q(\alpha s_2 m_2, \alpha' s_2' m_2'; \theta\phi) \times (\chi(s_1' m_1') | O(\mathbf{i}') | \chi(s_2' m_2')). \quad (2.5)$$

Complete statistical information on the spin states of a particle of spin i may be obtained by specifying the elements of the density matrix or alternatively by specifying the values of the irreducible spin tensor moments T_κ^q ($|\kappa| \leq q \leq 2i$). Note that there are $(2i+1)^2$ independent elements in each description. It is especially convenient to use the tensor moments since the low-ranking tensors correspond naturally to such physical quantities as the differential cross section and polarization and since these elements transform covariantly under rotation. As in I, all tensor operators are defined so as to agree with the definitions given by Racah[5] and are so normalized that

$$T_0^q = P_q(i_z' / [i'(i'+1)]^{\frac{1}{2}}). \quad (2.6)$$

A partial listing of these operators is given in Sec. V. Since the tensor operators and the elements of the density matrix form complete sets, it should be possible to expand the density matrix in terms of the initial tensor moments. This expansion is performed in Appendix A. For the case of an unpolarized target nucleus the result is

$$\rho(s_1 m_1; s_2 m_2) = \frac{[(2s_1+1)(2s_2+1)]^{\frac{1}{2}}}{(2I+1)} (-1)^{m_2 - i + I} \times \sum_q \frac{(2i)!(2q+1)^{\frac{1}{2}}}{[(2i-q)!(2i+q+1)!]^{\frac{1}{2}}} (s_1 s_2 - m_1 m_2 | s_1 s_2 q - \kappa) \times \frac{W(i s_1 i s_2; Iq)}{P_q([i/(i+1)]^{\frac{1}{2}})} T_\kappa^q, \quad (2.7)$$

[4] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1951).

[5] G. Racah, Phys. Rev. **62**, 442 (1942); references to this paper will be denoted by R.

where the tensor moments $T_\kappa{}^q$ are with respect to the incoming particle of spin i. Note that $\kappa=m_1-m_2$. Hence, for a definite input tensor moment, all dependence on the magnetic quantum numbers is contained in a Clebsch-Gordan coefficient.

The operator $O(\mathbf{i}')$ will now be chosen to be any spin tensor operator $T_{\kappa'}{}^{q'}$. The reduction of the resultant spin matrix element $(\chi(s_1'm_1')|T_{\kappa'}{}^{q'}|\chi(s_2'm_2'))$ has been performed previously in Appendix A of I. It was shown in I (A.2) that

$$\begin{aligned}(\chi(s_1'm_1')|T_{\kappa'}{}^{q'}|\chi(s_2'm_2'))\\ =(-1)^{I'-\kappa'-i'+m_1'}(2q'+1)^{-\frac{1}{2}}[(2i')!]^{-1}\\ \times[(2s_1'+1)(2s_2'+1)(2i'-q')!(2i'+q'+1)!]^{\frac{1}{2}}\\ \times W(i's_1'i's_2';I'q')(s_1's_2'-m_1'm_2'|s_1's_2'q'-\kappa')\\ \times P_{q'}([i'/(i'+1)]^{\frac{1}{2}}).\quad(2.8)\end{aligned}$$

If we combine Eqs. (2.3), (2.5), (2.7), and (2.8), as well as Eq. I (2.8) for the product of two spherical harmonics, the following result is obtained:

$$\begin{aligned}\langle T_{\kappa'}{}^{q'}\rangle=&\frac{(\pi)^{\frac{1}{2}}\lambda_\alpha{}^2(2i)!(2q+1)^{\frac{1}{2}}[(2i'-q')!(2i'+q'+1)!]^{\frac{1}{2}}}{2(2I+1)(2i')!(2q'+1)^{\frac{1}{2}}[(2i-q)!(2i+q+1)!]^{\frac{1}{2}}}\\ &\times\frac{P_{q'}([i'/(i'+1)]^{\frac{1}{2}})}{P_q([i/(i+1)]^{\frac{1}{2}})}\sum(-1)^{\kappa-\kappa'+i-i'-I+I'}i^{l_2-l_1-l_2'+l_1'}\\ &\times(2l_1+1)(2l_2+1)(2l_1'+1)(2l_2'+1)(2s_1+1)\\ &\times(2s_2+1)(2s_1'+1)(2s_2'+1)/(2r'+1)]^{\frac{1}{2}}\\ &\times R(\alpha l_1s_1,\alpha'l_1's_1';J_1\pi_1)^*R(\alpha l_2s_2,\alpha'l_2's_2';J_2\pi_2)\\ &\times W(is_1is_2;Iq)W(i's_1'i's_2';I'q')(l_1'l_2'00|l_1'l_2'r'0)\\ &\times Y_{r',\kappa'-\kappa}(\theta\phi)T_\kappa{}^q\sum_{m_1}\sum_{m_1'}[(s_1s_2-m_1m_2|s_1s_2q-\kappa)\\ &\times(s_1's_2'-m_1'm_2'|s_1's_2'q'-\kappa')\\ &\times(l_1'l_2'-\mu_1'\mu_2'|l_1'l_2'r'\kappa'-\kappa)(l_1s_10m_1|l_1s_1J_1m_1)\\ &\times(l_2s_20m_2|l_2s_2J_2m_2)(l_1's_1'\mu_1'm_1'|l_1's_1'J_1m_1)\\ &\times(l_2's_2'\mu_2'm_2'|l_2's_2'J_2m_2)],\quad(2.9)\end{aligned}$$

where the first sum is over $l_1l_2l_1'l_2's_1s_2s_1's_2'J_1J_2\pi_1\pi_2$ and the auxiliary variable r'.

It has been assumed in this expression that there is only a single input tensor moment $T_\kappa{}^q$. (The effects of several input moments are additive.) Note that only two magnetic quantum numbers are still independent, since $\kappa=m_1-m_2$ and $\kappa'=m_1'-m_2'$. The remaining geometrical sums are eliminated in the next section.

III. ELIMINATION OF MAGNETIC SUMS AND GENERAL EXPRESSION FOR TENSOR MOMENTS

The sums over m_1 and m_1' may be eliminated by the repeated use of some Racah identities which have been summarized in a review paper by Biedenharn *et al.*[6] In particular, the identities BBR (1), (14), (18), (19), and the properties of the Fano X coefficient [reference 3 and also I, Appendix B] are needed. The details of this procedure, which is laborious but not illuminating, will not be given here, since the specific steps for a similar reduction were given in I.

The magnetic sum can be shown to reduce to the following expression

$$\begin{aligned}\sum_{r,L}&(-1)^{s_2'-s_2+l_2'+l_2}(2J_1+1)(2J_2+1)[(2q+1)(2q'+1)\\ &\times(2r'+1)(2r+1)]^{\frac{1}{2}}(l_1l_200|l_1l_2r0)\\ &\times(rq0\kappa|rqL\kappa)(r'q'\kappa-\kappa'\kappa'|r'q'L\kappa)\\ &\times X(J_1l_1s_1;Lrq;J_2l_2s_2)X(J_1l_1's_1';Lr'q';J_2l_2's_2'),\quad(3.1)\end{aligned}$$

where X is the coefficient defined by Fano,[3] and L and r are auxiliary variables. This expression is the generalization of I (3.1).

If Eq. (3.1) is substituted in Eq. (2.9), there results an expression for the final tensor moment $T_{\kappa'}{}^{q'}$. The tensor moment $T_{\kappa'}{}^{q'}$, however, is measured with respect to the coordinate system with z axis along the direction of the incident particle. A more convenient and symmetrical form results if the final tensor moment is measured with respect to a coordinate system along the direction of the final particle. If the incident beam direction is denoted by $\mathbf{k}$ and the scattered direction by $\mathbf{k}'$, let us choose a new coordinate system with z' axis along $\mathbf{k}'$ and with the y' axis along $\mathbf{k}\times\mathbf{k}'$. The Euler angles of this rotation are then $(\phi,\theta,0)$ relative to the original coordinate system.

The spin tensor operator $T_{\mu'}{}^{q'}$ in the new system is then related to the spin tensor operators $T_{\kappa'}{}^{q'}$ in the original coordinate system by the relation[7]

$$T_{\mu'}{}^{q'}=\sum_{\kappa'}D_{\kappa',\mu'}{}^{(q')}(\phi,\theta,0)T_{\kappa'}{}^{q'},$$

where $D_{\kappa',\mu'}{}^{(q')}$ is an element of the three-dimensional rotation group. The complete dependence on κ' of the expression for $T_{\kappa'}{}^{q'}$ is now contained in the factors

$$(-1)^{-\kappa'}(r'q'\kappa-\kappa'\kappa'|r'q'L\kappa)Y_{r',\kappa'-\kappa}(\theta,\phi).$$

Hence, the new tensor moment contains the transformed factors

$$\sum_{\kappa'}(-1)^{-\kappa'}D_{\kappa',\mu'}{}^{(q')}(\phi,\theta,0)(r'q'\kappa-\kappa'\kappa'|r'q'L\kappa)\times Y_{r',\kappa'-\kappa}(\theta,\phi).$$

The sum over κ' may be performed by expressing $Y_{r',\kappa'-\kappa}$ as a D function and using a relation for the product of two D functions [see the paragraph in I

[6] Biedenharn, Blatt, and Rose, Revs. Modern Phys. 24, 249 (1952); references to this paper will be designated by BBR.

[7] E. Wigner, *Gruppentheorie* (F. Vieweg, Braunschweig, 1931), p. 165.

following I (6.5)]. The transformed factors become

$$(-1)^{\kappa}(r'q'0\mu'|r'q'L\mu')D_{\kappa,\mu'}^{(L)}(\phi,\theta,0)[(2r'+1)/4\pi]^{\frac{1}{2}}.$$

The complete expression for the final spin tensor moment $T_{\kappa'}^{q'}$ (measured with respect to the scattered axis) resulting from an initial spin tensor moment T_{κ}^{q} (measured with respect to the incident axis) is obtained by substituting Eq. (3.1) in Eq. (2.9) and making the transformation described in the previous paragraph. The result is

$$\langle T_{\kappa'}^{q'}\rangle=\frac{\lambda_\alpha^2(2i)![(2i'-q')!(2i'+q'+1)!]^{\frac{1}{2}}(2q+1)^{\frac{1}{2}}P_{q'}([i'/(i'+1)]^{\frac{1}{2}})}{4(2I+1)(2i')![(2i-q)!(2i+q+1)!]^{\frac{1}{2}}(2q'+1)^{\frac{1}{2}}P_q([i/(i+1)]^{\frac{1}{2}})}\sum R^*(\alpha l_1s_1,\alpha'l_1's_1';J_1\pi_1)$$
$$\times R(\alpha l_2s_2,\alpha'l_2's_2',J_2\pi_2)W(is_1is_2;Iq)W(i's_1'i's_2';I'q')(-1)^{s_2+s_2'+i-I+i'-I'+\kappa'}D_{\kappa,\kappa'}^{(L)}(\phi,\theta,0)T_\kappa^q$$
$$\times G_\kappa^*(J_1l_1s_1,L\quad q,J_2l_2s_2)G_{\kappa'}(J_1l_1's_1',L\quad q',J_2l_2's_2'),\quad(3.2)$$

where the sum is over $l_1l_2l_1'l_2's_1s_2s_1's_2'J_1J_2\pi_1\pi_2L$, and where the G function is defined in terms of the X function as (note that this definition has been changed somewhat from that given in a previous communication[8]):

$$\begin{aligned}G_\kappa(J_1l_1s_1,L\quad q,J_2l_2s_2)\\=[(2l_1+1)(2l_2+1)(2s_1+1)(2s_2+1)]^{\frac{1}{2}}\\\times[(2J_1+1)(2J_2+1)(2q+1)(2L+1)]^{\frac{1}{2}}\\\times i^{l_1+l_2}\textstyle\sum_r(l_1l_200|l_1l_2r0)(qL\kappa-\kappa|qLr0)\\\times X(J_1l_1s_1,Lrq,J_2l_2s_2).\quad(3.3)\end{aligned}$$

Equation (3.2) is the most general result derived in this paper. All selection rules are contained in the G coefficient. These selection rules are discussed in the next section. A short summary of the properties of the G function is contained in Appendix B. In addition, two alternative expressions for G are given in Eqs. (B.7) and (B.8). Although these expressions are less symmetric than Eq. (3.3), they are much simpler for computational purposes and for many algebraic reductions, such as occur in changes of representation.

IV. SELECTION RULES AND DEGREE OF ANGULAR COMPLEXITY

The selection rules for any nuclear problem, corresponding to specific choices of q, q', κ, and κ', can easily be read off from the properties of the G coefficient. Some of these are mentioned in the next section. A few general statements may also be made.

(a) If the subscripts 1 and 2 are interchanged in Eq. (3.2), the resulting expression may be restored to its original form by the use of the symmetry properties of the G function and BBR (14). The only new factor appearing after this process is the phase factor $(-1)^{q+q'}$. Hence, if $q+q'$ is even, the elements of the R matrix in a calculation can always be written in the form $R^*(1)R(2)+R(1)R^*(2)$. If $q+q'$ is odd, an interference term $R^*(1)R(2)-R(1)R^*(2)$ always results.

It was pointed out in I (footnote 13) that the elements of the S matrix resulting from the use of perturbation theory or the Born approximation are all real. Hence, in these approximations, there will be no contribution to a process in which $q+q'$ is odd.

(b) If both κ and κ' are zero, Eq. (B.4) along with conservation of parity requires that $q+q'$ be even. Thus, for example, there can be no transitions from $q=0$, $\kappa=0$ to $q=1$, $\kappa=0$. This is the reason why the polarization produced in a nuclear reaction lies along the tangent plane.

(c) The maximum complexity of the angular dependence is given by the largest permissible value of L. If there is a largest incident orbital angular momentum $l_{\max}$, total angular momentum $J_{\max}$ or final orbital momentum $l'_{\max}$, the largest value of L is given by the simultaneous conditions

$$L\leq\begin{cases}2l_{\max}+q\\2l_{\max}+q-1&(\kappa'=0,\kappa\neq0,q+q'\text{ odd})\\2J_{\max}\\2l'_{\max}+q'\\2l'_{\max}+q'-1&(\kappa=0,\kappa'\neq0,q+q'\text{ odd}).\end{cases}$$

These rules are easily seen to follow from Eqs. (B.2), (B.3), and (B.4).

V. APPLICATIONS TO SPECIFIC PROCESSES

An expression for a given reaction measurement is obtained by a particular choice of q, q', κ, and κ' in Eq. (3.2). For example, an initially unpolarized beam is specified by $T_\kappa^q=1$ for $q=\kappa=0$ and zero for all other values. It is convenient to list the values of some of the simpler tensor moments in terms of spin operators $\mathbf{s}$:

$$T_0^0=1,\quad T_0^1=\mathbf{s}_z/[s(s+1)]^{\frac{1}{2}},$$
$$T_{\pm1}^1=\mp(\mathbf{s}_x\pm i\mathbf{s}_y)/[2s(s+1)]^{\frac{1}{2}},$$
$$T_0^2=[3\mathbf{s}_z^2-s(s+1)]/[2s(s+1)],$$
$$T_{\pm1}^2=\mp(\sqrt{6}/4)[(\mathbf{s}_x\pm i\mathbf{s}_y)\mathbf{s}_z+\mathbf{s}_z(\mathbf{s}_x\pm i\mathbf{s}_y)]/[s(s+1)],$$
$$T_{\pm2}^2=(\sqrt{6}/4)(\mathbf{s}_x\pm i\mathbf{s}_y)(\mathbf{s}_x\pm i\mathbf{s}_y)/[s(s+1)].$$

Angular Distribution of Nuclear Reactions

The angular distribution resulting from an unpolarized initial beam corresponds to $q'=q=\kappa'=\kappa=0$. The result given in BB (3.16), (4.5), and (4.6) is obtained

[8] A. Simon, Phys. Rev. **90**, 991 (1953). Equation (1) of this letter has a misprint in it. The phase factor should be $(-1)^\kappa$ rather than $(-1)^{\kappa'}$.

from Eq. (3.2) by use of Eq. (B.9) and BBR (30).

$$\frac{d\sigma}{d\Omega}=\frac{\lambda_\alpha^2}{4(2i+1)(2I+1)}\sum(-1)^{s'-s}R^*(\alpha s l_1,\alpha' s' l_1';J_1\pi_1)\times R(\alpha s l_2,\alpha' s' l_2';J_2\pi_2)Z(l_1J_1l_2J_2,sL)\times Z(l_1'J_1l_2'J_2;s'L)P_L(\cos\theta),\quad(5.1)$$

where the sum is over $ss'l_1l_2l_1'l_2'J_1J_2\pi_1\pi_2$ and L.

Polarized Particles From Nuclear Reactions

The result given in I(3.2) for the polarization resulting from a reaction initiated by unpolarized particles is easily obtained by setting $q'=1$, $\kappa'=\pm1$ and $q=\kappa=0$. Note that I(3.2) is the expectation value of the operator $\mathbf{i}'/[i'(i'+1)]^{\frac{1}{2}}$. This is related to f, the percentage polarization in the final state, by

$$d|\mathbf{P}|=[i'/(i'+1)]^{\frac{1}{2}}fd\sigma,\quad(5.2)$$

where $d\sigma$ is the differential cross section for the reaction.

Detection of Polarized Particles of Spin 1/2

The most convenient way to detect the polarization of a spin-$\frac{1}{2}$ particle is to observe the angular distribution of a subsequent reaction initiated by this particle. A general expression for this cross section can be derived.

Let $\mathbf{k}_1$ be the direction of an initially unpolarized beam, which produces a polarized particle of spin i $(i=\frac{1}{2})$. The intermediate particle i is taken off in the direction $\mathbf{k}_2$, which is specified by the angles θ, ϕ with respect to $\mathbf{k}_1$, and allowed to bombard an unpolarized nucleus of spin I. The final reaction product which is of spin i' is observed in the direction $\mathbf{k}_3$, which has the angles $\theta'\phi'$ relative to $\mathbf{k}_2$. The spin of the final residual nucleus is I'. (Note that *only* the intermediate particle i need be of spin $\frac{1}{2}$.)

The initial system in state 1, being unpolarized, will contain only the tensor moment $q=0$, $\kappa=0$. The intermediate system then contains the three tensor moments T_0^0, T_1^1, and T_{-1}^1. (Note that T_0^1 is excluded by selection rule (b) of Sec. IV; all tensor moments are measured with respect to the beam axis.) These tensors contribute additively to the final angular distribution, which is the expectation value of T_0^0 in the final state.

One component of the final measurement can be written symbolically as $T_0^0(1)\rightarrow T_0^0(2)\rightarrow T_0^0(3)$, which is clearly equivalent to $d\sigma(\theta)d\sigma(\theta')$. Here $d\sigma(\theta)$ is the differential cross section for the first reaction, and $d\sigma(\theta')$ is what the cross section would be for the second reaction *if the intermediate beam were unpolarized.* The remaining components are simplified if one verifies that by the symmetry properties of the R matrix, the properties of the D function,[7] and Eq. (3.2)

$$T_{\pm1}^1(2)\rightarrow T_0^0(3)=3(2i'+1)(2I'+1)/[(2i+1)(2I+1)]\times[T_0^0(3)\rightarrow T_{\mp1}^1(2)].\quad(5.3)$$

Now by Eq. (5.2) the last bracket in the above expression can be written essentially as

$$[i/(i+1)]^{\frac{1}{2}}f(3\rightarrow2)d\sigma(3\rightarrow2),\quad(5.4)$$

where $f(3\rightarrow2)$ is the percentage polarization which would result if the second reaction were reversed in time, with the system in state 3 being unpolarized. But clearly

$$(2i'+1)(2I'+1)d\sigma(3\rightarrow2)=(2i+1)(2I+1)d\sigma(2\rightarrow3)\equiv(2i+1)(2I+1)d\sigma(\theta').\quad(5.5)$$

If we combine Eqs. (5.2), (5.3), (5.4), and (5.5), the second component of the final angular distribution can be written as

$$T_0^0(1)\rightarrow T_{\pm1}^1(2)\rightarrow T_0^0(3)=\mathbf{n}_1\cdot\mathbf{n}_2f(1\rightarrow2)f(3\rightarrow2)d\sigma(\theta)d\sigma(\theta'),$$

where $\mathbf{n}_1\sim\mathbf{k}_1\times\mathbf{k}_2$ and $\mathbf{n}_2\sim\mathbf{k}_2\times\mathbf{k}_3$ are unit vectors along the indicated normals to the scattering planes.

The observed angular distribution has the complete form

$$d\sigma(\theta)d\sigma(\theta')[1+\mathbf{n}_1\cdot\mathbf{n}_2f(1\rightarrow2)f(3\rightarrow2)].\quad(5.6)$$

Hence, the polarization is directly related to the azimuthal asymmetry of the second reaction. A similar result was given by Lepore[9] for the special case of the double elastic scattering of spin $\frac{1}{2}$ particles by spin zero nuclei. With these restrictions $f(3\rightarrow2)=f(2\rightarrow3)$, and Lepore's result follows.

Radiations from Polarized Nuclei

The problem of the decay of an arbitrarily polarized initial nucleus can be regarded as a special case of the general result given in Eq. (3.2). If the radiating nucleus is taken to be in the state designated by $\alpha J\pi$, the specialization is made by the following recipe:

(a) Replace $S(\alpha l_1s_1,\alpha' l_1' s_1',J_1\pi_1)$ with

$$S(\alpha0J,\alpha' l_1' s_1';J\pi)\delta(l_1,0)\delta(s_1,J).$$

(*b*) Set $i=J$ and $I=0$.

(*c*) Divide Eq. (3.2) by $\pi\lambda_\alpha^2$.

This recipe assures that there will be only a single initial state of the entire system of angular momentum J. In addition, the amplitude of this state is chosen to be unity.

The application of this recipe to Eq. (3.2) results in the following expression

$$T_{\kappa'}^{q'}=\frac{(2J)![(2i'-q')!(2i'+q'+1)!(2q+1)]^{\frac{1}{2}}P_{q'}([i'/(i'+1)]^{\frac{1}{2}})}{4\pi(2i')![(2J-q)!(2J+q+1)!(2q'+1)]^{\frac{1}{2}}P_q([J/(J+1)]^{\frac{1}{2}})}R^*(\alpha0J,\alpha' l_1' s_1';J\pi)R(\alpha0J,\alpha' l_2' s_2';J\pi)\times W(i's_1'i's_2';I'q')(-1)^{s_2'+i'-I'+\kappa'+\kappa}D_{\kappa,\kappa'}^{(q)}(\phi,\theta,0)G_{\kappa'}(Jl_1's_1',q\ \ q',Jl_2's_2')T_\kappa^q,\quad(5.7)$$

[9] J. V. Lepore, Phys. Rev. 79, 139 (1950).

where the sum is only over $l_1'l_2's_1'$ and s_2'. Here $T_\kappa{}^q$ is a spin tensor moment of the arbitrarily polarized initial nucleus. The final tensor moment $T_{\kappa'}{}^{q'}$ of the particle i' is measured with respect to its direction of motion $\mathbf{k}'$. The angles θ, ϕ relate $\mathbf{k}'$ to the initial axis of quantization of J.

Expressions for the angular distribution ($q'=\kappa'=0$) and polarization ($q'=1$, $\kappa'=\pm1$) of the emitted particles are easily obtained from Eq. (5.7). Up to the present time, chief interest has been centered on gamma radiation by polarized nuclei. In this case, the channel spin representation is not convenient; and, instead, one must use a representation involving the multipole expansions of the gamma-ray field. This point is treated in the next section. It is worth noting that, when the nuclear polarization has been accomplished by magnetic or electric fields, the resultant cylindrical symmetry requires that only initial tensor moments with $\kappa=0$ appear. The angular dependence then simplifies since

$$D_{0,\kappa}{}^{(q)}(\phi,\theta,0)=(-1)^{(|\kappa|-\kappa)/2}\times[(q-|\kappa|)!/(q+|\kappa|)!]^{\frac{1}{2}}P_q{}^\kappa(\theta),$$

where $P_q{}^\kappa$ is the usual associated Legendre function.

VI. GAMMA RAYS IN NUCLEAR REACTIONS

The S-matrix formalism may be extended to include the possibility of gamma rays in nuclear reactions. In this case, the vector potential $\mathbf{A}$ plays the role of the "wave function" for the particle. A formal proof of this equivalence which will not be given here requires the procedure of second quantization.[10]

The vector potential field is customarily expanded in terms of electric and magnetic multipoles, $\mathbf{A}(L,M,p)$. The total angular momentum of the multipole and the z component are denoted by L and M. The "parity" symbol p, which is defined to have the value 0 for magnetic radiation and 1 for electric radiation, is related to the actual parities in the vector field by

$$\begin{aligned}\text{parity}&=(-1)^L \quad \text{for } p=0\text{, magnetic}\\ &=(-1)^{L+1} \quad p=1\text{, electric.}\end{aligned}$$

The specific definition of the multipoles which will be used in this section is as follows:

$$\mathbf{A}(L,M,p)=-\sqrt{2}\sum_{\kappa,l}{}'(l\,1\,M-\kappa\,\kappa|l\,1\,L\,M)\times(L\,1\,-1\,1|L\,1\,l\,0)i^{l-L}\mathbf{u}_\kappa Y_{l,M-\kappa}(\theta,\phi)\times\exp[i(kr-l\pi/2)], \quad (6.1)$$

where the prime on the summation symbol indicates that only the term $l=L$ is to be included in the sum if $p=0$ and only the terms $l=L\pm1$ if $p=1$. The symbol $\mathbf{u}_\kappa$ denotes an irreducible component of the vector part of $\mathbf{A}$ and is defined in terms of the unit vectors $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ along the coordinate axis as

$$\mathbf{u}_{\pm1}=\mp(\mathbf{e}_1\pm i\mathbf{e}_2)/\sqrt{2}, \quad \mathbf{u}_0=\mathbf{e}_3.$$

Note that $\mathbf{A}(L,M,p)$ as written above consists of outgoing spherical waves only. The incoming spherical waves have the same definitions, but with a reversed sign in the exponential.

The normalization of $\mathbf{A}(L,M,p)$ has been chosen so that its absolute square integrates to unity over the solid angle. The first Clebsch-Gordan factor has the algebraic values:

$$(L\,1\,-1\,1|L\,1\,L\,0)=-1/\sqrt{2};$$
$$(L\,1\,-1\,1|L\,1\,L+1\,0)=[L/2(2L+1)]^{\frac{1}{2}};$$
$$(L\,1\,-1\,1|L\,1\,L-1\,0)=[(L+1)/2(2L+1)]^{\frac{1}{2}};$$

and in this form the multipoles may be recognized as essentially those given by Goertzel.[11] We prefer to keep this factor in the form of a Clebsch-Gordan coefficient, since a single algebraic expression will then suffice for all multipoles. The multipoles as defined above are everywhere transverse to the radial direction, as may be easily verified. The remaining longitudinal multipoles, which are excluded in the case of the electromagnetic field, are obtained immediately by replacing

$$(L\,1\,-1\,1|L\,1\,l\,1) \quad \text{by} \quad (L\,1\,0\,0|L\,1\,l\,0)$$

in Eq. (6.1).

The composite system consisting of a gamma-ray and a residual (or target) nucleus of spin I is now specified by the quantum numbers αJm_JLp, where J and m_J are the total angular momentum and z component of the entire system. This representation is very similar to the representation in which the angular momentum and spin of a bombarding "particle" (in this case a gamma ray of spin 1) are combined to form a definite j (which is the L of the multipole), which then combines with the nuclear spin I to form a total state specified by J and m_J. Our previous formulas for the particle reactions may now be extended to include gamma rays by simply changing from the channel spin to the multipole representation. Let us consider first the case of a reaction in which a particle comes in and a gamma ray comes out.

Particle In and Gamma Ray Out

For this reaction we wish to use a mixed representation for the R matrix in which the initial states are still in the channel spin representation, and the final states are in the multipole representation. By the usual transformation theory,

$$R(\alpha ls,\alpha'l's',J\pi)\rightarrow\sum_{L'p'}R(\alpha ls,\alpha'L'p';J\pi)\times(L'p'J\pi\|l's'J\pi). \quad (6.2)$$

Racah[12] has shown that the transformation matrix,

[10] See, e.g., J. A. Spiers, *Directional Effects in Radioactivity* (National Research Council of Canada, Ontario, 1949), Appendix II.

[11] G. Goertzel, Phys. Rev. **70**, 905 (1946).

[12] G. Racah, Phys. Rev. **63**, 367 (1943).

which represents the recoupling of three angular moments (l, 1, and I) must have the form of a Racah function. It is easy to show that the exact relation is

$$(L'p'J\pi\|l's'J\pi) = -\sqrt{2}[(2L'+1)(2s'+1)]^{\frac{1}{2}}W(l'\,1\,J\,I';L's') \times(L'\,1\,-1\,1|L'\,1\,l'\,0)i^{L'-l'}\delta(l',p'), \quad (6.3)$$

where the δ symbol vanishes if l' does not have the proper parity corresponding to p' and has the value unity otherwise. If Eqs. (6.2) and (6.3) are now substituted in the general expression of Eq. (3.2), the sums over s_1', s_2', l_1', and l_2' may then be performed without any detailed knowledge of the R matrix. (Note that $i'=1$.) The sum over s_2' is performed first by the use of BBR (17). Then this identity is used again to obtain the sum over s_1'.

We prefer not to do the sum on l_1' and l_2' explicitly in order that a single expression be valid for all choices of the multipole "parity." The final result is

$$T_{\kappa'}{}^{q'} = \frac{\lambda_\alpha{}^2(2i)![(2-q')!(3+q')!]^{\frac{1}{2}}(2q+1)^{\frac{1}{2}}P_{q'}(\sqrt{\tfrac{1}{2}})}{12(2I+1)[(2i-q)!(2i+q+1)!(2q'+1)]^{\frac{1}{2}}P_q([i/i+1]^{\frac{1}{2}})} \sum R^*(\alpha l_1 s_1, \alpha' L_1' p_1'; J_1\pi_1)$$
$$\times R(\alpha'_2 s_2, \alpha' L_2' p_2'; J_2\pi_2)(L_1'\,1\,-1\,1|L_1'\,1\,l_1'\,0)(L_2\;1\,-1\,1|L_2'\,1\,l_2'\,0)\delta(l_1',p_1')\delta(l_2',p_2')$$
$$\times[(2J_1+1)(2J_2+1)]^{\frac{1}{2}}i^{L_2'-L_1'+l_1'-l_2'}(-1)^{J_2+s_2+L_1'+q'-L+i-I-I'+\kappa'}W(is_1is_2;Iq)W(J_1L_1'J_2L_2';I'L)$$
$$\times D_{\kappa,\kappa'}{}^{(L)}(\phi\theta 0)\,T_\kappa{}^q\,G_\kappa{}^*(J_1l_1s_1;L\quad q;J_2l_2s_2)G_{\kappa'}(L_1'l_1'1;L\quad q';L_2'l_2'1), \quad (6.4)$$

where the sum is over $l_1l_2s_1s_2J_1J_2\pi_1\pi_2L_1'L_2'p_1'p_2'l_1'$ and l_2'. It is worth reminding the reader that the δ symbols signify that for magnetic multipole radiation, $p=0$, we have $l=L$. For electric multipole radiation $p=1$, we have $l=L\pm1$.

The meaning of the final spin tensor moments $T_{\kappa'}{}^{q'}$ for the gamma-ray case requires some clarification. It is clear that a pure gamma-ray state requires only two states U_1 and U_{-1}, corresponding to right and left circular polarization respectively for its description. Hence the statistical density matrix will have only four independent components, rather than nine as would be the case for the usual particle of spin one. Since our selection rules appear to allow all tensor moments up to $q'=2$, it is also clear that only four of these moments may be independent in the gamma-ray case.

A comparison with the listing of spin operators at the beginning of Sec. V resolves the ambiguity. First of all, it is easy to recognize that the tensor moments $T_1{}^1$, $T_{-1}{}^1$, $T_1{}^2$, and $T_{-1}{}^2$ must be identically zero. To see this, note that these four operators are proportional to the spin operators s_+ or s_- which have nonzero matrix elements only between states whose magnetic components differ by unity. The only available states for the gamma ray are U_1 and U_{-1} which differ by 0 or 2. In addition, the operator $T_0{}^2$ is equivalent to $T_0{}^0$ since both these states have $s_z{}^2=1$. Hence, the four independent operators are $T_0{}^0$, $T_0{}^1$, $T_2{}^2$, and $T_{-2}{}^2$. These have simple relations to the physical measurements which are listed below:

$T_0{}^0$	intensity	
$\sqrt{2}\,T_0{}^1$	circular polarization intensity	
$4\sqrt{\frac{2}{3}}\,T_2{}^2$	$Ae^{-i2\phi}$	where A is the intensity of the linear polarization, and ϕ is its azimuthal angle relative to the normal to the scattering plane.
$4\sqrt{\frac{2}{3}}\,T_{-2}{}^2$	$Ae^{i2\phi}$	

These parameters are essentially the Stoke's parameter description of the polarization of a beam of light as has been clearly pointed out by Fano.[3] A more complete discussion may be found in Appendix A of this reference.

The relations given above between the polarization of a gamma-ray and its tensor moments allows several selection rules to be read off from Eq. (6.4). These are:

(a) Circularly polarized light cannot result from a nuclear reaction initiated by unpolarized particles (i.e., no $T_0{}^0 \rightarrow T_0{}^1$). This is a special case of the selection rule given in Sec. IV (b).

(b) Linearly polarized light cannot be the product of a nuclear reaction initiated by s-wave particles of spin $\frac{1}{2}$.[13] To see this, note that the "tetrad" condition on the first G function ($l_1\,l_2\,L\,q$) requires that L be either 1 or 0. However, $\kappa'=\pm2$ for linear polarization. Hence, the second G function must vanish by Eq. (B.5).

Gamma Ray In and Particle Out

The derivation of this result follows in a manner similar to the previous case. The only added difficulty is that one must recall that incident gamma-rays of right and left circular polarizations, upon expansion into multipoles, have opposite phases for the amplitudes of the electric components. In particular, if one expands a plane wave along the z axis of circular polarization P ($P=\pm1$) into multipoles, the probability amplitude of the state of multipolarity L and parity p ($=0$ or 1) is $(P)^p i^{L+1}[(2L+1)\pi/2]^{\frac{1}{2}}\lambda$, which differs from the particle case by the extra factor $(P)^p$.

[13] This result was also obtained by Biedenharn, Rose, and Arfken, Phys. Rev. **83**, 683 (1951), by invoking Lloyd's theorem. The use of Lloyd's theorem, which is only an approximate relation [see F. Coester, Phys. Rev. **89**, 620 (1953)], is unnecessary in this problem since the result is an exact selection rule. To see this, note that the sums over m and M in Eq. (3) of Biedenharn *et al.* can be performed and reduce the anisotropic term to zero identically. The alternative proof of Lloyd's theorem given in footnote 5 of this reference is then invalid.

Now if the initial density matrix is expressed in terms of the averaged amplitudes of the states of circular polarization $a(+1)$ and $a(-1)$, we have:

$$D_{1,1}=|a(+1)|^2,$$
$$D_{-1,-1}=|a(-1)|^2,$$
$$D_{1,-1}=a^*(+1)a(-1),$$
$$D_{-1,1}=a(+1)a^*(-1).$$

As we have just seen, however, the terms $a(-1)$ will have the factors $(-1)^p$ in the multipole expansion. In addition, it is clear from the spin operator definition of the tensor moments that

$$T_0^0=D_{1,1}+D_{-1,-1},$$
$$T_0^1\sim D_{1,1}-D_{-1,-1},$$
$$T_2^2\sim D_{1,-1},$$
$$T_{-2}^2\sim D_{-1,1}.$$

Hence, the proper initial tensor moments for the gamma-ray case are:

$$T_0^0[1+(-1)^{p_1+p_2}]/2,$$
$$T_0^1[1-(-1)^{p_1+p_2}]/2,$$
$$T_2^2(-1)^{p_2},$$
$$T_{-2}^2(-1)^{p_1}.$$

In addition, the new expression must be multiplied by $\frac{3}{2}$ to correct for the statistical factor. The final expression for the case of a gamma-ray incident and a particle emerging is then:

$$T_{\kappa'}{}^{q'}=\frac{\lambda_\gamma^2[(2i'-q')!(2i'+q'+1)!(2q+1)]^{\frac{1}{2}}P_{q'}([i'/i'+1]^{\frac{1}{2}})}{2(2I+1)(2i')![(2-q)!(3+q)!(2q'+1)]^{\frac{1}{2}}P_q(1/\sqrt{2})}\sum R^*(\alpha L_1p_1,\alpha' l_1's_1';J_1\pi_1)R(\alpha L_2p_2,\alpha' l_2's_2';J_2\pi_2)$$
$$\times(L_1\,1\,-1\,1|L_1\,1\,l_1\,0)(L_2\,1\,-1\,1|L_2\,1\,l_2\,0)\delta(l_1,p_1)\delta(l_2,p_2)[(2J_1+1)(2J_2+1)]^{\frac{1}{2}}\,i^{L_1-L_2-l_1+l_2}$$
$$\times(-1)^{J_2+s_2'+L_1+q-L+i'-I'-I+\kappa'}W(J_1L_1J_2L_2;IL)W(i's_1'i's_2';I'q')D_{\kappa,\kappa'}{}^{(L)}(\phi,\theta,0)$$
$$\times G_\kappa^*(L_1\,l_1\,1;L\quad q;L_2\,l_2\,1)G_{\kappa'}(J_1l_1's_1';L\quad q';J_2l_2's_2')f(p_1p_2q\kappa)T_\kappa{}^q,\quad(6.5)$$

where

$$f(p_1p_2q\kappa)=[1+(-1)^{p_1+p_2}]/2\quad\text{for } q=\kappa=0,$$
$$=[1-(-1)^{p_1+p_2}]/2\qquad q=1,\ \kappa=0,$$
$$=(-1)^{p_2}\qquad q=2,\ \kappa=2,$$
$$=(-1)^{p_1}\qquad q=2,\ \kappa=-2.$$

Gamma Ray In and Gamma Ray Out

This expression follows immediately from the previous results.

$$T_{\kappa'}{}^{q'}=\frac{\lambda_\gamma^2[(2-q')!(3+q')!(2q+1)]^{\frac{1}{2}}P_{q'}(1/\sqrt{2})}{6(2I+1)[(2-q)!(3+q)!(2q'+1)]^{\frac{1}{2}}P_q(1/\sqrt{2})}$$
$$\times R^*(\alpha L_1p_1,\alpha' L_1'p_1';J_1\pi_1)R(\alpha L_2p_2,\alpha' L_2'p_2';J_2\pi_2)$$
$$\times(L_1\,1\,-1\,1|L_1\,1\,l_1\,0)(L_2\,1\,-1\,1|L_2\,1\,l_2\,0)$$
$$\times(L_1'\,1\,-1\,1|L_1'\,1\,l_1'\,0)(L_2'\,1\,-1\,1|L_2'\,1\,l_2'\,0)$$
$$\times\delta(l_1p_1)\delta(l_2p_2)\delta(l_1'p_1')\delta(l_2'p_2')(2J_1+1)(2J_2+1)$$
$$\times i^{L_1-L_2-l_1+l_2-L_1'+L_2'+l_1'-l_2'}(-1)^{I-I'+L_1'+L_1+q+q'+\kappa'}$$
$$\times W(J_1L_1J_2L_2;IL)W(J_1L_1'J_2L_2';I'L)$$
$$\times D_{\kappa,\kappa'}{}^{(L)}(\phi,\theta,0)T_\kappa{}^qG_\kappa^*(L_1l_11;L\quad q;L_2l_21)$$
$$\times G_{\kappa'}(L_1'l_1'1;L\quad q';L_2'l_2'1)f(p_1p_2q\kappa),\quad(6.6)$$

where f has the same definition as in Eq. (6.5).

Angular Distribution of Nuclear Reactions Involving Gamma Rays

The most frequent use of the previous general results are in the analysis of angular distributions of nuclear reactions. For this reason, it is convenient to list the results for the special case of the angular distribution ($q'=\kappa'=0$) resulting from a reaction initiated by unpolarized beams ($q=\kappa=0$). The results are expressed in terms of the Z coefficient defined in BB (4.3).

Particle In and Gamma Ray Out

$$d\sigma=(\lambda^2/2)(2i+1)^{-1}(2I+1)^{-1}$$
$$\times\sum R^*(\alpha l_1s,\alpha' L_1'p_1';J_1\pi_1)$$
$$\times R(\alpha l_2s,\alpha' L_2'p_2';J_2\pi_2)(L_1'\,1\,-1\,1|L_1'\,1\,l_1'\,0)$$
$$\times(L_2'\,1\,-1\,1|L_2'\,1\,l_2'\,0)\delta(l_1',p_1')\delta(l_2',p_2')$$
$$i^{L_2'-L_1'+l_1'-l_2'}(-1)^{s-I'-L+L_1'-L_2'+1}$$
$$\times W(J_1L_1'J_2L_2';I'L)Z(l_1J_1l_2J_2;sL)P_L(\theta)$$
$$\times Z(l_1'L_1'l_2'L_2';1\,L)$$
$$\times[(2J_1+1)(2J_2+1)]^{\frac{1}{2}}d\Omega,\quad(6.7)$$

where the sum is over $l_1l_2L_1'L_2'l_1'l_2'p_1'p_2'sJ_1J_2\pi_1\pi_2$ and L.

Gamma Ray In and Particle Out

$$d\sigma=(\lambda^2/4)(2I+1)^{-1}\sum R^*(\alpha L_1p,\alpha' l_1's';J_1\pi_1)$$
$$\times R(\alpha L_2p,\alpha' l_2's';J_2\pi_2)(L_1\,1\,-1\,1|L_1\,1\,l_1\,0)$$
$$\times(L_2\,1\,-1\,1|L_2\,1\,l_2\,0)\delta(l_1p)\delta(l_2p)$$
$$\times[(2J_1+1)(2J_2+1)]^{\frac{1}{2}}i^{L_2-L_1+l_1-l_2}$$
$$\times(-1)^{s'-I-L+L_1-L_2+1}W(J_1L_1J_2L_2;IL)$$
$$\times Z(l_1L_1l_2L_2;1\,L)Z(l_1'J_1l_2'J_2;s'L)$$
$$\times P_L(\theta)d\Omega,\quad(6.8)$$

where the sum is over $L_1L_2l_1l_2J_1J_2\pi_1\pi_2l_1'l_2's'p$ and L.

Gamma Ray In and Gamma Ray Out

$$d\sigma=(\lambda\!\!\!^{-2}/2)(2I+1)^{-1}\sum R^*(\alpha L_1p,\alpha' L_1'p_1';J_1\pi_1)R(\alpha L_2p,\alpha' L_2'p_2';J_2\pi_2)(L_1\,1\,-1\,1|L_1\,1\,l_1\,0)$$
$$\times(L_2\,1\,-1\,1|L_2\,1\,l_2\,0)(L_1'\,1\,-1\,1|L_1'\,1\,l_1'\,0)(L_2'\,1\,-1\,1|L_2'\,1\,l_2'\,0)\delta(l_1p)\delta(l_2p)$$
$$\times\delta(l_1',p_1')\delta(l_2',p_2')(2J_1+1)(2J_2+1)\mathrm{i}^{L_1-L_2-l_1+l_2-L_1'+L_2'+l_1'-l_2'}(-1)^{I-I'+L_1'+L_1-L_2'-L_2}W(J_1L_1J_2L_2;IL)$$
$$\times W(J_1L_1'J_2L_2';I'L)Z(l_1L_1l_2L_2;1L)Z(l_1'L_1'l_2'L_2';1L)P_L(\theta)d\Omega,\quad(6.9)$$

where the sum is over $L_1L_2L_1'L_2'l_1l_2l_1'l_2'J_1J_2\pi_1\pi_2pp_1'p_2'$ and L.

Gamma Radiation from Polarized Nuclei

The general result for gamma radiation is obtained by applying the recipe given under the last subheading of Sec. V to Eq. (6.4) .The result is

$$T_{\kappa'}{}^{q'}=\frac{(2J+1)![(2-q')!(3+q')!(2q+1)]^{\frac{1}{2}}P_{q'}(\sqrt{\tfrac{1}{2}})}{12\pi[(2J-q)!(2J+q+1)!(2q'+1)]^{\frac{1}{2}}P_q([J/J+1]^{\frac{1}{2}})}\sum R^*(\alpha 0J,\alpha' L_1'p_1';J\pi)R(\alpha 0J,\alpha' L_2'p_2';J\pi)$$
$$\times(L_1'\,1\,-1\,1|L_1'\,1\,l_1'\,0)(L_2'\,1\,-1\,1|L_2'\,1\,l_2'\,0)\delta(l_1',p_1')\delta(l_2',p_2')\mathrm{i}^{L_2'-L_1'+l_1'-l_2'}$$
$$\times(-1)^{L_1'+J-I'+q'+q+\kappa+\kappa'}D_{\kappa,\kappa'}{}^{(q)}(\phi,\theta,0)T_\kappa{}^qW(JL_1'JL_2';I'q)G_{\kappa'}(L_1'l_1'1;q\quad q';L_2'l_2'1),\quad(6.10)$$

where the sum is over $L_1'L_2'l_1'l_2'p_1'$ and p_2'. Note that the polarized nucleus has spin J and parity π. Its initial tensor moments are $T_\kappa{}^q$. If polarization is achieved by electric or magnetic fields, only the terms with $\kappa=0$ are present.

If we specialize the above expression, we obtain several results already given in the literature.[14]

VII. POLARIZED TARGET NUCLEI

The previous results in this paper have all been specialized to the case of a polarized beam of particles of spin i incident upon an unpolarized target nucleus of spin I. The more general problem is one in which the target nucleus is polarized as well. (We assume, however, that the incident particle and nucleus are incoherent.) The expansion of the density matrix in terms of tensor operators is no longer that given in Eq. (2.7) but rather has the form:

$$\rho(s_1m_1;s_2m_2)=\sum_q(-1)^{s_1-s_2-q+m_2}(s_1s_2-m_1m_2|s_1s_2q-\kappa)\mathfrak{T}_\kappa{}^q,\quad(7.1)$$

where

$$\mathfrak{T}_\kappa{}^q=\sum_{a,b,\lambda}\frac{(2i)!(2I)!(2a+1)(2b+1)[(2s_1+1)(2s_2+1)]^{\frac{1}{2}}}{[(2i-a)!(2i+a+1)!(2I-b)!(2I+b+1)!]^{\frac{1}{2}}}\times\frac{(-1)^{s_2}(ab\lambda\kappa-\lambda|abq\kappa)}{P_a([i/i+1]^{\frac{1}{2}})P_b([I/I+1]^{\frac{1}{2}})}\times X(iai;s_1qs_2;IbI)T_\lambda{}^a(i)T_{\kappa-\lambda}{}^b(I).\quad(7.2)$$

This result is derived in Appendix A. Equation (7.2) expresses the generalized tensor moment for the initial system in terms of the individual tensor moments $T_\lambda{}^a(i)$ and $T_{\kappa-\lambda}{}^b(I)$, defined in the usual manner, of the incident particle and target nucleus respectively.

All previous results in this paper are generalized to the case of polarized target nuclei by replacing $T_\kappa{}^q$ with $\mathfrak{T}_\kappa{}^q$ along with a change of normalization which is obvious from a comparison of Eqs. (7.1) and (2.7).

The author wishes to express his thanks to Dr. Theodore A. Welton for many valuable discussions and suggestions.

APPENDIX A. EXPANSION OF THE DENSITY MATRIX IN TERMS OF TENSOR OPERATORS

An element of the density matrix in the channel spin representation may be written as an average of the expectation value of an operator over the statistical ensemble. If $\chi(sm)$ is a channel spin wave function of channel spin s and z component m, an element of the density matrix is

$$\rho(s_1m_1;s_2m_2)=\langle\chi(s_1m_1)\chi^*(s_2m_2)\rangle_{\mathrm{Av}}.\quad(\mathrm{A}.1)$$

The channel spin wave function is defined in terms of the spin wave functions for the incident particle i and target nucleus I as

$$\chi(sm)=\sum_{m_i}(i\,I\,m_i\,m-m_i|i\,I\,s\,m)\chi(im_i)\chi(I\,m-m_i).\quad(\mathrm{A}.2)$$

If Eq. (A.2) is substituted in Eq. (A.1) and if it is assumed that wave functions of the target nucleus and incident particle are incoherent, one immediately obtains

$$\rho(s_1m_1;s_2m_2)=\sum_{m_im_i'}(i\,I\,m_i\,m_1-m_i|i\,I\,s_1\,m_1)\times(i\,I\,m_i'\,m_2-m_i'|i\,I\,s_2\,m_2)\times\rho_i(m_i,m_i')\rho_I(m_1-m_i,m_2-m_i'),\quad(\mathrm{A}.3)$$

[14] N. R. Steenberg, Proc. Phys. Soc. (London) **65**, 791 (1952) and **66**, 391 (1953); H. A. Tolhoek and J. A. M. Cox, Physica **29**, 101 (1953). These authors treat the angular distribution and polarization of gamma rays from aligned nuclei for the case of pure multipoles only. The correspondence to the notation of Tolhoek and Cox is as follows:

$$T_0{}^0=1;\quad T_0{}'=P\xi_3/\sqrt{2};\quad T_2{}^2+T_{-2}{}^2=-\sqrt{6}P\xi_1/4;$$
$$\mathrm{i}(T_2{}^2-T_{-2}{}^2)=\sqrt{6}P\xi_2/4.$$

where ρ_i and ρ_I are the density matrices for the individual particles.

The general covariance properties of ρ may be obtained directly from Eq. (A.1), however. If P_r denotes the operation of a rotation r of the coordinate system, there follows immediately:

$$P_r\rho(s_1m_1;s_2m_2)=\sum_{\lambda,\lambda'} D_{\lambda,m_1}{}^{(s_1)}(r)D_{\lambda',m_2}{}^{*(s_2)}(r) \times\langle\chi(s_1\lambda)\chi^*(s_2\lambda')\rangle_{\text{Av}}, \quad (\text{A.4})$$

where D is an element of the rotation group. By Eq. (16a) of reference 7, the product of two D coefficients may be expressed as a single D coefficient. If we use this, Eq. (A.4) may be written

$$=\sum_{\lambda-\lambda',q}(-1)^{-m_2}(s_1\,s_2\,m_1-m_2|s_1\,s_2\,q\,m_1-m_2) \times D_{\lambda-\lambda',m_1-m_2}{}^{(q)}(r)\mathfrak{T}_{\lambda-\lambda'}{}^q, \quad (\text{A.5})$$

where the sum on λ and λ' has been replaced by a sum on $\lambda-\lambda'$ and λ', and where

$$\mathfrak{T}_{\lambda-\lambda'}{}^q=\sum_{\lambda'}(-1)^{\lambda'}(s_1\,s_2\,\lambda-\lambda'|s_1\,s_2\,q\,\lambda-\lambda') \times\langle\chi(s_1\lambda)\chi^*(s_2\lambda')\rangle_{\text{Av}}. \quad (\text{A.6})$$

It is clear from Eq. (A.5) that $\mathfrak{T}_{\lambda-\lambda'}{}^q$ transforms under rotation as a tensor of rank q. Hence in the limit of zero rotation, Eq. (A.5) goes into the desired expansion.

$$\rho(s_1m_1;s_2m_2)=\sum_q(-1)^{-m_2} \times(s_1\,s_2\,m_1-m_2|s_1\,s_2\,q\,m_1-m_2)\mathfrak{T}_{m_1-m_2}{}^q. \quad (\text{A.7})$$

By the unitary property of the Clebsch-Gordon coefficient, this may be written

$$\mathfrak{T}_{m_1-m_2}{}^q=\sum_{m_2}(-1)^{m_2}(s_1\,s_2\,m_1-m_2|s_1\,s_2\,q\,m_1-m_2) \times\rho(s_1\,m_1;s_2\,m_2). \quad (\text{A.8})$$

If we substitute Eq. (A.3) in the above, there results

$$\mathfrak{T}_{m_1-m_2}{}^q=\sum(-1)^{m_2}(s_1\,s_2\,m_1-m_2|s_1\,s_2\,q\,m_1-m_2) \times(i\,I\,m_i\,m_1-m_i|i\,I\,s_1\,m_1)(i\,I\,m_i'\,m_2-m_i'|i\,I\,s_2\,m_2) \times\rho_i(m_i,m_i')\rho_I(m_1-m_i,m_2-m_i'), \quad (\text{A.9})$$

where the sum is over m_2, m_i, and m_i'.

It is now necessary to express the particle density matrices in terms of their tensor moments. Once again this follows from the operation P_r upon the density matrix for the particle i, say. The analogous result to (A.7) is

$$\rho_i(m_i,m_i')=\sum_a(-1)^{-m_i'} \times(i\,i\,m_i-m_i'|i\,i\,a\,m_i-m_i')\mathfrak{T}_{m_i-m_i'}{}^a, \quad (\text{A.10})$$

where

$$\mathfrak{T}_\kappa{}^a=\sum_\lambda(-1)^{\lambda'}(i\,i\,\lambda-\lambda'|i\,i\,a\,\kappa)\langle\chi(i\lambda)\chi^*(i\lambda')\rangle_{\text{Av}}.$$

Since $\mathfrak{T}_\kappa{}^a$ transforms as an irreducible tensor of rank a under rotation it must differ from the usual spin tensor operator $T_\kappa{}^a$, defined in Eq. (2.6), by a normalization factor only. In order to determine this factor, let us evaluate $\mathfrak{T}_\kappa{}^a$ and $T_\kappa{}^a$ for the special case of an ensemble having only the single state $\chi(i,i)$ and for $\kappa=0$. Then

$$\mathfrak{T}_0{}^a=(-1)^i(i\,i\,i-i|i\,i\,a\,0).$$

From the definition of $T_0{}^a$ given in Eq. (2.6),

$$T_0{}^a=P_a([i/i+1]^{\frac{1}{2}}).$$

Hence, in general,

$$\mathfrak{T}_\kappa{}^a=\frac{(-1)^i(i\,i\,i-i|i\,i\,a\,0)}{P_a([i/i+1]^{\frac{1}{2}})}T_\kappa{}^a. \quad (\text{A.11})$$

If we combine Eq. (A.11) with a relation obtained from BBR(1) and R(16),

$$(i\,i\,i-i|i\,i\,a\,0) =(2i)!(2a+1)^{\frac{1}{2}}[(2i-a)!(2i+a+1)!]^{-\frac{1}{2}}, \quad (\text{A.12})$$

and substitute in (A.10), we have

$$\rho_i(m_i,m_i')=\sum_a(-1)^{i-m_i'}\frac{(2i)!(2a+1)^{\frac{1}{2}}}{[(2i-a)!(2i+a+1)!]^{\frac{1}{2}}} \times\frac{(i\,i\,m_i-m_i'|i\,i\,a\,m_i-m_i')}{P_a([i/i+1]^{\frac{1}{2}})}T_{m_i-m_i'}{}^a. \quad (\text{A.13})$$

An analogous formula holds for ρ_I. If Eq. (A.13) is substituted in (A.9), we have an expression for the generalized tensor moment of the system in terms of the individual particle tensor moments.

$$\mathfrak{T}_{m_1-m_2}{}^q=\sum(-1)^{i+I}\frac{(2i)!(2a+1)^{\frac{1}{2}}(2I)!(2b+1)^{\frac{1}{2}}}{[(2i-a)!(2i+a+1)!(2I-b)!(2I+b+1)!]^{\frac{1}{2}}}\frac{T_{m_i-m_i'}{}^a(i)T_{m_1-m_2-m_i+m_i'}{}^b(I)}{P_a([i/i+1]^{\frac{1}{2}})P_b([I/I+1]^{\frac{1}{2}})}$$
$$\times\sum[(i\,i\,m_i-m_i'|i\,i\,a\,m_i-m_i')(i\,I\,m_i\,m_1-m_i|i\,I\,s_1\,m_1) \times(i\,I\,m_i'\,m_2-m_i'|i\,I\,s_2\,m_2)(s_1\,s_2\,m_1-m_2|s_1\,s_2\,q\,m_1-m_2) \times(I\,I\,m_1-m_i\,m_i'-m_2|I\,I\,b\,m_1-m_2+m_i'-m_i)], \quad (\text{A.14})$$

where the first sum is over a, b, and m_i-m_i'. The second sum is over m_2 and m_i.

The details of the elimination of the second sum will not be given here since several similar reductions have been illustrated before. The final result is in the form of an X function, as might have been anticipated. The entire second sum becomes

$$=(-1)^{s_2+i+I}[(2a+1)(2b+1)(2s_1+1)(2s_2+1)]^{\frac{1}{2}} \times(a\, b\, m_i-m_i'\, m_1-m_2+m_i'-m_i|a\, b\, q\, m_1-m_2) \times X(i\, a\, i;\, s_1\, q\, s_2;\, I\, b\, I). \quad \text{(A.15)}$$

If we substitute Eqs. (A.15) and (A.14) in Eq. (A.7) we obtain the result given in Eq. (7.2).

The special case of an unpolarized initial nucleus corresponds to $b=0$. Equation (2.7) of the text then follows immediately from Eq. (7.2) by the use of I(B.1).

APPENDIX B. SOME PROPERTIES OF THE G FUNCTION

The G function may be defined in terms of the Fano X function[15] as

$$G_\kappa\begin{pmatrix} a & b & c \\ d & & f \\ g & h & i \end{pmatrix}=[(2a+1)(2b+1)(2c+1)(2d+1) \times(2f+1)(2g+1)(2h+1)(2i+1)]^{\frac{1}{2}} i^{b+h} \times\sum_e\left[(b\, h\, 0\, 0|b\, h\, e\, 0) \times(f\, d\, \kappa-\kappa|f\, d\, e\, 0) X\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right]. \quad \text{(B.1)}$$

By the use of BBR (1), (5), and the symmetry properties of the X function, it is clear that the interchange of the two outer rows of the G function or of the two outer columns simply multiplies the G function by the phase factor $(-1)^{a+d+g+c+f+i}$.

The "triad" conditions for the nonvanishing of the X function also immediately yield the condition that the elements of each of the outer rows and columns of the G function must form possible vector triads in order that G shall not vanish:

$$(abc),\ (adg),\ (ghi),\ (cfi)\ \text{"triads."} \quad \text{(B.2)}$$

In addition, the two "triad" conditions involving e yield a less restrictive condition on the G function which is that the four central elements $(d\, b\, f\, h)$ must form a possible vector "tetrad:"

$$(d\, b\, f\, h)\ \text{"tetrad."} \quad \text{(B.3)}$$

In the special case of $\kappa=0$, this last condition becomes more restrictive. By BBR (5) we must have $b+h+e$ = even integer and $f+d+e$ = even integer. Hence,

$$b+h+f+d=\text{even integer for } \kappa=0. \quad \text{(B.4)}$$

From the properties of the Clebsch-Gordon coefficients, it is also clear that d and f must satisfy the conditions:

$$d\geq\kappa,\quad f\geq\kappa. \quad \text{(B.5)}$$

Two alternative expressions for the G functions are often useful. These are obtained by first substituting in Eq. (B.1) the expression for X in terms of Racah functions.[15] The resultant expression is

$$G_\kappa(abc;d\quad f;ghi) =i^{b+h}[(2a+1)(2b+1)(2c+1)(2d+1) \times(2f+1)(2g+1)(2h+1)(2i+1)]^{\frac{1}{2}} \times\sum_{e,z}[(-1)^{a+b+c+d+e+f+g+h+i}(2z+1)^{\frac{1}{2}} \times W(bdcg;za)W(dbfh;ze)W(gchf;zi) \times(b\, h\, 0\, 0|b\, h\, e\, 0)(f\, d\, \kappa-\kappa|f\, d\, e\, 0)]. \quad \text{(B.6)}$$

For simplicity in printing, we have written the arguments of G on a single line. The two expressions for G are now obtained by performing the sum on e by the use of BBR (18) in either of two possible orderings of b, h and f, d. The two expressions are

$$G_\kappa(abc;d\quad f;ghi) =i^{b+h}[(2a+1)(2b+1)(2c+1)(2d+1)(2f+1) \times(2g+1)(2h+1)(2i+1)]^{\frac{1}{2}}(-1)^{a+g+c+i+\kappa} \times\sum_z[(-1)^z(f\, h\, \kappa\, 0|f\, h\, z\, \kappa)(d\, b\, \kappa\, 0|d\, b\, z\, \kappa) \times W(bdcg;za)W(gchf;zi)], \quad \text{(B.7)}$$

and

$$G_\kappa(abc;d\quad f;ghi) =i^{b+h}[(2a+1)(2b+1)(2c+1)(2d+1)(2f+1) \times(2g+1)(2h+1)(2i+1)]^{\frac{1}{2}}(-1)^{d+f+\kappa} \times\sum_z[(-1)^z(f\, b\, \kappa\, 0|f\, b\, z\, \kappa)(d\, h\, \kappa\, 0|d\, h\, z\, \kappa) \times W(hdia;zg)W(aibf;zc)]. \quad \text{(B.8)}$$

The G function reduces to a simpler form when one of the elements is zero. The following forms are obtained from Eqs. (B.1), (B.4), and (B.5) by the use of BBR (30) and I(B.1):

$$G_0(abc;d\quad 0;ghi) =\delta_{ci}(-1)^{c-g}(2c+1)^{\frac{1}{2}} i^d Z(bahg;cd); \quad \text{(B.9)}$$

$$G_\kappa(a0c;d\quad f;ghi) =\delta_{ac} i^h[(2i+1)(2a+1)(2g+1)(2d+1)(2f+1)]^{\frac{1}{2}} \times(f\, d\, \kappa-\kappa|f\, d\, h\, 0)W(dahi;gf); \quad \text{(B.10)}$$

$$G_\kappa(0bc;d\quad f;ghi) =\delta_{bc}\delta_{dg} i^{b+h}[(2h+1)(2d+1)(2f+1)]^{\frac{1}{2}}(-1)^{d+i} \times(f\, i\, \kappa-\kappa|f\, i\, b\, 0)(d\, h\, \kappa\, 0|d\, h\, i\, \kappa), \quad \text{(B.11)}$$

where the Z coefficient is defined in BB (4.3). The remaining forms follow by the use of the symmetry conditions.

[15] For the properties of the X function see reference 3 and also Appendix B of I.