Nuclear Forces from *P*-Wave Mesons

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The nucleon-nucleon potential from meson exchange is related to a matrix describing the scattering of virtual and real mesons by nucleons. This meson-nucleon scattering matrix is calculated for p-wave mesons, using the model of Chew, which approximates experimental phase shifts for real mesons. The corrections to the g^2 and g^4 perturbation theory nuclear forces are evaluated. States involving the simultaneous existence of three or more mesons, none of which are absorbed by the same nucleon that emitted them, have been omitted in this treatment. Comparison of these and other results implies that this neglect is unjustified.

INTRODUCTION

An understanding of the nucleon-nucleon potential depends upon a correct description of the π-meson nucleon interaction. The analysis of recent meson-nucleon scattering experiments yields information on the behavior of free mesons and nucleons. The scattering of virtual mesons, which enters in an important way into nuclear force calculations, can be extrapolated from these experiments only for a specific model.

Perturbation theory fails to account for important features of the large p-wave real meson scattering. In particular the observed dominance of the scattering in the isotopic spin 3/2 state is absent. Chew[1] has suggested an approach which gives a scattering of free p-wave mesons in fair agreement with observation. In the spirit of this calculation we have investigated the nuclear force which results from the p-wave interaction of virtual mesons.

Section I is a demonstration of a relation between a nucleon-nucleon scattering matrix, which can be computed by the standard Feynman[2]-Dyson[3] prescriptions, and the nuclear force. In Sec. II we relate this nucleon-nucleon scattering matrix to the scattering of real and virtual mesons. Section III treats the scattering of real and virtual p-wave mesons by nucleons. Section IV gives the explicit results for the nuclear force calculation. The difference between these results and those of Watson and Brueckner[4] are discussed.

I. RELATION OF POTENTIAL TO NUCLEON-NUCLEON SCATTERING MATRIX

It is convenient to work with a scattering matrix for the two nucleon system which can, in principle, be calculated by the Feynman-Dyson techniques. In this section we derive a prescription for calculating the nuclear potential from this matrix.

We consider two free nucleons in the center-of-mass system, whose momentum energy vectors are $(\mathbf{p}_0, E_0)$ and $(-\mathbf{p}_0, E_0)$. The standard Feynman-Dyson rules give a power series expansion of the matrix element for the scattering of these nucleons into the states $(\mathbf{p}_1, E_0)$. Let this scattering matrix (whose leading term is proportional to g^2) be $\mathcal{R}(\mathbf{p}_1, \mathbf{p}_0)$.

Since we shall be concerned only with nonrelativistic nucleons, we have $E_0 = M + p_0^2/2M$. The $\mathcal{R}$ matrix is calculated for a scattering from $\mathbf{p}_0$ to any momentum $\mathbf{p}_1$, but the same fourth component $E_0 (p_1^2 \neq p_0^2)$.

Let $V(\mathbf{p}', \mathbf{p})$ be the solution of

$$V(\mathbf{p}', \mathbf{p}) = i\mathcal{R}(\mathbf{p}', \mathbf{p}) - i\int d\mathbf{p}'' V(\mathbf{p}', \mathbf{p}'') \frac{M}{p''^2 - p^2 + i\delta} \mathcal{R}(\mathbf{p}'', \mathbf{p}), \quad (6)$$

and $\phi(\mathbf{p}, E)$ a solution of

$$\left(\frac{p^2}{M} - E\right)\phi(\mathbf{p}, E) = \int d\mathbf{p}' V(\mathbf{p}, \mathbf{p}')\phi(\mathbf{p}', E). \quad (7)$$

A solution of (7) for $E = p_0^2/M$ is

$$\phi(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p}_0) + \frac{M}{p^2 - p_0^2 + i\delta} i\mathcal{R}(\mathbf{p}, \mathbf{p}_0). \quad (8)$$

The spin indices have been suppressed for initial and final states, but introduce no complication. $\mathcal{R}(\mathbf{p}, \mathbf{p}_0)$ has been constructed to give the correct amplitude for the scattering when $p^2 = p_0^2$. Therefore, for a given angular momentum state the solution (8) has the asymptotic form in coordinate space:

$$\psi(r \to \infty) \sim Y_{l,m}(\theta, \phi)(kr)^{-1}[\exp\{-i(kr - (l+1)\pi/2\} + S(k, l)\exp\{i(kr - (l+1)\pi/2\}]. \quad (9)$$

$S(k, l)$ is the Heisenberg S matrix[5] for the angular momentum l. [If the spin indices are included Eq. (9) holds for singlet states, and there is an analogous expression in terms of m, J, and $S(k, J)$ for triplet states.] If $S(k, l)$ can be analytically continued as a function of k to the negative imaginary axis, bound states correspond to zeros of S.[6] When $S = 0$,

$$\psi(r \to \infty) \sim (|k|r)^{-1}\exp(-|k|r). \quad (10)$$

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3 F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).
4 K. A. Brueckner and K. M. Watson, preceding paper [Phys. Rev. **92**, 1023 (1953)].
5 W. Heisenberg, Z. Physik **120**, 513, 673 (1943); C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **23**, No. 1 (1945).
6 C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **22**, No. 19 (1946).

Therefore, where a zero of $S(-ik', l)$, $k'>0$, corresponds to a bound state, solutions of Eq. (7), which are asymptotic to $r^{-1}\exp[-r(M|E|)^{\frac{1}{2}}]$, give the energy levels of the bound system.[7] The Schrödinger Eq. (7) then gives a correct description of scattering and bound states when $V(\mathbf{p}, \mathbf{p}')$ is a sufficiently short range potential. In the following the nucleon-nucleon potential is, therefore, taken to be $V(\mathbf{p}, \mathbf{p}')$ as defined by Eq. (6).

The Feynman-Dyson prescription gives $\mathcal{R}(\mathbf{p}_1, \mathbf{p}_0)$ as a power series in g^2, where g is the meson-nucleon coupling parameter. In practice this series is approximated by including only a finite number of terms. If $\mathcal{R}$ includes terms up to g^{2m} only, it is not sensible to solve for V in Eq. (6) to a higher power than $2m$.

II. NUCLEON-NUCLEON AND MESON-NUCLEON SCATTERING

The range of the nucleon-nucleon interaction depends upon the number of mesons exchanged between nucleons. We shall consider only those terms in the $\mathcal{R}$ matrix where at most two mesons are exchanged although a meson may be scattered an arbitrary number of times by each nucleon. In general, these scatterings have only a slight effect upon the range of forces. The correction to the nuclear potential from these multiple scatterings can be expressed in terms of the matrix element for meson-nucleon scattering:

$$r(\boldsymbol{\sigma}_a; E_1, \mathbf{p}_1; E_0, \mathbf{p}_0; w_1, \mathbf{k}_1; w_0, \mathbf{k}_0), \tag{11}$$

see Fig. 1(a), where $(\mathbf{k}_0, w_0)$ and $(\mathbf{k}_1, w_1)$ are the 4-momenta of the incident and scattered meson, respectively. $(E_0, \mathbf{p}_0)$ and $(E_1, \mathbf{p}_1)$ are the 4-momenta of the incident and scattered nucleon, and $\boldsymbol{\sigma}_a$ is the Pauli spin operator which acts on the incident nucleon spin wave function. Four-momentum conservation gives

$$E_1 = E_0 + w_0 - w_1 \quad \text{and} \quad \mathbf{p}_1 = \mathbf{p}_0 + \mathbf{k}_0 - \mathbf{k}_1, \tag{12}$$

but none of the 4-momenta are assumed to be those of free particles.

In terms of Eq. (11) the $\mathcal{R}^{(2)}$ matrix for the exchange of two mesons [see Fig. 1(b), (c)] is

$$\begin{aligned}\mathcal{R}^{(2)}(\mathbf{p}_1, \mathbf{p}_0) = -\frac{1}{2}\left(\frac{1}{2\pi}\right)^4 \int r(\boldsymbol{\sigma}_a; E_0, \mathbf{p}_1; E_0, \mathbf{p}_0;\\ w, \mathbf{k}; w, \mathbf{p}_1 - \mathbf{p}_0 + \mathbf{k})\{r(\boldsymbol{\sigma}_b; E_0, -\mathbf{p}_1; E_0, -\mathbf{p}_0;\\ -w, -\mathbf{k}; -w, -\mathbf{p}_1 + \mathbf{p}_0 - \mathbf{k}) + r(\boldsymbol{\sigma}_b; E_0, -\mathbf{p}_1;\\ E_0, -\mathbf{p}_0; w, \mathbf{p}_1 - \mathbf{p}_0 + \mathbf{k}; w, \mathbf{k})\}(w^2 - k^2 - \mu^2 + i\delta)^{-1}\\ \times[w^2 - (\mathbf{p}_1 - \mathbf{p}_0 + \mathbf{k})^2 - \mu^2 + i\delta]^{-1} d\mathbf{k}dw. \end{aligned} \tag{13}$$

$\mathcal{R}^{(1)}(\mathbf{p}_1, \mathbf{p}_0)$, the nucleon scattering amplitude when only one mesons is exchanged between different nucleons, can also be expressed in terms of Eq. (11). However, it is necessary to give an explicit form to the coupling between meson and nucleon fields. If, for

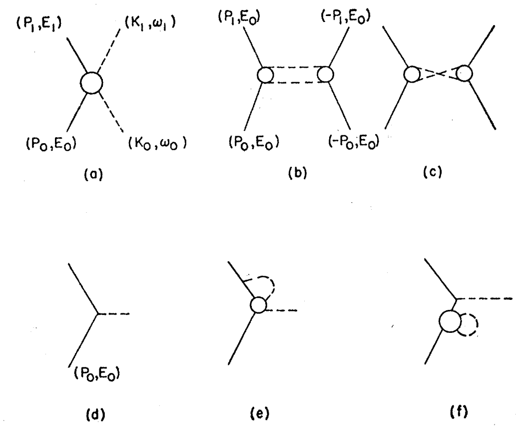

FIG. 1. (a) Representation of meson-nucleon scattering matrix; (b) and (c) contributions to the nucleon-nucleon scattering matrix in terms of (a); (d) lowest-order meson nucleon interaction; (e) and (f) all corrections to (d) expressed in terms of (a).

example, this is taken to be $(g/\mu)\int\psi^*\boldsymbol{\sigma}\cdot\nabla\sum_{i=1}^3\Phi_i\tau_i\psi d\mathbf{r}$ for symmetrical pseudoscalar mesons, then

$$\begin{aligned}\mathcal{R}^{(1)}(\mathbf{p}_1, \mathbf{p}_0) = -i\Big[(g/\mu)\boldsymbol{\sigma}_a\cdot\mathbf{k}_0\boldsymbol{\tau}_a\cdot\boldsymbol{\phi}_0\\ -(g/\mu)(2\pi)^{-4}\int dw' d\mathbf{k}'\sigma_a\cdot\mathbf{k}'\boldsymbol{\tau}_a\cdot\boldsymbol{\phi}'\\ \times r(\boldsymbol{\sigma}_a; \boldsymbol{\tau}_a; E_0, \mathbf{p}_1; E_0, \mathbf{p}_0; w', \mathbf{k}', \boldsymbol{\phi}'; 0, \mathbf{k}_0, \boldsymbol{\phi}_0)\\ \times(w'^2 - k'^2 - \mu^2 + i\delta)^{-1}(p_0^\nu\gamma^\nu + k_0^\nu\gamma^\nu - k'^\nu\gamma^\nu - M)^{-1}\Big]\\ \times[\text{as above, with } a\to b, \mathbf{k}_0\to -\mathbf{k}_0, \mathbf{p}_0\to -\mathbf{p}_0, \mathbf{p}_1\to -\mathbf{p}_1]\\ \times(k_0^2 + \mu^2)^{-1}, \end{aligned} \tag{14}$$

where $\mathbf{k}_0 = \mathbf{p}_1 - \mathbf{p}_0$, and $\boldsymbol{\phi}$ is a unit vector in charge space, specifying the charge of the meson.

The simple meson-nucleon interaction Fig. 1(d) is corrected in Eq. (14) by terms represented by Fig. 1(e). Contributions from Fig. 1(f) have been omitted since they are the interactions of a free nucleon ($E_0 = M + p_0^2/2M$) with its own meson field, which should be cancelled by a proper renormalization program for the free nucleon properties. Similar terms from Fig. 1(e), which are included in Eq. (14), will also be cancelled. (Appendix II.)

From Eq. (6), the potential is given in terms of Eq. (13) and Eq. (14) as

$$\begin{aligned} V &\approx V^{(1)} + V^{(2)},\\ V^{(1)} &= i\mathcal{R}^{(1)}(\mathbf{p}_1, \mathbf{p}_0),\\ V^{(2)} &= i\mathcal{R}^{(2)}(\mathbf{p}_1, \mathbf{p}_0)\\ &\quad + \int d\mathbf{p}\mathcal{R}^{(1)}(\mathbf{p}_1, \mathbf{p})\frac{M}{p^2 - p_0^2 + i\delta}\mathcal{R}^{(1)}(\mathbf{p}, \mathbf{p}_0). \end{aligned} \tag{15}$$

In general, for infinitely heavy nucleons, $V^{(2)}$ is real

[7] R. Jost and W. Kohn, Phys. Rev. 87, 977 (1952).

and finite although $\mathcal{R}^{(2)}$ is infinite, even for a renormalized meson theory. $V^{(1)}$ and $V^{(2)}$ are series containing all powers of g^2, but correspond, respectively, to the exchange of only one and two mesons between two nucleons.

III. SCATTERING OF REAL AND VIRTUAL MESONS

In this section we derive an approximation to the r matrix for the scattering of real and virtual mesons in the limit of infinite nucleon mass.

In perturbation theory, the r matrix for the scattering of p-state mesons symmetrically coupled to nonrelativistic nucleons is

$$r=-i\frac{g^2}{\mu^2}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_1\boldsymbol{\sigma}\cdot\mathbf{k}_0\boldsymbol{\tau}\cdot\boldsymbol{\phi}_1\boldsymbol{\tau}\cdot\boldsymbol{\phi}_0}{w_0}+i\frac{g^2}{\mu^2}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_0\boldsymbol{\sigma}\cdot\mathbf{k}_1\boldsymbol{\tau}\cdot\boldsymbol{\phi}_0\boldsymbol{\tau}\cdot\boldsymbol{\phi}_1}{w_1}, \quad (16)$$

where $\boldsymbol{\phi}_0$ and $\boldsymbol{\phi}_1$ are unit vectors in charge space specifying the charge of the incident and scattered meson. Equation (16) is valid for virtual mesons, where $w_1^2 \neq k_1^2+\mu^2$ and $w_0^2\neq k_0^2+\mu^2$. Important higher-order effects can be computed by treating Eq. (16) as the Born approximation scattering from a meson-nucleon potential. The higher-order terms which are then included involve intermediate states which may have energies close to that of the incident meson and so can give large contributions. The sum r of all order scatterings from this potential satisfies the integral equation:

$$\begin{aligned}
&r(\boldsymbol{\sigma};\boldsymbol{\tau};\boldsymbol{\phi}_1,w_1,\mathbf{k}_1;\boldsymbol{\phi}_0,w_0,\mathbf{k}_0)\\
&=-i\frac{g^2}{\mu^2}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_1\boldsymbol{\sigma}\cdot\mathbf{k}_0\boldsymbol{\tau}\cdot\boldsymbol{\phi}_1\boldsymbol{\tau}\cdot\boldsymbol{\phi}_0}{w_0}+i\frac{g^2}{\mu^2}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_0\boldsymbol{\sigma}\cdot\mathbf{k}_1\boldsymbol{\tau}\cdot\boldsymbol{\phi}_0\boldsymbol{\tau}\cdot\boldsymbol{\phi}_1}{w_1}\\
&+\frac{g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int d\mathbf{k}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_1\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\tau}\cdot\boldsymbol{\phi}_1\boldsymbol{\tau}\cdot\boldsymbol{\phi}}{2\omega w_0(w_0-\omega+i\delta)}\\
&\times r(\boldsymbol{\sigma};\boldsymbol{\tau};\boldsymbol{\phi},\omega,\mathbf{k};\boldsymbol{\phi}_0,w_0,\mathbf{k}_0)+\frac{g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int d\mathbf{k}\\
&+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{k}_1\boldsymbol{\tau}\cdot\boldsymbol{\phi}\boldsymbol{\tau}\cdot\boldsymbol{\phi}_1}{2\omega(w_0-\omega-w_1+i\delta)(w_0-\omega+i\delta)}\\
&\times r(\boldsymbol{\sigma};\boldsymbol{\tau};\boldsymbol{\phi},\omega,\mathbf{k};\boldsymbol{\phi}_0,w_0,\mathbf{k}_0). \quad (17)
\end{aligned}$$

Since the nucleon is fixed in space, the dependence of r on $\mathbf{p}$ and E is suppressed. The dependence of r on the charge of the mesons and nucleon, omitted from Eq. (11), is included in Eq. (16). w_1 and w_0 are fixed parameters unrelated to k_0 and k_1, but $\omega^2=k^2+\mu^2$.

The r matrix defined by Eq. (17) is the sum of the scattering represented by Feynman diagrams (a) and (b) of Fig. 2, and by arbitrary combinations of iterations of (a) and (b) such as (c), (d) and (e) of Fig. 2. Before a solution is attempted for all angular momentum and isotopic spin states certain mass and charge renormalizations will be performed, which alter parts of Eq. (17). For this purpose it is convenient to separate the matrix Eq. (17) into four integral equations for the scattering amplitudes $r_J{}^I$, with $J=1/2$, $3/2$ and $I=1/2$, $3/2$ giving, respectively, the total angular momentum and isotopic spin for the meson-nucleon system. Then

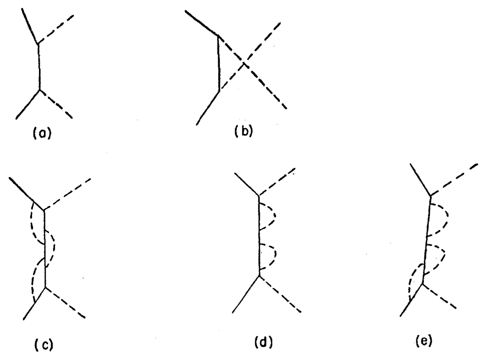

FIG. 2. (a), (b) Feynman diagrams for meson-nucleon scattering in lowest order perturbation theory; (c), (d), (e) higher order terms included in Eq. (17).

$$\begin{aligned}
r=&(A\boldsymbol{\sigma}\cdot\mathbf{k}_1\boldsymbol{\sigma}\cdot\mathbf{k}_0+B\boldsymbol{\sigma}\cdot\mathbf{k}_0\boldsymbol{\sigma}\cdot\mathbf{k}_1)(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1\boldsymbol{\tau}_0\cdot\boldsymbol{\phi}_0)\\
&+(C\boldsymbol{\sigma}\cdot\mathbf{k}_1\boldsymbol{\sigma}\cdot\mathbf{k}_0+D\boldsymbol{\sigma}\cdot\mathbf{k}_0\boldsymbol{\sigma}\cdot\mathbf{k}_1)(\boldsymbol{\tau}_0\cdot\boldsymbol{\phi}_0\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1), \quad (18)
\end{aligned}$$

where

$$\begin{aligned}
A&=r_3{}^3/12+r_3{}^1/6+r_1{}^3/6+r_1{}^1/3,\\
B&=r_3{}^3/4+r_1{}^3/2,\\
C&=r_3{}^3/4+r_3{}^1/2,\\
D&=3r_3{}^3/4.
\end{aligned} \quad (19)$$

From Eqs. (17), (18) and (19)

$$\begin{aligned}
r_3{}^3(w_1,w_0)=&\,i\frac{4g^2}{3w_1\mu^2}+\frac{4}{3}\frac{g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\\
&\times\int\frac{k^2d\mathbf{k}}{2\omega(w_0-\omega-w_1+i\delta)(w_0-\omega+i\delta)}r_3{}^3(\omega,w_0), \quad (20)
\end{aligned}$$

$$\begin{aligned}
r_1{}^3(w_1,w_0)=r_3{}^1(w_1,w_0)=&-i\frac{2}{3}\frac{g^2}{w_1\mu^2}\\
&-\frac{2}{3}\frac{g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}r_3{}^1(\omega,w_0)}{2\omega(w_0-\omega-w_1+i\delta)(w_0-\omega+i\delta)}, \quad (21)
\end{aligned}$$

$$\begin{aligned}
r_1{}^1(w_1,w_0)=&-i\frac{3g^2}{w_0\mu^2}+3\frac{g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\\
&\times\int\frac{k^2d\mathbf{k}}{2\omega(w_0-\omega+i\delta)w_0}r_1{}^1(\omega,w_0)+i\frac{g^2}{3w_1\mu^2}\\
&+\frac{1}{3}\frac{g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}}{2\omega(w_0-\omega+i\delta)(w_0-\omega-w_1+i\delta)}\\
&\times r_1{}^1(\omega,w_0). \quad (22)
\end{aligned}$$

The potential obtained from Eq. (16) is sufficiently singular that Eqs. (20)–(22) have no sensible solution.

If, however, Eq. (16) is assumed to vanish for $k>k_{\max}$, then solutions do exist.[8]

The quantities r_3^3, r_3^1, and r_1^3 contain contributions from Feynman diagrams of type (b) and (c) Fig. 1 only. On the other hand, r_1^1 arises from a variety of different kinds of Feynman diagrams [*viz*, Fig. 2 (a), (d), (e)], parts of which involve nucleon mass and charge renormalization.

To apply the renormalization prescriptions to r_1^1, Eq. (22) it is convenient to neglect the two terms with coefficient $\frac{1}{3}$ next to those with 3 and later to introduce them as a perturbation. The solution of the resulting integral equation represents the sum of diagrams of the type (a) and (d) of Fig. 2. [After renormalization this sum corresponds to the replacement of S_F by S_F' in Fig. 2(a).]

The self-energy of a free nucleon of infinite mass, Fig. 3(a), is

$$3\frac{g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}}{2\omega^2}. \tag{23}$$

Therefore, the nucleon mass M, in terms of the observed mass $M_{\exp}$, is given by

$$M=M_{\exp}-\frac{3g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}}{2\omega^2}. \tag{24}$$

If the Hamiltonian for the meson-nucleon system is written in terms of $M_{\exp}\rightarrow\infty$, then we also have a contribution represented by Fig. 3(d) from $M-M_{\exp}$. The matrix element represented by Fig. 3(c), when the incident nucleon is free, is

$$-3\frac{g^3}{\mu^3}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}}{2\omega(\omega-w_0-i\delta)}\frac{1}{w_0}\boldsymbol{\sigma}\cdot\mathbf{k}_0. \tag{25}$$

The sum of Figs. 3(b), (c), (d) thus gives

$$\frac{g}{\mu}\left[1-3\frac{g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}}{2\omega^3}-\frac{9}{2}\Delta'(w_0)\right]\boldsymbol{\sigma}\cdot\mathbf{k}_0, \tag{26}$$

with

$$\Delta'(w_0)=\frac{2}{3\pi}\frac{g^2}{4\pi}\frac{w_0}{\mu^2}\frac{1}{4\pi}\int_0^{k_{\max}}\frac{k^2d\mathbf{k}}{\omega^3(\omega-w_0-i\delta)}. \tag{27}$$

A further correction occurs when the self-energy part, Fig. 3(a), occurs for a free nucleon before the interaction $\boldsymbol{\sigma}\cdot\mathbf{k}_0$. When both nucleon legs are free ($w_0=0$), the total contribution of the self-energy part is included by putting $g\boldsymbol{\sigma}\cdot\mathbf{k}_0\rightarrow g_p\boldsymbol{\sigma}\cdot\mathbf{k}_0$ with

$$g_p=g\left(1-6\frac{g^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}}{2\omega^3}\right). \tag{28}$$

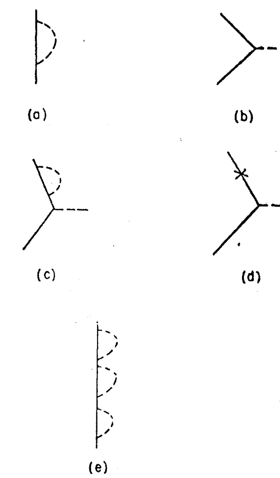

FIG. 3. Feynman diagrams relevant to the discussion of mass and charge renormalization.

Hence, if the experimental mass, $M_{\exp}$, and coupling constant, g_p, are used in the Hamiltonian, then the contribution of the self-energy part vanishes on free nucleon legs. Furthermore, for virtual nucleon propagation we obtain a correction from successive self-energy parts [Fig. 3(e)], which is given by the factor:

$$\sum_{n=0}^{\infty}\left[-\frac{9}{2}\Delta'(w_0)\right]^n=\left[1+\frac{9}{2}\Delta'(w_0)\right]^{-1}. \tag{29}$$

Insertion into Fig. 2(a) gives

$$-\frac{3ig_p^2}{w_0\mu^2}\frac{1}{1+(9/2)\Delta'(w_0)}. \tag{30}$$

Adding the two omitted terms of (22), we have

$$r_1^1\approx i\frac{g^2}{\mu^2}\left[-\frac{8[1-(9/4)\Delta'(w_0)]}{3w_0[1+(9/2)\Delta'(w_0)]}-\frac{g^2}{\mu^2w_0(2\pi)^3}\times\int\frac{k^2d\mathbf{k}}{2\omega(w_0-\omega)(w_0-\omega+w_1)[1+(9/2)\Delta'(w_0)]}+\frac{w_0-w_1}{3w_0w_1}\right]. \tag{31}$$

For a consistent treatment of the renormalization of the r_J^I matrix it is also necessary to replace g by g_p and S_F by S_F' in Figs. 2(b) and 2(c). Otherwise self-energy corrections would arise even from the free nucleon legs.

After this substitution for g and S_f has been made r_J^I can be written in terms of the two parameters g_p and $k_{\max}$ only. Insertion of the correction Eq. (29) into Eq. (20) for r_3^3 gives

$$r_3^3(w_1,w_0)=\frac{4ig_p^2}{3w_1\mu^2}\left[1+\frac{9}{2}\Delta'(-w_1)\right]^{-1}+\frac{4}{3}\frac{g_p^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}[1+(9/2)\Delta'(w_0-\omega)]^{-1}[1+(9/2)\Delta'(w_0-\omega-w_1)]^{-1}}{2\omega(w_0-\omega-w_1+i\delta)(w_0-\omega+i\delta)}r_3^3(\omega,w_0); \tag{32}$$

[8] This is equivalent to the source considered by Chew (see reference 1).

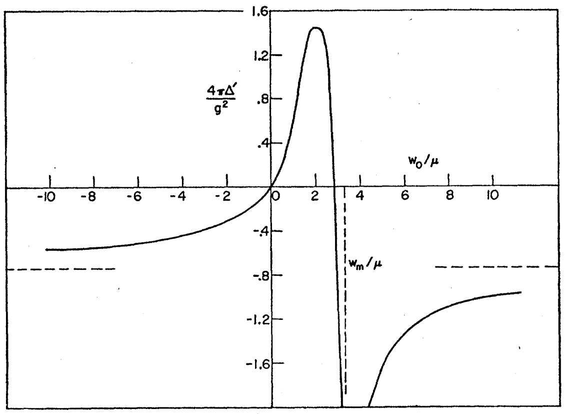

FIG. 4. Real part of $\Delta'(w_0)$ for $\omega_{\max}=3.2\mu$.

then, (appendix I)

$$r_3{}^3(w_1,w_0)\approx\frac{4ig_p{}^2}{3w_1\mu^2}\frac{[1-2\Delta'(w_0-w_1)]}{[1-2\Delta'(w_0)][1+(9/2)\Delta'(-w_0)]}. \tag{33}$$

Similarly

$$r_1{}^3(w_1,w_0)=r_3{}^1(w_1,w_0)\approx-\frac{2ig_p{}^2}{3w_1\mu^2}\frac{[1-\Delta'(w_1-w_0)]}{[1+\Delta'(w_0)][1+(9/2)\Delta'(-w_0)]}. \tag{34}$$

For the scattering of real p-wave mesons $w_1=w_0=(k_0{}^2+\mu^2)^{\frac{1}{2}}$ and $\Delta'(w_1-w_0)=0$. Equations (32) and (33) are then almost the same as those of Chew[1] for $\mu\leq w_0<w_{\max}$. However the factor $[1+(9/2)\Delta'(-w_0)]^{-1}$ was not included by him in obtaining a best fit to the experimental data, which occurred for

$$g_p{}^2/4\pi\approx0.2\quad\text{and}\quad w_{\max}\approx3.2\mu. \tag{35}$$

(The neglected factor will increase $r_3{}^3$, $r_1{}^3$, and $r_3{}^1$ by about 25 percent for free meson scattering and thus necessitates a small change in $g_p{}^2/4\pi$ and $w_{\max}$.)

The function $\Delta'(w_0)$ is a measure of the departure of the matrix $r_J{}^I$ from that given in perturbation theory. The real part of $\Delta'(w_0)$, Eq. (27), is given by

$$\mathrm{Re}\Delta'(w_0)=\frac{g^2}{4\pi}\frac{2}{3\pi}\frac{1}{\mu^2}\left\{w_0{}^2\ln\left(\frac{k_{\max}+\omega_{\max}}{\mu}\right)+w_0k_{\max}-\frac{k_{\max}\mu^2}{\omega_{\max}}-\frac{\mu^3}{w_0}\cos^{-1}\frac{\mu}{\omega_{\max}}-R\right\},$$

$$R=\frac{(w_0{}^2-\mu^2)^{\frac{3}{2}}}{w_0}\ln\left|\frac{(w_0{}^2-\mu^2)^{\frac{1}{2}}k_{\max}+w_0\omega_{\max}-\mu^2}{\mu(\omega_{\max}-w_0)}\right|\quad\text{for}\quad w_0{}^2\geq\mu^2, \tag{36}$$

$$R=\frac{(\mu^2-w_0{}^2)^{\frac{3}{2}}}{w_0}\cos^{-1}\left[\frac{\mu^2-w_0\omega_{\max}}{\mu(\omega_{\max}-w_0)}\right]\quad\text{for}\quad w_0{}^2\leq\mu^2,$$

and is plotted in Fig. 4. Asymptotically, for $|w_0|\rightarrow\infty$

$$\mathrm{Re}\Delta'(w_0)=-\frac{g^2}{4\pi}\frac{2}{3\pi}\frac{w_0}{\mu^2}\int_\mu^{\omega_{\max}}\frac{k^3d\omega}{\omega^2w_0}=-\frac{g^2}{4\pi}\frac{2}{3\pi}\frac{1}{\mu^2}\left\{k_{\max}\left(\frac{\omega_{\max}}{2}+\frac{\mu^2}{\omega_{\max}}\right)-\tfrac{3}{2}\mu^2\ln\left(\frac{\omega_{\max}+k_{\max}}{\mu}\right)\right\}. \tag{37}$$

For $-w_{\max}\leq w_0\leq\mu$, $(g_p{}^2/4\pi)\Delta'(w_0)$ is small and $r_J{}^I$ is close to the perturbation result.

IV. THE NUCLEAR FORCE

The potentials, $V^{(1)}$ and $V^{(2)}$ can be found by substitution of Eqs. (31), (33), and (34) into Eqs. (18) and (19) to yield r. From Eqs. (13) and (14), $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ can then be found, and Eq. (15) gives $V^{(1)}+V^{(2)}$. These calculations are performed in Appendix (II). The potential is approximately (see reference 14)

$$V=V^{(1)}+V^{(2)},$$

$$V^{(1)}\approx-\frac{g^2}{4\pi}\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}\boldsymbol{\sigma}_b\cdot\boldsymbol{\nabla}\frac{e^{-\mu x}}{\mu^2x}[1+0.07], \tag{38}$$

$$\begin{aligned}V^{(2)}\approx&\left(\frac{g^2}{4\pi}\right)^2\lim_{x_0\rightarrow x_1}\sum_{i,j=1}^{3}[\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_1\tau_a{}^i,\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_0\tau_a{}^j]\\&\times[\boldsymbol{\sigma}_b\cdot\boldsymbol{\nabla}_1\tau_b{}^i,\boldsymbol{\sigma}_b\cdot\boldsymbol{\nabla}_0\tau_b{}^j]\frac{(x_0+x_1)K_0(\mu x_0+\mu x_1)}{2\pi x_0x_1\mu^4}\\&+\left(\frac{g^2}{4\pi}\right)^2\lim_{x_0\rightarrow x_1}0.18\{\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_1,\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_0\}\{\boldsymbol{\sigma}_b\cdot\boldsymbol{\nabla}_1,\boldsymbol{\sigma}_b\cdot\boldsymbol{\Delta}_0\}\\&\times\frac{e^{-\mu(x_0+x_1)}}{\mu^5x_0x_1}=\left(\frac{g^2}{4\pi}\right)^2\mu\frac{8}{\pi}\left\{\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b\left[\frac{3}{\mu^3x^3}K_0(2\mu x)\right.\right.\\&\left.+\left(\frac{2}{\mu^2x^2}+\frac{3}{\mu^4x^4}\right)K_1(2\mu x)\right]-\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b\left[\left(\frac{1}{\mu x}+\frac{23}{4\mu^3x^3}\right)\right.\\&\left.\times K_0(2\mu x)+\left(\frac{3}{\mu^2x^2}+\frac{23}{4\mu^4x^4}\right)K_1(2\mu x)\right]\\&-\left[\frac{3\boldsymbol{\sigma}_a\cdot\mathbf{x}\boldsymbol{\sigma}_b\cdot\mathbf{x}}{x^2}-\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b\right]\left[\frac{3}{\mu^3x^3}K_0(2\mu x)\right.\\&\left.\left.+\left(\frac{1}{\mu^2x^2}+\frac{15}{4\mu^4x^4}\right)K_1(2\mu x)\right]\right\}+\left(\frac{g^2}{4\pi}\right)^2\mu0.7\left\{\frac{1}{2\mu^2x^2}\right.\\&\left.+\frac{4}{\mu^3x^3}+\frac{12}{\mu^4x^4}+\frac{12}{\mu^5x^5}+\frac{6}{\mu^6x^6}\right\}e^{-2\mu x}.\end{aligned} \tag{39}$$

In Eq. (39), the subscripts 1 or 0 on the $\boldsymbol{\nabla}$-operator means that the differentiation is to be carried out only with respect to x_1 or x_0, respectively; [,] is the commutator bracket, and { , } is the anticommutator bracket. These potentials are plotted in Fig. 5 for deuterium.

The first terms on the right-hand side of Eqs. (38) and (39) are the usual second- and fourth-order perturbation theory potentials. The correction to $V^{(1)}$ has the same shape and spin dependence as the perturbation result with approximately 7 percent of its magnitude. The dominant correction to $V^{(2)}$ is independent of spin and isotopic spin. It gives a short-range repulsion, about 20 percent of the perturbation theory central force in the ^3S state. The large departure from perturbation theory which is found for the scattering of real mesons is not reflected in the nuclear force because of the very different behavior of real and virtual mesons with the same momentum.[9]

The first term in Eq. (39) has also been obtained by Taketani, Machida, and Onuma,[10] Klein,[11] Feynman and Lopes,[12] and others, but differs in an important way from that of Brueckner and Watson,[4] who have analyzed this difference in detail. Its origin lies in the treatment of states with at most one meson at any time. The $\mathcal{R}$ matrix, Eq. (13), has been calculated for the exchange of two meson with any time ordering. That part of it which is relevant to this discussion is represented in Fig. (6a), when $\min(t_1, t_2) > \max(t_3, t_4)$. Substitution of this $\mathcal{R}$ into Eq. (6) gives a contribution to the potential of

$$\left(\frac{g^2}{4\pi}\right)^2 \frac{1}{4\pi^4}(3-2\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b)\int d\mathbf{k}d\mathbf{k}' \times\frac{\boldsymbol{\sigma}_a\cdot\mathbf{k}\boldsymbol{\sigma}_a\cdot\mathbf{k}'\boldsymbol{\sigma}_b\cdot\mathbf{k}\boldsymbol{\sigma}_b\cdot\mathbf{k}'e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}}{\omega^3\omega'^2\mu^4}. \quad (40)$$

This term is the difference between the potential $V^{(2)}$, Eq. (39), and that obtained by Brueckner and Watson. It is included in Eq. (39) and is big enough to change the ^3S attraction of Brueckner and Watson to a repulsion. [Addition of a repulsive core nullifies the effect of the very strong singular part of (40) at small x.] Brueckner and Watson point out that, in the Tamm-Dancoff formalism, this is one of a series of terms that comes from an expansion of the "velocity dependent" part of the one meson interaction. When the entire series of terms are included (no expansion), the contribution is much smaller than (40), and, indeed, quite

[9] The perturbation result for $V^{(2)}$ was valid because the important values of w_0 do not lie between μ and $w_{\max}$ where $\Delta'(w_0)$ contributed significantly. However, the production of a real meson in a nucleon-nucleon collision can be altered in an important way. The simplest Feynman diagram for this process is the production of the meson by one nucleon and the scattering of this meson (with energy as well as momentum transfer) by the other. This scattering can be expressed in terms of Eqs. (31), (33) and (34). The w_0 is that of the final free meson in the c.m. system of the meson and the nucleon from which it scattered. Since $w_{\max} > w_0 > \mu$, the correction from $\Delta'(w_0)$ are not negligible. Scattering is increased in the $I=J=\frac{3}{2}$ state, otherwise decreased. Therefore π^+ production from *p-p* collisions will be increased relative to π^- production from *p-n* collisions as observed.

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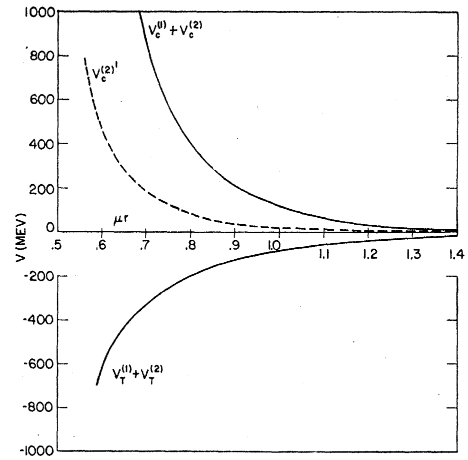

FIG. 5. $V^{(1)}+V^{(2)}$ for the deuteron $V_c^{(2)'}$ is the correction to the perturbation theory result. The central forces V_c are repulsive. V_t are the tensor forces.

negligible.[4] In this sense, the potential without (40) may be expected to be a better approximation than that including it. On the other hand, for neutral scalar mesons with fixed sources[13] a similar argument can be presented, but the required cancellation of the g^4 potential comes only when the term corresponding to Fig. (6a) is included.

The neutral scalar theory contains certain features similar to the case of interest, and we shall examine it following the method suggested by Brueckner and Watson. The Schrödinger equation, including the static approximation to those parts of the potential arising from two meson states, but treating the one meson contribution without approximation is

$$\left(\frac{p^2}{M}-\epsilon\right)\phi(\mathbf{p})=\int\bar{V}(\mathbf{k})\phi(\mathbf{p}-\mathbf{k})d\mathbf{k} +\frac{g^2}{4\pi}\int\frac{T_{\mathbf{p}}+T_{\mathbf{p}-\mathbf{k}}-\epsilon}{\omega^2(\omega+T_{\mathbf{p}}+T_{\mathbf{p}-\mathbf{k}}-\epsilon)}\phi(\mathbf{p}-\mathbf{k})d\mathbf{k}, \quad (41)$$

where $T_p = p^2/2M$, and $\bar{V}(\mathbf{k})$ is the Fourier transform of

$$-\frac{g^2}{4\pi x}e^{-\mu x}\left[1+\frac{g^2}{4\pi}\frac{2K_0(2\mu x)}{\pi}\right]-\text{repulsive core.} \quad (42)$$

In this case, the correct potential is obtained by omitting the second term in (42). For sufficiently small repulsive core, or for large g^2, the second term of (42) contributes the major part of the attractive potential [as is the case with the g^4 potential, Eq. (39), with the parameters (35)]. If the second term on the right-hand side of Eq. (41) is expanded in powers of $(T_{\mathbf{p}}+T_{\mathbf{p}-\mathbf{k}}-\epsilon)$, then the leading term of this expansion exactly cancels the second term of (42). Since the leading term is large, the expansion is, of course, suspect. On the other hand,

[13] This problem can be solved exactly and gives $V=(g^2/4\pi)e^{-\mu x}/x$.

if Eq. (41) is solved without this second term of (42) for a bound state ($\epsilon<0$), then the contribution to the binding of the neglected term is

$$\frac{g^2}{4\pi}\int\frac{dk}{\omega^2}\phi^*(\mathbf{k})\left[1-\frac{\omega}{\omega+T_{\mathbf{p}}+T_{\mathbf{p-k}}-\epsilon}\right]\phi(\mathbf{p}-\mathbf{k}) \leq\frac{g^2}{4\pi}\int\psi^*(\mathbf{x})\frac{e^{-\mu x}}{x}\psi(\mathbf{x})d\mathbf{x}. \quad (43)$$

This is less than the contribution to the binding of the first term of (42). Therefore, for appropriate g^2 and core radius, (43) may be negligible next to the second term of (42), even though the first term in the expansion of (43) is large. But, if the term giving (43) is neglected, then the potential is given by (42), with the inclusion of the second term, which, in this example, should be omitted.

The entire term (43) includes contributions to V from the exchange of more than 2 mesons, but only one at a time [Fig. 6(b)]. Since the inclusion of all such terms almost cancels the $e^{-\mu x}K_0(2\mu x)$ contribution from [Fig. 6(a)], these higher order terms are not

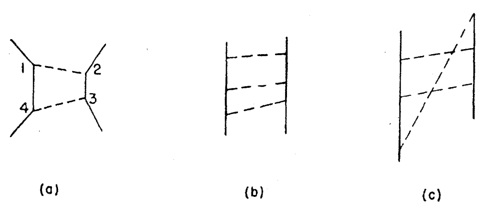

FIG. 6. (a) Feynman diagram for two meson exchange between space-time points 1, 2, 3, 4. (b) (c) Two types of meson exchange giving a g^6 term in $\mathcal{R}$.

negligible. But for the neutral scalar theory, the $\mathcal{R}$ from all diagrams representing the exchange of 3 mesons in any time sequence gives a zero contribution to the potential. Therefore, when Fig. 6(b) contributes significantly, then states with 3 mesons Fig. 6(c) will be of comparable magnitude.

The treatment of the neutral scalar theory thus suggests that states with three mesons which have been omitted by both Brueckner and Watson and by ourselves, are important enough to qualitatively change both potentials.

CONCLUSIONS

Corrections to the usual second and fourth order perturbation theory results of the nuclear force arise in two ways: (a) a virtual meson may be scattered an arbitrary number of times by one of the nucleons before being absorbed by it or by the other nucleon; (b) three or more mesons may exist at one time, no one of which are emitted and absorbed by the same nucleon.

The correction (a) is found to be small, but the neglect of (6) is unjustified and probably changes even the qualitative features of the usual[10–12] g^2+g^4 perturbation theory potential. With $(g^2/4\pi)$ adjusted to give a satisfactory description of real meson-nucleon scattering phase shifts, the calculated potential gives unsatisfactory deuteron properties.

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APPENDIX I

Since we shall primarily be concerned with $r_J{}^I(w_0, w_0)$, it is convenient to first find the solution to Eq. (32) when $w_1=w_0$. This is

$$\bar{r}_3{}^3=i\frac{4}{3w_0}\frac{g_p{}^2}{\mu^2}\left[1+\frac{9}{2}\Delta'(-w_0)\right]^{-1}[1-2\Delta''(w_0)]^{-1},$$

where

$$\Delta''(w_0)=\frac{2}{3\pi}\frac{g_p{}^2}{4\pi}\frac{w_0}{\mu^2}\frac{1}{4\pi}\int_0^{k_{\max}}\frac{k^2d\mathbf{k}}{\omega^3(\omega-w_0-i\delta)} \times\left[\frac{1+(9/2)\Delta'(-w_0)}{[1+(9/2)\Delta'(w_0-\omega)][1+(9/2)\Delta'(-\omega)]^2}\right].$$

$\Delta''(w_0)$ differs from $\Delta'(w_0)$ by the factor within the bracket. Now $\Delta''(w_0)$ is large for $w_0\sim 2\mu$. Furthermore, in the interval of integration for ω, $\Delta'(w_0-\omega)\ll 1$ for $g_p{}^2/4\pi\approx 0.2$ [see Fig. (3)]. Also $\Delta'(-\omega)$ is small in this interval $[\Delta'(-2\mu)\approx -0.06]$. Therefore near the maximum for Δ'' the bracket is close to 1 so we shall approximate $\Delta''\approx\Delta'$. This can be a bad approximation for $w_0\sim\omega_{\max}$ but these w_0 do not play an important role in the nuclear force for $x>\hbar/2\mu c$. However, the prime justification for putting $\Delta''\approx\Delta'$ is that an appreciable error in Δ' does not greatly change the nuclear potential. If we define $s_3{}^3$,

$$s_3{}^3\equiv r_3{}^3-\bar{r}_3{}^3,$$

then the solution for $s_3{}^3$ is

$$s_3{}^3(w_1, w_0)\approx 2\Delta'(w_0-w_1)\bar{r}_3{}^3(w_1, w_0) -\frac{4}{3}\frac{g_p{}^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}w_1s_3{}^3(\omega, w_0)}{2\omega^2(w_0-\omega+i\delta)(w_0-\omega-w_1+i\delta)}$$

and

$$s_3{}^3(w_0, w_0) \approx+\frac{8}{3}\frac{g_p{}^2}{\mu^2}w_0\left(\frac{1}{2\pi}\right)^3\bar{r}_3{}^3(w_0, w_0)\int\frac{k^2d\mathbf{k}\Delta'(w_0-\omega)}{2\omega^3(w_0-\omega)}.$$

In the region $-2\mu<w_0<2\mu$, which is important for the virtual scattering contribution to the extra-core part of the nuclear force, $s_3{}^3(w_0, w_0)$ is less than $0.1r_3{}^3$.

It is therefore neglected in the text and only $\bar{r}_3{}^3$ is used in computing the nuclear force.

APPENDIX II

The potentials $V^{(1)}$ and $V^{(2)}$ will be found below by the method outlined in Sec. IV.

In Eq. (14) the correction to the perturbation result for $V^{(1)}$ contains $r(w_1, \mathbf{k}_1; 0, \mathbf{k})$. However, it is not appropriate to use expression (18) for r. When introduced into Eq. (14), the entire expression (18) includes terms from Feynman diagrams which give no contribution after renormalization. In particular, that part of $r_1{}^1$ which arises from the $1/w_0$ term of Eq. (22) (replacement of S_F by S_F') is to be neglected. When substituted into Eq. (14), the only part of $r_1{}^1$ which will be kept is $ig^2/3w_1\mu^2$. On the other hand, $r_3{}^3$, $r_1{}^3$, and $r_3{}^1$ are unaltered by these considerations.

When the integration over w' is performed in Eq. (14), the term $g_p{}^2\Delta'(w'-w_0)$ of Eqs. (24) and (25) is evaluated at $w_0=0$ and $w'=-(k^2+\mu^2)^{\frac{1}{2}}$, the relevant pole of the meson Green's function. Since $g_p{}^2\Delta'(-(k^2+\mu^2)^{\frac{1}{2}})\ll 1$ for all $\mathbf{k}$, this correction term can be neglected. Since furthermore, $\Delta'(0)=0$, $r_3{}^3$, $r_3{}^1$, and $r_1{}^3$ reduce to the perturbation theory result. Inclusion of $r_1{}^1$ then gives

$$V^{(1)}(\mathbf{k}_0)=\frac{g_p{}^2}{\mu^2}\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b\frac{\boldsymbol{\sigma}_a\cdot\mathbf{k}_0\boldsymbol{\sigma}_b\cdot\mathbf{k}_0}{k_0{}^2+\mu^2}\times\left[1+\frac{1}{3}\frac{g_p{}^2}{\mu^2}\left(\frac{1}{2\pi}\right)^3\int\frac{k^2d\mathbf{k}}{2\omega^3}\right]. \quad \text{(a)}$$

In coordinate space, this is

$$V^{(1)}(x)=-[1+0.07]\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b\frac{g^2}{4\pi}\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\sigma}\cdot\nabla\frac{e^{-\mu x}}{\mu^2 x}, \quad \text{(b)}$$

which differs from the perturbation potential by the numerical factor [1+0.07] in front of Eq. (b).

In evaluating $V^{(2)}$ it is useful to rewrite r of Eq. (18) as

$$r(\mathbf{k}_1,\mathbf{k}_0)=\sum_{i,j=1}^{3}(E\{\boldsymbol{\sigma}\cdot\mathbf{k}_1,\boldsymbol{\sigma}\cdot\mathbf{k}_0\}+F[\boldsymbol{\sigma}\cdot\mathbf{k}_1,\boldsymbol{\sigma}\cdot\mathbf{k}_0])\{\tau_i,\tau_j\}+\sum_{i,j=1}^{3}(G\{\boldsymbol{\sigma}\cdot\mathbf{k}_1,\boldsymbol{\sigma}\cdot\mathbf{k}_0\}+H[\boldsymbol{\sigma}\cdot\mathbf{k}_1,\boldsymbol{\sigma}\cdot\mathbf{k}_0])[\tau_i,\tau_j], \quad \text{(c)}$$

where { , } is the anticommutator and [,] the commutator of the quantities enclosed. Also, we have

$$E=r_3{}^3/3+r_1{}^3/3+r_1{}^1/12,$$
$$F=G=-r_3{}^3/6+r_1{}^3/12+r_1{}^1/12,$$
$$H=r_3{}^3/12-r_1{}^3/6+r_1{}^1/12.$$

Also, Eq. (13) may be rewritten as

$$\mathcal{R}^{(2)}=-\left(\frac{1}{2\pi}\right)^4\frac{1}{4}\int(r^+(\mathbf{k}_1,\mathbf{k}_0)+r^-(\mathbf{k}_0,\mathbf{k}_1))_a(r^+(\mathbf{k}_0,\mathbf{k}_1)+r^-(\mathbf{k}_1,\mathbf{k}_0))_b\frac{d\mathbf{k}_0d\mathbf{k}_1dw_0\delta(\mathbf{p}_1-\mathbf{p}_0+\mathbf{k}_1-\mathbf{k}_0)}{(w_0{}^2-k_0{}^2-\mu^2+i\delta)(w_0{}^2-k_1-\mu^2+i\delta)}, \quad \text{(d)}$$

where $r^+\equiv r(w_0,w_0)$ and $r^-\equiv r(-w_0,-w_0)$. In writing Eq. (d), use has been made of the fact that only terms symmetrical in w_0 contribute to $\mathcal{R}^{(2)}$. Substitution of (c) into (d) gives for $\mathcal{R}^{(2)}$

$$\mathcal{R}^{(2)}=-\left(\frac{1}{2\pi}\right)^4\frac{1}{4}\int\{[3(E^++E^-)^2-2(G^+-G^-)^2\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b]\times\{\boldsymbol{\sigma}_a\cdot\mathbf{k}_1,\boldsymbol{\sigma}_a\cdot\mathbf{k}_0\}\{\boldsymbol{\sigma}_b\cdot\mathbf{k}_1,\boldsymbol{\sigma}_b\cdot\mathbf{k}_0\}-[3(F^+-F^-)^2-2(H^++H^-)^2\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b][\boldsymbol{\sigma}_a\cdot\mathbf{k}_1,\boldsymbol{\sigma}_a\cdot\mathbf{k}_0][\boldsymbol{\sigma}_b\cdot\mathbf{k}_1,\boldsymbol{\sigma}_b\cdot\mathbf{k}_0]\}\times\frac{d\mathbf{k}_1d\mathbf{k}_0dw_0\delta(\mathbf{p}_1-\mathbf{p}_0+\mathbf{k}_1-\mathbf{k}_0)}{(w_0{}^2-k_0{}^2-\mu^2+i\delta)(w_0{}^2-k_1{}^2-\mu^2+i\delta)}, \quad \text{(e)}$$

where $E^+\equiv E(w_0,w_0)$ and $E^-\equiv E(-w_0,-w_0)$. $V^{(2)}$ is obtained by substituting $\mathcal{R}^{(2)}$ into Eq. (15). $\mathcal{R}^{(2)}$ is first calculated for a large but finite nucleon mass. On passing to the limit of infinite mass in Eq. (15), $V^{(2)}$ is a finite and well-defined integral of the form

$$\int\{\alpha_3{}^2(\chi_3{}^{3+}\pm\chi_3{}^{3-})^2+\alpha_0{}^2(\chi_1{}^{3+}\pm\chi_1{}^{3-})^2+\alpha_1{}^2(\chi_1{}^{1+}\pm\chi_1{}^{1-})^2+2\alpha_3\alpha_0(\chi_3{}^{3+}\pm\chi_3{}^{3-})(\chi_1{}^{3+}\pm\chi_1{}^{3-})+2\alpha_3\alpha_1(\chi_3{}^{3+}\pm\chi_3{}^{3-})(\chi_1{}^{1+}\pm\chi_1{}^{1-})+2\alpha_1\alpha_0(\chi_1{}^{3+}\pm\chi_1{}^{3-})(\chi_1{}^{1+}\pm\chi_1{}^{1-})\}\times\frac{d\mathbf{k}_1d\mathbf{k}_0dw_0\delta(\mathbf{p}_1-\mathbf{p}_0+\mathbf{k}_1-\mathbf{k}_0)}{(w_0{}^2-k_0{}^2-\mu^2+i\delta)(w_0{}^2-k_1{}^2-\mu^2+i\delta)}, \quad \text{(f)}$$

where $\alpha_i(i=0, 1, \text{or } 3)$ is a constant. In this form of writing $V^{(2)}$, the effect of the second term of Eq. (15) is to remove the infinities which occur from poles at $w_0=0$. The function $\chi_J{}^I$ is related to $r_J{}^I$ by just the proper subtraction of these infinities.

As a typical term of (f) consider $\alpha_1{}^2[\chi_1{}^{1+}]^2$. The integration over w_0 can be performed exactly.[14] It is convenient to convert it to contour integration, choosing a contour that does not contain the branch points and poles of $1/[1+4\Delta'(w_0)]$.

It can be shown, that the poles and branch points of $F(w_0)\equiv 1/[1+a\Delta'(w_0)]$ always lie in the second and fourth quadrant of the complex plane, close to the real

[14] In the computation of contributions of $r_1{}^1$ to $V^{(2)}$, Eq. (31) was approximated by $r_1{}^1$:

$$r_1{}^1\approx -i\frac{8}{3}\frac{g^2}{\mu^2 w_0}\frac{1}{1+4\Delta'(w_0)}.$$

axis. Thus, if $w_0=x+iy$, then $F(w_0)$ is

$$F(w_0)\equiv\frac{1}{1+a\Delta'(w_0)}$$

$$=\frac{1}{1+a\dfrac{2}{3\pi}\dfrac{1}{4\pi}\left(\dfrac{g^2}{4\pi}\right)\dfrac{1}{\mu^2}\displaystyle\int\dfrac{k^2d\mathbf{k}}{\omega^3}\dfrac{(x+iy)(\omega-x+iy+i\delta)}{[(\omega-x)^2+(y+\delta)^2]}}. \quad (g)$$

Necessary, but not sufficient conditions for $F(w_0)$ to have poles is that the imaginary part of the denominator be zero and that the real part of the integral be one. Thus,

$$\int\frac{k^2d\mathbf{k}}{\omega^3}\frac{\delta x+y\omega}{[(\omega-x)^2+(y+\delta)^2]}=0,$$

$$a\frac{2}{3\pi}\frac{1}{4\pi}\left(\frac{g^2}{4\pi}\right)\frac{1}{\mu^2}\int\frac{x\omega-x^2-y^2-\delta y}{[(\omega-x)^2+(y+\delta)^2]}\frac{k^2d\mathbf{k}}{\omega^3}=-1. \quad (h)$$

For negative a the conditions (h) are possible only if x is positive and y is negative and small; for positive a, both conditions may be satisfied if x and y have opposite signs. In addition to poles, $F(w_0)$ has branch points at $w_0=\mu$ and $\omega_{\max}$. For Chew's value of $g_p^2/4\pi=0.2$ and $\omega_{\max}=3.2\mu$, the poles occur at $w_0\sim\omega_{\max}$ and only for r_1^3, r_3^1 and r_1^1.

For the term $\alpha_1^2[\chi_1^{1+}]^2$ of (f) all singularities of $F(w_0)$ occur in the fourth quadrant, and can be completely avoided by choosing a contour which encloses the upper half plane. Then

$$\frac{\alpha_1^2}{2\pi i}\int[\chi_1^{1+}]^2\frac{dw_0}{(w_0^2-\mu^2-k_0^2+i\delta)(w_0^2-k_1^2-\mu^2+i\delta)}$$

$$=-\frac{g^4}{\mu^4}\frac{32\alpha_1^2}{9(k_0^2+\mu^2)^{\frac{3}{2}}}\left[\frac{1}{k_0^2-k_1^2+i\delta}\times\left(\frac{1}{1+4\Delta'(-(k_0^2+\mu^2)^{\frac{1}{2}})}\right)^2\right]+\text{a similar term with } k_0\to k_1,\ k_1\to k_0. \quad (i)$$

For the term $\alpha_1^2[\chi_1^{1-}]^2$ of (f), the same result is obtained by integrating around the lower half plane. Most of the other terms in (f) cannot be integrated so simply. Thus, for $\chi_1^{1+}\chi_1^{1-}$, it is impossible to avoid including the branch points of $F(w_0)$, since they occur symmetrically above and below the real axis. These integrals were estimated by approximating $F(w_0)$ in the region $-\omega_{\max}\lesssim w_0\lesssim-\mu$, which contains all singularities of $F(-w_0)$. It can be shown that a good approximation to the integral obtains if the approximation is accurate in this region. A best fit to $F(w_0)$ of the form $A+Bw_0+Cw_0^2$ is used in this region. For $-\omega_{\max}\lesssim w_0\lesssim-\mu$, $\Delta'(w_0)$ is a slowly varying function (see Fig. 4), and this approximation should be sufficient. The best fits with the constants of Eq. (35) are

$$\frac{1}{1-2\Delta'(w_0)}\approx1.0165-0.0748w_0/\mu-0.0080w_0^2/\mu^2,$$

$$\frac{1}{1+\Delta'(w_0)}\approx0.9885+0.0289w_0/\mu+0.0034w_0^2/\mu^2,$$

$$\frac{1}{1+4\Delta'(w_0)}\approx\frac{1}{1+(9/2)\Delta'(w_0)}\approx0.9477+0.0928w_0/\mu+0.0118w_0^2/\mu^2. \quad (j)$$

With this approximation, the integrals over w_0, $\mathbf{k}_0$, and $\mathbf{k}_1$ can be evaluated directly. The integral over w_0 gives for the coordinate space representation of $V^{(2)}$:

$$V^{(2)}\approx\text{constant}\frac{g^4}{\mu^4}\lim_{r_0\to r_1}\int\left\{\frac{e^{i(\mathbf{k}_0\cdot\mathbf{r}_0+\mathbf{k}_1\cdot\mathbf{r}_1)}d\mathbf{k}_0d\mathbf{k}_1}{(k_0^2+\mu^2)^{\frac{3}{2}}(k_0^2-k_1^2+i\delta)}\right.$$
$$\times[(E_1-G_1\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b)\{\boldsymbol{\sigma}_a\cdot\mathbf{k}_1,\boldsymbol{\sigma}_a\cdot\mathbf{k}_0\}\{\boldsymbol{\sigma}_b\cdot\mathbf{k}_1,\boldsymbol{\sigma}_b\cdot\mathbf{k}_0\}$$
$$-(F_1-H_1\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b)[\boldsymbol{\sigma}_a\cdot\mathbf{k}_1,\boldsymbol{\sigma}_a\cdot\mathbf{k}_0][\boldsymbol{\sigma}_b\cdot\mathbf{k}_1,\boldsymbol{\sigma}_b\cdot\mathbf{k}_0]]$$
$$\left.+\text{terms symmetric to above with }\mathbf{k}_0\to\mathbf{k}_1,\ \mathbf{k}_1\to\mathbf{k}_0\right\}$$
$$+\text{constant}\frac{g^4}{\mu^4}\lim_{r_0\to r_1}\int\frac{e^{i(\mathbf{k}_0\cdot\mathbf{r}_0+\mathbf{k}_1\cdot\mathbf{r}_1)}dk_0dk_1}{(k_0^2+\mu^2)(k_1^2+\mu^2)}$$
$$\times[(10-1.4\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b)\{\boldsymbol{\sigma}_a\cdot\mathbf{k}_1,\boldsymbol{\sigma}_a\cdot\mathbf{k}_0\}\{\boldsymbol{\sigma}_b\cdot\mathbf{k}_1,\boldsymbol{\sigma}_b\cdot\mathbf{k}_0\}$$
$$-(2.2-1.5\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b)[\boldsymbol{\sigma}_a\cdot\mathbf{k}_1\boldsymbol{\sigma}_a\cdot\mathbf{k}_0][\boldsymbol{\sigma}_b\cdot\mathbf{k}_1,\boldsymbol{\sigma}_b\cdot\mathbf{k}_0]], \quad (k)$$

where

$$E_1=1.2\omega_0/\mu+0.68\omega_0^2/\mu^2,$$
$$F_1=40-0.45\omega_0/\mu+0.02\omega_0^2/\mu^2,$$
$$G_1=60-0.67\omega_0/\mu+0.03\omega_0^2/\mu^2,$$
$$H_1=0.67\omega_0/\mu+0.04\omega_0^2/\mu^2,$$

with $\omega_0=(k_0^2+\mu^2)^{\frac{1}{2}}$. Integration over $\mathbf{k}_0$ and $\mathbf{k}_1$ then gives Eq. (39) for $V^{(2)}$. Only the dominant correction to the perturbation result has been kept; ω_0^2/μ^2 terms are very small and have been neglected.