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Nuclear Forces in Pseudoscalar Meson Theory*

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The nuclear forces in pseudoscalar meson theory are evaluated using a nonrelativistic approximation to the relativistic interaction. The potential is obtained in a form which allows explicit evaluation of the contribution due to the multiple scattering of the virtual mesons between the two nucleons. An approximate expression for the potential, including the multiple scattering of a single meson, is obtained in closed form. For $r<0.5\hbar/\mu c$, the multiple-scattering terms predominate and a power series expansion of the potential is non-convergent. On the other hand, for $r>0.5\hbar/\mu c$ the potential obtained can be approximated by the second- and fourth-order terms as obtained from perturbation theory. With these latter two terms and a phenomenological "repulsive core," quite satisfactory results are obtained for the low-energy properties of the two-nucleon system.

I. INTRODUCTION

RECENT experimental results[1] on the scattering of pions by nucleons have led to a considerable improvement of the understanding of the nature of the pion-nucleon interaction. At the same time the approximate techniques of computation in field theory have been greatly improved over the weak- or strong-coupling approximations previously used almost universally. These developments are particularly apparent in the study of meson-nucleon scattering where the experimental results have been given a qualitatively correct interpretation by Chew[2] and Bethe and Dyson,[3] who considered pseudoscalar meson theory. In the theory of of the scattering, it was found that only a slight improvement over the weak-coupling perturbation theoretic methods of calculation was necessary to give a very considerable improvement in the results. Considerations of the photomesonic processes[4] also seem to indicate that a fairly good agreement between theory and experiment exists.

As a result of this clarification in the experimental and theoretical aspects of some of the problems of meson physics, it has seemed worthwhile to re-examine the meson theory of nuclear forces to see to what extent previous results need to be amended or extended. Although we shall consider the pseudoscalar form of the coupling in pseudoscalar theory, it will become apparent that in this problem the pseudovector form of the coupling can be used to give approximately the same results.

The potential which we shall evaluate has been constructed by methods recently considered by the authors.[5] It differs from that obtained by the S-matrix method or by the usual techniques of perturbation theory in that it does not result from a power series expansion in the coupling constant. The successive terms in the potential are modified from the usual expansion in that for a given term in the potential series, the energies of the virtual mesons are modified by the interaction with the nucleons which results from the terms in the potential of lower order. The effect is to take into account at all times, in evaluating a given contribution to the potential, the interaction of the meson which is already given by previously evaluated terms in the potential series. This form of the potential expansion shows that in the S-matrix expression almost all of the terms of high order in the coupling constant are associated with the multiple scattering of the virtual mesons between the two nucleons. For example, in the potential term which if expanded in a power series in g has as leading term the usual g^4 contribution, the virtual mesons are not emitted into and absorbed from plane wave states but rather from the multiple-scattering state which results from the g^2 interaction of the mesons with the nucleons. This contribution to the potential also is more general than the usual g^4 result in that it includes an important subset of the g^6 contributions together with a subset of all contributions of still higher order. Evaluation of the two leading terms in the potential therefore includes not only all g^2, g^4, and the important g^6 contributions but also indicates the effects of a variety of additional terms of higher order.

In Sec. II we shall summarize the results of the calculation of the potential and of the low-energy scattering parameters; in Sec. III we shall obtain the formal expression for the first two terms in the potential; in Sec. IV the single and multiple scattering problem for the virtual mesons will be evaluated; in Sec. V the explicit form of the potential will be obtained, and finally in Sec. VI we shall consider the non-adiabatic corrections to the potential. Some concluding remarks will be made in Sec. VII.

II. SUMMARY OF RESULTS

The considerations of Secs. III, IV, and V indicate that the potential given by pseudoscalar theory is represented to a fairly good approximation by the

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[1] Anderson, Fermi, Nagle, and Yodh, Phys. Rev. **86**, 793 (1952).
[2] G. F. Chew, Phys. Rev. **89**, 591 (1953).
[3] H. Bethe and F. J. Dyson, Phys. Rev. **90**, 372 (1953).
[4] K. A. Brueckner and K. M. Watson, Phys. Rev. **86**, 923 (1952); B. T. Feld, Phys. Rev. **89**, 330 (1953).
[5] K. A. Brueckner and K. M. Watson, Phys. Rev. **90**, 699 (1953). This paper will hereafter be referred to as I.

second- and fourth-order potentials calculated throughout in the adiabatic limit using the nonrelativistic limit to the pseudoscalar coupling. In obtaining this result, the following procedure has been used: first, the relativistic coupling has been reduced to the well-known nonrelativistic form[6] in which the leading term in g is the usual pseudovector coupling and the next term is a pair coupling of the meson field. Previous estimates of radiative effects[7–9] on the pair formation of nucleons which is associated with the meson-pair term indicate that the term is probably considerably overestimated in the perturbation calculation; this effect is indicated by multiplying the pair term by a parameter which is estimated to have a value roughly $(1+g^2/4\pi)^{-1}$ but is left arbitrary in the potential evaluation. The calculation is then made treating the nucleons adiabatically. The qualitative features of the potential are the following: the contributions to the potential which arise from the meson pair term acting once or twice nearly cancel for $r>0.8\hbar/\mu c$ and give a repulsive core to the potential for $r<0.8\hbar/\mu c$, the exact strength and range depending on the extent to which the radiative effects are taken into account. Higher-order contributions from the pair term alone are also small if the pair damping is taken into account, in agreement with the exact nonrelativistic evaluation of the pair term by Wentzel.[8] The remaining contributions to the potential, which do not involve the pair term (or nucleon pair formation), then include all g^2, g^4, and g^6 contributions together with a subset of all higher-order terms. The evaluation of this potential in the adiabatic limit and the estimate of the terms of higher order than g^4 center about the calculation of the matrices for the scattering of the virtual mesons off the energy shell and the solution of the multiple scattering equations for the propagation of the meson between the two nucleons between emission and absorption. This calculation shows that the scattering by the nucleons individually is very weak off the energy shell and is fairly well described by the Born approximation, this result contrasting markedly with the scattering on the energy shell where the well-known resonance effects occur.[2,3] Using this result, an approximate evaluation of the multiple-scattering equation shows that the effect of higher orders so taken into account is attractive and sets in strongly only near the repulsive core radius $(r=0.3\hbar/\mu c)$ and is less than a 30-percent correction outside $r=0.6\hbar/\mu c$, decreasing very rapidly for increasing r. It seems fairly clear from this result that the potential is rather uncertain near the core since the higher-order terms set in so strongly; it is also this region where radiative effects can be expected to start to modify the gradient-coupling terms appreciably. These uncertainties are reflected in the partially phenomenological insertion of a repulsive core of radius only roughly given by the theory; an increase in the potential strength near the core can of course be compensated by a small increase in the repulsive radius. The conclusion of this investigation into the higher-order multiple-scattering effects is that the potential is given to a fairly good approximation outside $r=0.6\hbar/\mu c$ by the g^2+g^4 contributions from the gradient coupling and inside this radius the strength is somewhat arbitrary and must be obtained phenomenologically by adjustments of the core radius.

A further estimate of the validity of the potential can be made by determining the errors of the adiabatic treatment. This has been done by evaluating the expectation value of the non-adiabatic corrections to the g^2 potential using the wave function obtained from the solution to the potential problem, as discussed in Sec. VI. This is a valid procedure if the nonadiabatic potential is a small perturbation. The result confirms this assumption since the expectation value of this correction is only about $\frac{1}{10}$ Mev compared to 20 Mev for the static potential itself.

The final evaluation of the problem has been made using the g^2+g^4 potentials which, as discussed above, are a good approximation to the more general potential considered in Secs. III and IV outside $r=0.6\hbar/\mu c$. Inside this region the arbitrariness in the potential strength is represented by an adjustable core radius which is expected to lie in the range 0.3–$0.5\hbar/\mu c$. The remaining arbitrary parameters of the theory are the coupling constant and the strength of the pair term in the meson-nucleon coupling. These two parameters cannot, however, be regarded as completely unspecified since the analysis of meson-nucleon scattering indicates that $g^2/4\pi\simeq14$–15 and the estimate of radiative effects on the pair term suggests the pair coupling is quite weak. The results of the evaluation are given in Table I for the deuteron ground state and for the singlet low-energy scattering. The evaluation in the triplet state was done for us by Professor J. M. Blatt and Dr. M. H. Kalos at the University of Illinois using the Illiac (Illinois Automatic Computer); the results for the singlet state are those of Taketani *et al.*,[10] who used a potential very nearly identical with ours for the singlet states. The results are given for two choices of the strength of the pair-coupling term, first unmodified from the perturbation value, and second with the pair terms negligible corresponding to the full strength of the damping. The agreement with the experimental values is considerably better in the latter case which, however, cannot be regarded as significant since the principal effect of the pair terms comes in the very uncertain region between $r=0.3\hbar/\mu c$ (the core radius) and $r=0.6\hbar/\mu c$ where the multiple scattering terms appear strongly.

[6] F. J. Dyson, Phys. Rev. **73**, 929 (1948); L. L. Foldy, Phys. Rev. **84**, 168 (1951).
[7] S. Drell and E. M. Henley, Phys. Rev. **88**, 1053 (1952).
[8] G. Wentzel, Phys. Rev. **86**, 802 (1953).
[9] Brueckner, Gell-Mann, and Goldberger, Phys. Rev. **90**, 476 (1953).
[10] Taketani, Machida, and Onuma, Prog. Theoret. Phys. (Japan) **7**, 45 (1952).

TABLE I. Low energy parameters of the nucleon-nucleon system. The data for the triplet-even (deuteron) state is given for two values of the parameter λ which determines the strength of the pair coupling. $\lambda=1$ corresponds to undamped pair formation; $\lambda=0$ corresponds to negligible contributions from pair formation. The singlet data (reference 10) is given for $\lambda=0$; the results are quite insensitive to the pair terms which give a net repulsive effect quite weak compared with the very strong central attraction. The parameters are adjusted to give correctly the deuteron binding energy and the singlet scattering length. For comparison, the experimental values are $Q=2.73\times10^{-28}$ cm², r_e(triplet)$=1.71\times10^{-13}$ cm, and r_e(singlet)$=2.7\pm0.5\times10^{-13}$ cm.

	Triplet-even		Singlet-even	
	$\lambda=1$	$\lambda=0$		
$g^2/4\pi$	19.5	15.4	13.3	16.0
Q[a]	3.43×10^{-28} cm²	2.83×10^{-28} cm²		
$P_D(\%)$	7.54	6.12		
Effective range	1.93×10^{-13} cm	1.73×10^{-13} cm	2.10×10^{-13} cm	2.585×10^{-13} cm
Core radius	0.300×10^{-13} cm	0.300×10^{-13} cm	0.328×10^{-13} cm	0.384×10^{-13} cm

[a] Due to the method of integration used, these values for the quadrupole moment are not accurate to better than 5 or 10 percent.

It is interesting to note that the coupling constant which gives the correct fit to the nuclear force data agrees very well with the value of 15 deduced from the meson-nucleon scattering. The excellent overall agreement results in part from our treatment of the nonadiabatic terms as discussed in Sec. VI; other treatments[10,11a] have estimated these using the wave function given by the g^2 potential (therefore corresponding to an infinite expectation value of the kinetic energy) and found an effective fourth-order static potential which was very strong and repulsive, changing the sign of the central triplet even state interaction. The treatment of the nonadiabatic terms is not unambiguous; it is felt, however, that a correct estimate must be based at least on a reasonable wave function which reflects approximately the known properties of the deuteron ground state.

III. CONSTRUCTION OF THE POTENTIAL

A. An Approximate Nonrelativistic Reduction of the Pseudoscalar Coupling

The pseudoscalar coupling term which we shall consider is

$$H_{ps}=ig\bar{\psi}\gamma_5\tau_\alpha\psi\phi_\alpha. \tag{1}$$

This interaction is dominated in the weak-coupling approximation by nucleon-pair formation for which the matrix elements of the relativistic operator γ_5 are of the order of unity. This feature of the coupling can be made more explicit by a variety of transformations[6] which exhibit more clearly the nonrelativistic features of the theory; the leading terms in powers of g which result are, in nonrelativistic approximation,

$$H_{ps}=h+h_p, \tag{2}$$

where

$$h=(g/2M)\boldsymbol{\sigma}\cdot\nabla(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\rho(r),\quad h_p=(g^2/2M)\phi^2\rho(r). \tag{3}$$

Here $\rho(r)$ is the nucleon source density. The first term is the usual nonrelativistic approximation to the pseudovector coupling; the meson-pair term arises from the creation and annihilation of a nucleon pair and appears to dominate the interaction since $g\simeq15$.

[11a] A. Klein, Phys. Rev. **90**, 1101 (1953).

It is evident from the observed interaction of pions with nucleons and scattering and production, which is predominantly in P states, that the strong S-state interaction which would appear to arise from the pair coupling term of Eq. (3) is rather unimportant. This suggests that for these phenomena at least the nucleon pair formation which gives rise to the pair-coupling term is suppressed. The explanation of this result has been given in a variety of ways:

1. In the scattering of mesons the potential which arises from the pair term is strongly repulsive and of rather short range,[3] with the result that it gives rise to quite weak scattering.

2. Drell and Henley[7] have shown that in a nonrelativistic treatment of the pseudoscalar coupling, a canonical transformation leads to a form of the theory in which the meson-pair term is very strongly damped so that the effective coupling is quite weak.

3. Wentzel[8] has obtained exact solutions considering only the meson-pair term and has found that its contribution to both nuclear forces and meson scattering is strongly damped, the coupling constant g^2 of the pair term being reduced approximately by a factor $[1+g^2/4\pi]^{-1}$.

4. Brueckner, Gell-Mann, and Goldberger[9] have considered the relativistic pseudoscalar theory and shown that the nucleon-pair formation is strongly damped by reactive effects associated with the strongly bound meson field. This differs somewhat from the results of the nonrelativistic theories in that the correct renormalization of the radiative effects shows that terms which do not involve nucleon pair formation are damped only weakly. The consequence of these considerations is that the coupling terms of Eq. (2) are better approximated by

$$H_{ps}\simeq h+[1+3g^2/16\pi^2]^{-1}h_p. \tag{2'}$$

These results suggest very strongly that the effects of nucleon pair formation are overestimated in the weak coupling treatment of pseudoscalar theory and that they may in fact play a rather unimportant role in the nuclear force problem, at least in the nonrelativistic region. It is felt that the various considerations (1 to 4 above) which give the strong pair damping can be used to estimate the effective pair coupling. We shall for the present leave the effect unspecified by writing the pair term as

$$h_p=\lambda(g^2/2M)\phi^2, \quad (4)$$

where λ expresses the effect of the damping and is accordingly probably rather small.

B. Formal Derivation

The two-nucleon potential is obtained by using the methods recently proposed by the authors.[5] The effects of nucleon recoil on the potential will be neglected in the first approximation. The corrections due to this effect (i.e., nonadiabatic corrections) will be considered at a later state (Sec. VI).

Using the methods of I, the potential is given as an infinite series of terms. The successive terms may each be classified according to the maximum number of virtual mesons present at once. We shall restrict ourselves to only those terms for which at most two mesons are simultaneously present in an intermediate state. This is a well-defined approximation and receives some justification when it is recalled that the "energy denominators" are larger when more mesons are present.

In terms of the interaction of Eq. (2), the first term in the potential series is

$$V_0=\text{D.P.}\Delta_0, \quad (5)$$

where

$$\Delta_0=h_p+h[E-H_0-h_p]^{-1}h. \quad (6)$$

The first term of Δ_0 contributes only to meson scattering; the second term gives both a nuclear force and a meson scattering contribution in which the virtual mesons emitted or absorbed by the linear coupling term h can be scattered by h_p. The notation "D.P." means taking just that part of the operator which is diagonal in meson occupation numbers; h and h_p are defined by Eq. (3). Following again the notation of I, we define $U_0(e)$ and $U_0(o)$ to be the nondiagonal parts (in terms of occupation numbers) of Δ_0 which produce or absorb an even and an odd number of meson pairs, respectively.

The second term in the potential series is

$$V_1=\text{D.P.}[U_0(o)(E-H_0-V_0-U_0(e))^{-1}{}_1U_0(o)]. \quad (7)$$

To make the evaluation of Eqs. (5) and (7) manageable we shall now adopt one further approximation; that is, we shall consider h_p to be a small perturbation and develop the potential

$$V=V_0+V_1 \quad (8)$$

in powers of h_p. In particular, we shall keep only terms linear and quadratic in h_p. The validity of this procedure depends on the assumption of strong damping of the pair term; i.e., on the parameter λ of Eq. (4) having a small value.

To facilitate a symbolic description of the terms in the potential we shall denote the creation of a single meson by an operator with a superscript "(+)" and the annihilation of a single meson by a superscript "(−)." Thus $h^{(+)}$ and $h^{(-)}$ represent the respective matrix elements of h for creation and absorption of a meson, respectively. h_p is bilinear in the meson-field variables, so we employ

$$h_p{}^{(++)},\quad h_p{}^{(+-)},\quad h_p{}^{(--)},$$

to denote the creation of two mesons, etc.

Then we may write Eq. (8) as

$$V=v(1p)+v(2p)+v_g. \quad (9)$$

These terms are classified according to the degree to which h_p occurs. Expanding Eqs. (5) and (7) in powers of h_p and retaining only g^2 and g^4 contributions in the small terms depending linearly and bilinearly on h_p, we easily obtain:

$$v_{2p}=h_p{}^{(--)}\frac{1}{a}h_p{}^{(++)}, \quad (10)$$

$$v(1p)=h^{(-)}\frac{1}{a}h_p{}^{(+-)}\frac{1}{a}h^{(+)}+h^{(-)}\frac{1}{a}h^{(-)}\frac{1}{a}h_p{}^{(++)}$$
$$+h_p{}^{(--)}\frac{1}{a}h^{(+)}\frac{1}{a}h^{(+)}, \quad (11)$$

and

$$v_g=h^{(-)}\left[a-h^{(-)}\frac{1}{a}h^{(+)}\right]^{-1}h^{(+)}, \quad (12)$$

where

$$a\equiv E-H_0. \quad (13)$$

The symbol a will be frequently used for brevity in what follows. The terms $v(1p)$ and $v(2p)$ of Eqs. (10) and (11) can be easily interpreted since they have the usual form given by perturbation theory; v_g[11b] has a more complex structure and will be discussed in more detail in the next section. It is seen to depend only upon the gradient coupling term in the original interaction of Eq. (3).

If we were to expand the operator $[a-h^{(-)}(1/a)h^{(+)}]^{-1}$ and keep only the first two terms, then

$$v_g\simeq h^{(-)}\frac{1}{a}h^{(+)}+h^{(-)}\frac{1}{a}h^{(-)}\frac{1}{a}h^{(+)}\frac{1}{a}h^{(+)}. \quad (14)$$

These represent the so-called second- and fourth-order contributions to the nuclear potential, as obtained from the gradient coupling. The terms given by Eqs. (10), (11) and (14) are evaluated and discussed in detail in Sec. IV. In the rest of the present section we shall consider corrections to Eq. (14) which result from a

[11b] This modified fourth-order potential is identical with that derived in the Tamm-Dancoff method, restricting the maximum number of mesons present to two. (Added in proof.)

more careful analysis of Eq. (12). The reader who is not interested in the rather involved mathematical details of these corrections (which have been qualitatively discussed in Sec. II) may turn immediately to Sec. V.

C. Derivation of the Multiple Scattering Equations

Returning to Eq. (12), we observe that

$$u \equiv h^{(-)}(1/a)h^{(+)}. \tag{15}$$

may be interpreted as a "potential" for the scattering of the meson which is produced by the first $h^{(+)}$ in Eq. (12) and absorbed by the last $h^{(-)}$. Indeed, v_g may be rewritten as

$$v_g = h^{(-)}w(1/a)h^{(+)}, \tag{16}$$

where w satisfies the equation

$$w = 1 + (1/a)uw, \tag{17}$$

which describes the multiple scattering of the virtual mesons by the two nucleons. The form of the "potential" u can be made more explicit if we make use of the form of h, which is

$$h_1 + h_2, \tag{18}$$

where h_i contains the field variables ϕ evaluated at the position $\mathbf{z}_i$ of nucleon "i." We then have

$$u = h_1^{(-)}\frac{1}{a}h_1^{(+)} + h_2^{(-)}\frac{1}{a}h_2^{(+)} + h_1^{(-)}\frac{1}{a}h_2^{(+)} + h_2^{(-)}\frac{1}{a}h_1^{(+)}. \tag{15$'$}$$

The last two terms contribute negligibly to the scattering if the two nucleons are not close together, since they correspond to the absorption of the meson at one nucleon and the emission at the other. The first two terms of Eq. (18) we shall call u_1 and u_2, respectively; they are the potentials for scattering the meson at nucleons (1) and (2). Dropping the last two terms, which will be discussed later in Sec. IV, we have

$$w = 1 + (1/a)(u_1 + u_2)w, \tag{17$'$}$$

which has the solution[12]

$$\begin{aligned} w &= 1 + (1/a)(t_1 w_1 + t_2 w_2), \\ w_1 &= 1 + (1/a)t_2 w_2, \\ w_2 &= 1 + (1/a)t_1 w_1. \end{aligned} \tag{19}$$

Equations (19) represent a system of simultaneous integral equations to be evaluated in terms of the scattering matrices t_1 and t_2 for mesons from nucleons "1" and "2," respectively. We can obtain t_1 and t_2 by solving the auxiliary equations

$$W_1 = 1 + (1/a)u_1 W_1, \quad W_2 = 1 + (1/a)u_2 W_2, \tag{20}$$

[12] K. M. Watson, Phys. Rev. **89**, 575 (1953).

which allow us to evaluate t_1 and t_2 in terms of u_1 and u_2, since we have from the definition of the scattering matrices

$$W_1 = 1 + (1/a)t_1, \quad W_2 = 1 + (1/a)t_2. \tag{21}$$

Equations (20) and (21) describe the scattering of a meson by either nucleon and so represent a two-body problem (i.e., one meson and one nucleon). The solution to Eq. (17) is reduced to a solution of Eqs. (19) and (20). Having found t_1 and t_2, the treatment of Eqs. (19) is made by the methods previously used by one of us.[13]

IV. SOLUTION OF THE MULTIPLE SCATTERING EQUATIONS

A. Determination of the Matrices t_1 and t_2

We need consider only one of the two sets of equations in Eqs. (20) and (21) since these differ only by an interchange of the nucleon indices "1" and "2." We shall therefore attempt to solve the equation referring to nucleon "1." In a momentum representation,

$$(\mathbf{q}'|u_1|\mathbf{q}) = -\frac{g^2}{8M^2}(2\pi)^{-3}\exp[i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{z}_1][w_q w_{q'}]^{-\frac{1}{2}} \times [w_q + w_{q'}]^{-1}[\boldsymbol{\sigma}_1\cdot\mathbf{q}\boldsymbol{\sigma}_1\cdot\mathbf{q}']\boldsymbol{\tau}_1\cdot\mathbf{U}_q\boldsymbol{\tau}_1\cdot\mathbf{U}_{q'}^{+}. \tag{22}$$

Here $w_q = (q^2+\mu^2)^{\frac{1}{2}}$, etc., and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\tau}_1$ are the spin and isotopic spin matrices, respectively, for nucleon "1." $\mathbf{U}_q$ is the absorption operator for the meson with momentum $\mathbf{q}$ and $\mathbf{U}_{q'}^{+}$, the creation operator for the meson with momentum $\mathbf{q}'$ (these quantities are vectors in charge-space). We have neglected the energy of the nucleons in Eq. (22). We shall also neglect it in solving Eqs. (19) and (20).

u_1 is diagonalized with respect to states of total spin and isotopic spin by means of projection operators.

$$E_{\frac{3}{2}} = \tfrac{1}{3}[i\boldsymbol{\tau}_1\cdot\mathbf{U}_q\times\mathbf{U}_{q'}^{+} + 2]$$

and

$$E_{\frac{1}{2}} = -\tfrac{1}{3}[i\boldsymbol{\tau}_1\cdot\mathbf{U}_q\times\mathbf{U}_{q'}^{+} - 1] \tag{23}$$

represent the projection operators on to states of isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$, respectively, for the single-meson single-nucleon system. When the meson is in an orbital P state with respect to the nucleon,

$$\begin{aligned} F_{\frac{3}{2}} &= \frac{1}{qq'}\frac{1}{4\pi}[i\boldsymbol{\sigma}_1\cdot\mathbf{q}\times\mathbf{q}' + 2\mathbf{q}\cdot\mathbf{q}'], \\ F_{\frac{1}{2}} &= -\frac{1}{qq'}\frac{1}{4\pi}[i\boldsymbol{\sigma}_1\cdot\mathbf{q}\times\mathbf{q}' - \mathbf{q}'\cdot\mathbf{q}] \end{aligned} \tag{24}$$

represent the respective projection operators on to states of total angular momentum $\frac{3}{2}$ and $\frac{1}{2}$.

Introducing

$$\alpha = \frac{g^2}{4\pi}\left[\frac{1}{3\pi M^2}\right], \tag{25}$$

[13] K. A. Brueckner, Phys. Rev. **89**, 834 (1953).

and using Eqs. (23) and (24), we can write Eq. (22) as

$$(\mathbf{q}'|u_1|\mathbf{q})=-\frac{\alpha}{4}\frac{qq'}{[w_q w_{q'}]^{\frac{1}{2}}}\frac{\exp[i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{z}_1]}{w_q+w_{q'}}\times[4E_{\frac{3}{2}}F_{\frac{3}{2}}+E_{\frac{1}{2}}F_{\frac{1}{2}}-2E_{\frac{3}{2}}F_{\frac{1}{2}}-2E_{\frac{1}{2}}F_{\frac{3}{2}}]. \quad (26)$$

This permits us to decompose the Eq. (20) for W_1 into four separate equations for the four eigenstates of spin and isotopic spin. The coefficient of $E_{\frac{3}{2}}F_{\frac{3}{2}}$, which we may call $u_1(\frac{3}{2},\frac{3}{2})$, in Eq. (26) is the strongest of the four potentials. We shall consider this state explicitly. The equation (20) for the $(\frac{3}{2},\frac{3}{2})$ state is

$$(\mathbf{k}|W_1(\tfrac{3}{2},\tfrac{3}{2})|\mathbf{k}_0)=\frac{1}{k_0^2}\delta(k-k_0)+\frac{\alpha}{w_k}\int q^2dq\times\frac{\exp[i(\mathbf{q}-\mathbf{k})\cdot\mathbf{z}_1]}{[w_kw_q]^{\frac{1}{2}}}\frac{kq}{w_k+w_q}(q|W_1(\tfrac{3}{2},\tfrac{3}{2})|\mathbf{k}_0). \quad (27)$$

To solve this integral equation, we shall make the reasonable approximation of replacing w_q+w_k by the larger of the two variables w_q and w_k.[14] The resulting equation for $W_1(\frac{3}{2},\frac{3}{2})$ is

$$(k|W_1(\tfrac{3}{2},\tfrac{3}{2})|k_0)=k_0^{-2}\delta(k-k_0)+\frac{\alpha k}{w_k}\left[\frac{\lambda_1(k)}{w_k}+\frac{\lambda_2(k)}{\mu}\right]\exp(-i\mathbf{k}\cdot\mathbf{z}_1), \quad (30)$$

where

$$\lambda_1(k)=\int_0^k\frac{q^3dq}{w_k^{\frac{1}{2}}}(q|W_1(\tfrac{3}{2},\tfrac{3}{2})|k_0)\exp(i\mathbf{q}\cdot\mathbf{z}_1),$$
$$\lambda_2(k)=\mu\int_k^\infty\frac{q^3dq}{w_k^{\frac{3}{2}}}(q|W_1(\tfrac{3}{2},\tfrac{3}{2})|k_0)\exp(i\mathbf{q}\cdot\mathbf{z}_1), \quad (31)$$

and α is defined by Eq. (25). Differentiating these expressions for λ_1 and λ_2 with respect to k and making use of Eq. (30) for $W_1(\frac{3}{2},\frac{3}{2})$, we finally obtain the coupled differential equations:

$$\frac{d\lambda_1}{dk}=\frac{\alpha k^4}{w_k^2}\left[\frac{\lambda_1}{w_k}+\frac{\lambda_2}{\mu}\right]+\frac{k_0}{w_0^{\frac{1}{2}}}\delta(k-k_0),$$
$$\frac{d\lambda_2}{dk}=-\frac{\alpha k^4\mu}{w_k^3}\left[\frac{\lambda_1}{w_k}+\frac{\lambda_2}{\mu}\right]-\frac{k_0\mu}{w_0^{\frac{3}{2}}}\delta(k-k_0), \quad (32)$$

with the boundary conditions that λ_1 vanishes at $k=0$ and that λ_2 vanishes at the cut-off momentum $k_{\max}$ which is introduced to approximate the recoil effects of the relativistic theory and which is necessary to give a finite result. These equations can readily be solved numerically; the following approximate method which gives a closed expression for the scattering matrix is also useful. A detailed comparison of the numerical results with the approximate solutions shows agreement to within the accuracy of the numerical calculations. In the nonrelativistic limit, it is easily shown that general solutions to Eq. (32) are

$$\lambda_1=C_2+C_1\alpha k^5/5\mu^3,\quad \lambda_2=C_1-\lambda_1, \quad (33)$$

where C_1 and C_2 are arbitrary constants. In the relativistic limit, the solutions are:

$$\lambda_2=\mu\alpha^{\frac{1}{2}}[C_3\cos\alpha^{\frac{1}{2}}k-C_4\sin\alpha^{\frac{1}{2}}k],$$
$$\lambda_1=C_3[\sin\alpha^{\frac{1}{2}}k-k\alpha^{\frac{1}{2}}\cos\alpha^{\frac{1}{2}}k]+C_4[\cos\alpha^{\frac{1}{2}}k+k\alpha^{\frac{1}{2}}\sin\alpha^{\frac{1}{2}}k], \quad (34)$$

where C_3 and C_4 are arbitrary constants. The solutions which take proper account of the boundary conditions and of the delta function discontinuity at k_0 can be constructed from these. The two solutions for small and large k are then joined at $k=\mu$ where both limiting forms have approximate validity. The resulting expressions are very lengthy but they take on a simple form in two limiting cases. If $\alpha^{\frac{1}{2}}\tan(\alpha^{\frac{1}{2}}k_{\max})$ is less than one, or equivalently, if $0<g^2/4\pi<20$, for $k_{\max}=M$, then the scattering matrix

$$(k|t_1|k_0)=-\alpha k\left[\frac{\lambda_1}{w_k}+\frac{\lambda_2}{\mu}\right]\exp(-i\mathbf{k}\cdot\mathbf{z}_1) \quad (35)$$

is fairly well given (to within a factor of two) by the Born approximation result,

$$-\alpha\frac{kk_0}{[w_kw_0]^{\frac{1}{2}}}\exp[i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{z}_1]\times\begin{cases}1/w_k, & w_k>w_0\\ 1/w_0, & w_k<w_0.\end{cases} \quad (36)$$

This dependence on w_k and w_0 results from our approximation to the energy denominator w_k+w_0.

If $\alpha^{\frac{1}{2}}\tan(\alpha^{\frac{1}{2}}k_{\max})$ is larger than one, then the largest term in the scattering matrix is approximately

$$-\alpha\frac{kk_0}{[w_kw_0]^{\frac{1}{2}}}\alpha^{\frac{1}{2}}\tan(\alpha^{\frac{1}{2}}k_{\max}), \quad (37)$$

which can be very large if $\alpha^{\frac{1}{2}}k_{\max}\simeq\pi/2$ or, taking $k_{\max}=M$, if $g^2/4\pi=3\pi^3/4=23.3$. This value of the coupling constant is considerably larger than is compatible with the observed pion scattering; in this case, Bethe and Dyson have found that $g^2/4\pi\simeq15$. To verify this result for our approximate methods of solution, we have calculated the scattering of mesons on the energy shell and found that for $g^2/4\pi=15$, the resonance in the scattering occurs at somewhat less than 140 Mev, showing that this value of the coupling constant is consistent with our method of cutoff for high momenta. Accordingly, we can conclude that the Born approximation result [Eq. (26)] is not qualitatively a bad approximation for the scattering. It is noteworthy that for somewhat larger values of the coupling constant than appear to be indicated by the scattering experi-

[14] A similar approximation is discussed by H. Bethe in *Proceedings of the Third Annual Rochester Conference on High Energy Physics.*

ments, the nuclear forces would show a remarkable deviation from the perturbation theory results since the scattering of the virtual mesons would then be very strong.

B. Evaluation of the Multiple Scattering

We have found that to a fair approximation we can take

$$(\mathbf{k}|t_1|\mathbf{k}_0)=(\mathbf{k}|u_1|\mathbf{k}_0), \tag{38}$$

as given by Eq. (26). With this result, we shall return to the multiple-scattering equations (19). These equations are difficult to solve in general. Consequently, we shall make a number of approximations by which, it is felt, that we can obtain a qualitatively correct result. The first approximation involves our choice of the scattering matrices t_1 and t_2. We have remarked that Eq. (38) seems to be correct to within a factor of no more than two over the momentum range, k, $k_0<M$. Actually, we shall not use Eq. (38), but another form [see Eq. (49)] for the t's. This latter form seems to be at least *no worse* an approximation than is Eq. (38).

Our next approximation is connected with the fact that the integrals occurring in Eqs. (19) must be cutoff at high momenta if an unambiguous result is to be obtained. Since we started with an approximate interaction [Eq. (2)] which treats the nucleons as infinitely heavy, we cannot entirely remove these divergences by renormalization. On the other hand, we can identify and first remove the renormalization terms before introducing a cutoff into the theory. This seems quite reasonable and can easily be done.

We first note that if the power series solution for w,

$$w=1+\frac{1}{a}(t_1+t_2)+\frac{1}{a}\left(t_1\frac{1}{a}t_2+t_2\frac{1}{a}t_1\right)+\cdots, \tag{39}$$

is substituted into Eq. (16), characteristic terms such as

$$h_1^{(-)}\frac{1}{a}t_1\frac{1}{a}h_1^{(+)} \tag{40}$$

appear. These evidently describe a self-energy. On the other hand, such a combination as

$$\cdots\frac{1}{a}t_2\frac{1}{a}t_1\frac{1}{a}h_1^{(+)} \tag{41}$$

includes a renormalization of the coupling constant, as is evident from Fig. 1.

FIG. 1. Coupling-constant renormalization. The emission and re-absorption of the meson with momentum $\mathbf{k}'$ leads to renormalization of the coupling constant for the emission of the meson $\mathbf{k}$ which is subsequently scattered.

To remove the renormalization terms which can occur only in association with the first or last scattering, since the iteration $t_1a^{-1}t_2a^{-1}t_1\cdots$ itself contains no such effects, it is only necessary to isolate the first and last scattering from the remainder of the multiple-scattering problem. For this purpose, we express w_1 and w_2 of Eq. (19) as

$$\begin{aligned} w_1&=1+G_{11}(1/a)t_1+G_{12}(1/a)t_2,\\ w_2&=1+G_{21}(1/a)t_1+G_{22}(1/a)t_2. \end{aligned} \tag{42}$$

Substitution into Eq. (19) shows that

$$G_{22}=(1/a)t_1G_{12} \quad\text{and}\quad G_{11}=(1/a)t_2G_{21}, \tag{43}$$

where G_{12} and G_{21} satisfy the uncoupled integral equations

$$G_{12}=1+\frac{1}{a}t_2\frac{1}{a}t_1G_{12}$$

and

$$G_{21}=1+\frac{1}{a}t_1\frac{1}{a}t_2G_{21}. \tag{44}$$

The solution of those equations of course involves no renormalization difficulties.

When Eqs. (42) are substituted into Eq. (19) for w, there results

$$w=1+\frac{1}{a}t+\frac{1}{a}t_2+\frac{1}{a}t_1G_{11}\frac{1}{a}t_2+\frac{1}{a}t_1G_{12}\frac{1}{a}t_2 +\frac{1}{a}t_2G_{21}\frac{1}{a}t_1+\frac{1}{a}t_2G_{22}\frac{1}{a}t_2. \tag{45}$$

This form is particularly useful since, as required above, the first and last scatterings which are the only points at which renormalization effect can occur, are isolated from the rest of the multiple scattering.

On substituting this equation for w into Eq. (16) for v_0, we see that the self-energies are of the form of Eq. (40) and result from the second and third terms only on the right side of Eq. (45). The only renormalizations of g^2 are of the form of Eq. (41) and can be removed if we express t_1 in terms of u_1 by using the identity

$$t_1\frac{1}{a}h_1^{(+)}=u_1\frac{1}{a}h_1^{(+)}+t_1\frac{1}{a}u_1\frac{1}{a}h_1^{(+)} \tag{46}$$

and carry out standard renormalization procedures on the divergent term

$$u_1(1/a)h_1^{(+)}. \tag{47}$$

We return now to the general multiple scattering problem. The potential

$$v_0=h^{(-)}w(1/a)h^{(+)}$$

can be described as due to the emission of a meson by $h^{(+)}$, its multiple scattering between the two nucleons, and finally its reabsorption by $h^{(-)}$. After effecting the coupling constant renormalization, it is evident that we will obtain a similar expression, but one in which $h^{(+)}$ and $h^{(-)}$ are modified by radiative corrections. Also in this modified expression, a meson created, for instance, at nucleon "1" must be first scattered by nucleon "2," etc.

For simplicity in obtaining numerical results we shall not include the radiative corrections to $h^{(+)}$ and $h^{(-)}$, as mentioned above (although this would present no particular difficulties). These give somewhat shorter range corrections and do not seem to be qualitatively important.

We may next [see Eq. (45)] divide v_g into two types of terms: those for which the meson is re-absorbed by the same nucleon which originally emitted it and those for which it is reabsorbed by the other nucleon. The calculation of both types of terms is essentially the same, since in any case the solution to Eqs. (44) is involved. We shall first calculate the terms of the latter type and later return to those omitted. Then we must evaluate

$$v^{(1)} \equiv h_2^{(-)}(1/a)t_1G_{12}(1/a)t_2(1/a)h_1^{(+)} \equiv h_2^{(-)}G_{21}(1/a)h_1^{(+)}. \tag{48}$$

(to which we must add the same expression with "1" and "2" interchanged).

To determine G_{21} we must solve Eq. (44). Rather than to treat the states of spin and isotopic spin in detail, which is straightforward but exceedingly laborious, we shall set the t matrices for all the spin and isotopic spin states equal to the *largest* one [i.e., that for the $(\frac{3}{2},\frac{3}{2})$ state]. This will presumably give us a reasonable upper limit on the magnitude of the multiple scattering effects. (We note that our final results would not have been qualitatively affected had we set all the submatrices t equal to the *smallest* one.)

On the basis of the conclusions of Part B of this section, we shall then set

$$(k|t_1|k_0) = -\frac{\Phi}{\mu^3}\frac{\mathbf{k}\cdot\mathbf{k}_0}{[w_kw_0]^{\frac{1}{2}}}\exp[i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{z}_1]. \tag{49}$$

The spin and isotopic spin matrices do not appear as a result of our setting the t's equal for the spin and isotopic spin substates, as mentioned above. The functional form in Eq. (49) is chosen as a compromise between Eqs. (36) and (37), Φ is taken to be independent of $\mathbf{k}$ and $\mathbf{k}_0$ and will later be assigned a magnitude to correspond as closely as possible to the strength of the t matrix for the $(\frac{3}{2},\frac{3}{2})$ state.

With the choice (59) for the t's, we can easily solve Eq. (44) exactly. We first obtain from Eq. (49):

$$\left(\mathbf{k}\left|t_1\frac{1}{a}t_2\right|\mathbf{k}_0\right) = -\frac{2\pi^2\Phi^2}{\mu^6}\frac{\exp[i(\mathbf{k}_0\cdot\mathbf{z}_2-\mathbf{k}\cdot\mathbf{z}_1)]}{[w_kw_0]^{\frac{1}{2}}}\times[\mathbf{k}\cdot\mathbf{k}_0L+\mathbf{k}_0\cdot\mathbf{R}\mathbf{k}\cdot\mathbf{R}R^{-2}N]. \tag{50}$$

In this equation, $\mathbf{R}=\mathbf{z}_1-\mathbf{z}_2$ and

$$L=\frac{1}{R}\frac{d}{dR}\left[\frac{e^{-\mu R}}{R}\right],\quad N=R\frac{dL}{dR}. \tag{51}$$

Substituting Eq. (50) into Eq. (43), we obtain

$$(\mathbf{k}|G_{21}|\mathbf{k}_0)=\delta(\mathbf{k}-\mathbf{k}_0)-\frac{2\pi^2\Phi^2}{w_k^{\frac{1}{2}}\mu^6}\exp(i\mathbf{k}\cdot\mathbf{z}_1)\times\int\frac{d^3q}{w_q^{\frac{1}{2}}}\exp(i\mathbf{q}\cdot\mathbf{z}_2)[\mathbf{k}L+\mathbf{k}\cdot\mathbf{R}\mathbf{R}R^{-2}N)\cdot\mathbf{q}(\mathbf{q}|G_{21}|\mathbf{k}_0). \tag{52}$$

Defining

$$\mathbf{\Lambda}=\int\mathbf{q}\frac{\exp(i\mathbf{q}\cdot\mathbf{z}_2)}{w_q^{\frac{1}{2}}}(\mathbf{q}|G_{21}|\mathbf{k}_0)d^3q \tag{53}$$

[$\mathbf{\Lambda}$ appears implicitly on the right hand side of Eq. (52)], multiplying both sides of Eq. (52) by $\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{z}_2)[w_k]^{-\frac{1}{2}}$, and integrating over $\mathbf{k}$, we obtain an algebraic equation for $\mathbf{\Lambda}$:

$$\mathbf{\Lambda}=\mathbf{k}_0\frac{\exp(i\mathbf{k}_0\cdot\mathbf{z}_2)}{w_0^{\frac{1}{2}}}+\frac{2\pi^2\Phi^2}{\mu^6}\mathbf{\Lambda}\cdot\int d^3k(\mathbf{k}L+\mathbf{R}\mathbf{k}\cdot\mathbf{R}R^{-2})\times\mathbf{k}\frac{\exp(i\mathbf{k}\cdot\mathbf{R})}{w_k^2}. \tag{54}$$

Since L and N are functions only of R, the k integral is readily evaluated in terms of the derivatives of $e^{-\mu R}/R$. Solving Eq. (54) for $\mathbf{\Lambda}$ and substituting this into the right-hand side of Eq. (52), we obtain

$$(\mathbf{k}|G_{21}|\mathbf{k}_0)=\delta(\mathbf{k}-\mathbf{k}_0)+\eta\frac{\exp[i(\mathbf{k}_0\cdot\mathbf{z}_2-\mathbf{k}\cdot\mathbf{z}_1)]}{w_k^{\frac{1}{2}}w_0^{\frac{1}{2}}}\times\left\{\mathbf{k}\cdot\mathbf{k}_0L+\mathbf{k}\cdot\mathbf{R}\mathbf{k}_0\cdot\mathbf{R}R^{-2}\left(\frac{\lambda L+N}{1-\lambda}\right)\right\}. \tag{55}$$

Here

$$\eta=\frac{2\pi^2\Phi^2}{\mu^6}\left[1-\left[\frac{2\pi^2\Phi}{\mu^3}\right]^2L^2\right]^{-1} \tag{56}$$

and

$$\lambda=2\pi^2\eta R^2N[2L+N]. \tag{57}$$

We obtain the nuclear potential $v^{(1)}$ on substituting Eq. (55) into the second of Eqs. (48). The integrals are easily done. For the central potential of the deuteron state (i.e., the spin triplet, isotopic-spin singlet state), we obtain Eq. (48) has to be multiplied by a factor of two, since we must add to that equation the one resulting from an interchange of the two nucleons:

$$v^{(1)}=-\frac{g^2}{4\pi}\left(\frac{\mu}{2M}\right)^2\mu\left\{\frac{e^{-\mu R}}{\mu R}+2\pi^2\eta\times\left[\frac{e^{-\mu R}}{\mu R}L^2+\frac{1}{\mu^3}\left(\frac{2NL+\lambda L^2+N^2}{1-\lambda}\right)(L+N)\right]\right\}. \tag{58}$$

To choose Φ to agree as well as possible with Eq. (26), we have taken

$$\Phi = (2\pi)^{-3}(\mu^2/4M^2)(\mu R)g^2. \quad (59)$$

The factor μR is inserted to correct for the factor $2\mu/(w_k+w_0)$ which was dropped from the (correct) form of Eq. (36) in Eq. (49). This is, of course, only a qualitative correction for the lost factor. On the other hand, were we to use the functional form (37) for the matrix t, we should have set $(\mu R)=1$ in Eq. (59). In any case, the presence of the factor (μR) does not greatly modify our conclusions concerning the strength of the multiple-scattering corrections. This results from the fact that the multiple-scattering corrections become important so rapidly at small distances that their importance seems to be qualitatively unaffected by the details of the calculations.

We observe from Eq. (58) that the multiple-scattering effects appear as an additive correction to the g^2 potential, $e^{-\mu R}/\mu R$ (starting with a term of g^6). Their importance is indicated in Fig. 2, where the g^2 potential is compared with Eq. (58). The multiple-scattering corrections become important at $\mu R=0.8$ for $g^2/4\pi=15$. The importance of the multiple-scattering correction seems to be due to the singularity of the gradient coupling. To show this, we have everywhere in Eq. (58) replaced $e^{-\mu R}$ by $(e^{-\mu R}-e^{-2\mu R})$. The importance of the multiple-scattering corrections is indeed reduced, as is indicated in the dotted curve of Fig. 2. For convenient comparison, the modified $v^{(1)}$ has been multiplied by a factor $e^{-\mu R}(e^{-\mu R}-e^{-2\mu R})^{-1}$. The result is the dotted curve in Fig. 2.

The actual importance of the multiple-scattering corrections in Eq. (58) have been considerably over-

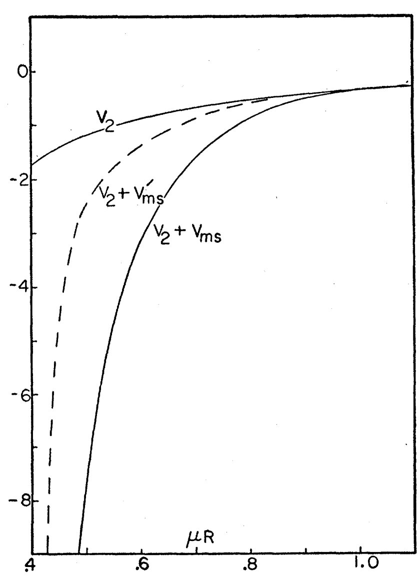

FIG. 2. Multiple-scattering corrections (starting as g^6) to the second-order potential. The dashed curve indicates the depression of the multiple scattering if radiative effects tend to decrease the singularity of the meson coupling. V_2 is taken to be $e^{-\mu R}/R$.

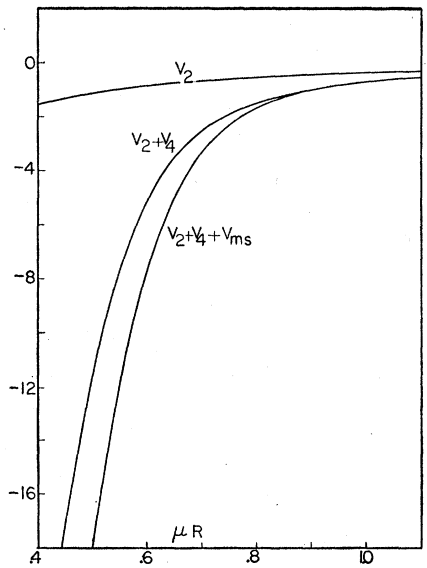

FIG. 3. Multiple-scattering corrections (starting as g^6) to the second- and fourth-order potentials. The notation is the same as in Fig. 2.

estimated in Fig. 2. This is due to our neglect of the fourth-order terms in the potential series, which is calculated in the next section. In Fig. 3 we present a comparison of the second order, the second plus fourth order, and the second plus fourth order plus the multiple-scattering terms (the latter, beginning with a sixth-order force term) for the central-force deuteron potential.

It is seen from Fig. 3 that for $\mu R \gg 1$ the second-order term predominates, while for $\mu R<1$ the higher-order terms are of more importance. However, the corrections to the fourth-order potential are not qualitatively important for $\mu R>0.5$. This is somewhat reassuring, especially when we recall that we have set the matrices t for the four states of spin and isotopic spin equal to the largest. On the other hand, had we set all the t's equal to the smallest, the multiple scattering would have become predominant for $\mu R=0.4$.

The results of Fig. 3 seem to be quite reasonable. The importance of higher-order corrections appear to set in at progressively smaller distances. However, for $\mu R<0.5$ we conclude that the power series expansion must break down completely. For $\mu R>0.5$ the sum of the second- and fourth-order potentials appears to be a valid approximation to the actual potential.

In this connection, the dotted curve of Fig. 2 suggests that if radiative corrections to the individual scattering processes should cut off the singular integrals which have been encountered, the expansion in terms of meson exchanges might be valid to somewhat shorter distances. Unless such a cutoff should occur, we conclude that a power series expansion of Eq. (58) is quite misleading (and divergent) for $\mu R<0.5$.

We must now say something about the terms which have been omitted in our calculation of Eq. (58). Besides the terms (48) which we have calculated, there are also terms of the form

$$v^{(2)}=h_1^{(-)}G_{11}(1/a)h_1^{(+)}=h_1^{(-)}(1/a)t_2G_{21}(1/a)h_1^{(+)},$$

etc. The lowest-order correction to the curves of Fig. 3 which are obtained from this type are of $O(g^8)$. The multiple-scattering corrections are the same as before, since the same matrix G_{21} again appears. Because of the qualitative nature of our calculations of the multiple-scattering effects, there seems to be no point in explicitly calculating these latter terms.

We finally return to the two terms dropped in Eq. (15′) for the potential u. To estimate the importance of these, we have set

$$u=u_3\equiv h_1^{(-)}(1/a)h_2^{(+)}$$

and have calculated w [Eq. (17)]. Approximations similar to those already used were made. The resulting potential had a form very similar to that of Eq. (58). It was somewhat smaller in magnitude, but probably not significantly so.

It seems safe to conclude that in no sense have we obtained quantitative corrections to the fourth-order potential. On the other hand, we have seen the effects of higher-order meson exchanges set in at progressively small distances. It seems likely that in the region $\mu R<0.6$ the power series expansion breaks down completely. Outside this region it is not unlikely that the second plus fourth-order potentials may provide a reasonable approximation to the two-nucleon potential.

V. EVALUATION OF THE g^2 AND g^4 CONTRIBUTIONS TO THE POTENTIALS

Neglecting higher-order multiple-scattering contributions, as discussed in Sec. IV, we now evaluate the potential V of Eq. (9) using Eq. (14) for v_g. The terms of order g^2 and g^4 in v_g are denoted by v_2 and v_4, respectively. The diagrams representing these potentials are given in Fig. 4; the explicit potentials are

$$\begin{aligned}
v_2&=-(g/2M)^2\frac{\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{(2\pi)^3}\int \boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{w_k^2}d^3k,\\
v(\text{1-pair})&=-\frac{6\lambda(g^2/2M)^2}{(2\pi)^6}\int\frac{d^3kd^3k'}{w^2w'^2}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}},\\
v(\text{2-pair})&=-\frac{3\lambda^2(g^2/2M)^2}{(2\pi)^6}\int\frac{d^3kd^3k'}{ww'[w+w']}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}},\\
v_4&=-\frac{(g/2M)^4}{(2\pi)^6}\int\frac{d^3kd^3k'}{w^3w'}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\\
&\quad\times\bigg\{\bigg(\frac{3}{w'}+\frac{2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{w+w'}\bigg)\times(\mathbf{k}\cdot\mathbf{k}')^2\\
&\quad+\boldsymbol{\sigma}_1\cdot\mathbf{k}\times\mathbf{k}'\boldsymbol{\sigma}_2\cdot\mathbf{k}\times\mathbf{k}'\bigg(\frac{3}{w+w'}+\frac{2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{w'}\bigg)\bigg\}.
\end{aligned}\tag{60}$$

In these results we have not included the radiative corrections to the g^2 potential since explicit evaluation[15] shows that they are very small.

The evaluation of these integrals is straightforward; details are given in the appendix to reference 15. We find, with $\beta\equiv\mu[\mu/2M]^2g^2/4\pi$,

$$v_2=\tfrac{1}{3}\beta\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\bigg[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\frac{3+3x+x^2}{x^2}S_{12}\bigg]\frac{e^{-x}}{x},$$

$$v(\text{1-pair})=\beta\bigg[6\lambda\frac{g^2}{4\pi}\frac{\mu}{2M}\bigg]\bigg(\frac{1+x}{x^2}\bigg)^2e^{-2x},$$

$$v(\text{2-pair})=\beta\bigg[-3\lambda^2\frac{g^2}{4\pi}\bigg]\frac{1}{x^2}\frac{2}{\pi}K_1(2x),$$

$$\begin{aligned}
v_4=\beta\bigg[-\frac{g^2}{4\pi}\frac{\mu^2}{4M^2}\bigg]\bigg(\frac{1}{x^3}\bigg)\frac{2}{\pi}\bigg\{\bigg[\frac{4+4x+x^2}{x}\\
\times e^{-x}K_1(x)+(2+2x+x^2)K_0(x)e^{-x}\bigg]\\
\times(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)+\bigg[(23+4x^2)K_0(2x)\\
+\frac{23+12x^2}{x}K_1(2x)\bigg]\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\\
-2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\bigg[6K_0(2x)+\frac{6+4x^2}{x}K_1(2x)\bigg]\\
+\tfrac{2}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\bigg[K_0(x)e^{-x}(1+x)\\
+\frac{2+2x+x^2}{x}K_1(x)e^{-x}\bigg]\\
+\tfrac{1}{3}S_{12}\bigg[36K_0(2x)+\frac{45+12x^2}{x}K_1(2x)\\
-(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\bigg((1+x)K_0(x)e^{-x}\\
+\frac{5+5x+x^2}{x}K_1(x)e^{-x}\bigg)\bigg]\bigg\},
\end{aligned}$$

where $S_{12}=3(\boldsymbol{\sigma}_1\cdot\mathbf{r}\boldsymbol{\sigma}_2\cdot\mathbf{r}/r^2)-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$. The result for v_4 has been previously obtained by Taketani, Machida, and Onuma[10] who, however, used the S-matrix theory to obtain the potential and did not include the terms in $K_0(x)$ and $K_1(x)$. This result arises from a different treatment of the nonadiabatic correction to the g^2 potential; as discussed in detail in the next section.

[15] M. Lévy, Phys. Rev. 88, 725 (1952).

The potential which results from the formation of one or two nucleon pairs in intermediate states is plotted in Fig. 5 for various values of λ. It is weak for $r > \hbar/\mu c$ and becomes repulsive and strong for $r < 0.8\hbar/\mu c$. It is remarkable that the effect of changing λ from 1 to 0.2 is relatively small, having only the effect of slightly shifting the point at which the repulsion sets in strongly. This result differs markedly from previous results which did not take into account the effects of formation of one nucleon pair which give a strong spin independent repulsive potential, nearly cancelling the central attractive term resulting from the formation of two pairs.[16]

The contributions to the potential resulting from no pairs, i.e., those given by the linear nonrelativistic coupling $\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi}$, give (Fig. 6) for even states a strong attractive singlet force, a weaker triplet force, and a strong tensor force of the correct sign which however changes sign at small distances. The central repulsion associated with single-nucleon pair formation also gives a repulsive core which is dominant at small distances, $r \simeq \frac{1}{2}\hbar/\mu c$. For the odd states (Fig. 7), the singlet force is repulsive and very strong; the triplet force is weak and attractive; and the tensor force has a sign opposite to that in even states at large distances but changes sign at $r \simeq 0.6\hbar/\mu c$.

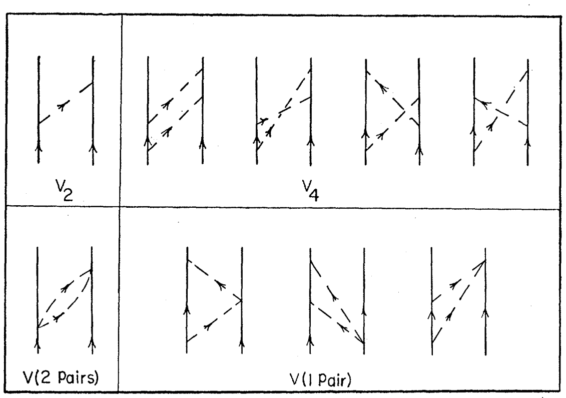

FIG. 4. Contributions to the various potentials. The time sequence of emissions and absorptions is indicated by the directions of the arrows.

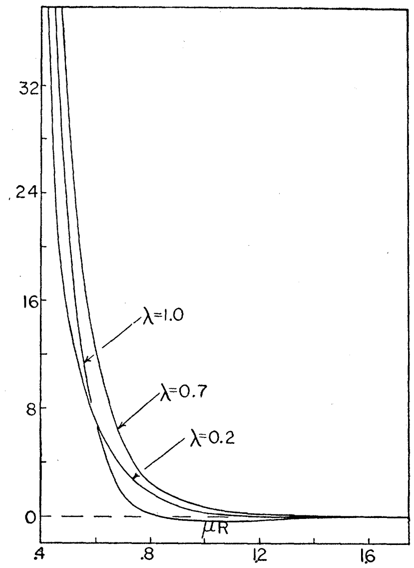

FIG. 5. The total-spin and isotopic-spin independent potential arising from the creation of one or two nucleon pairs in intermediate states. The parameter λ measures the extent of the damping of pair formation; $\lambda=1$ corresponds to no damping. $g^2/4\pi$ has been taken to be 15. The ordinate is in units of $(g^2/4\pi)[\mu/2M]^2\mu = 11.8$ Mev.

This very complicated result for the potential probably has an approximate validity in the non-relativistic region, for $r \geq 0.6\hbar/\mu c$. It is very likely, as discussed in Sec. IV, that higher-order effects such as the strong multiple scattering of the virtual mesons which seems to set in at $r \simeq \frac{1}{2}\hbar/\mu c$ will radically modify the potential in this region. It has become customary to approximate these effects by a repulsive core at small distances; it seems however that although this may be the correct qualitative behavior, the effective radii of repulsion may be different for all six potentials. The matching of these results to experiment should also be attempted only at low energies where the nonstatic effects, such as the strong increase in the scattering of the virtual mesons, are small.

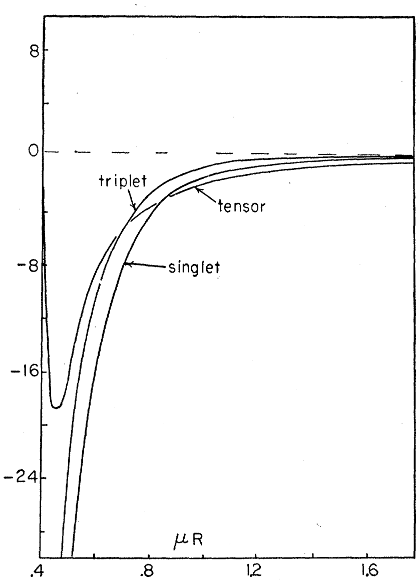

FIG. 6. The sum of potentials for even states which do not involve nucleon pair formation. The ordinate is in units of $(g^2/4\pi)[\mu/2M]^2\mu = 11.8$ Mev.

[16] The large corrections to V (2 pairs) which have been found here seem to be in qualitative agreement with the S-matrix calculations of K. M. Watson and J. V. Lepore, Phys. Rev. **76**, 1157 (1949), using the pseudoscalar coupling. This can be seen from Fig. 3 of that paper, where V (2 pairs) was referred to as the "nonrelativistic" approximation.

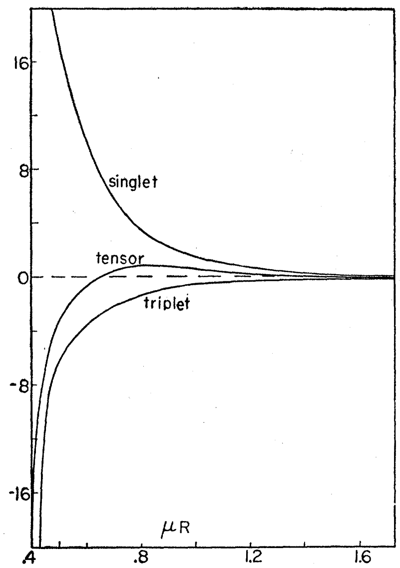

FIG. 7. The sum of potentials for odd states which do not involve nucleon pair formation. The ordinate is in units of $(g^2/4\pi)[\mu/2M]^2\mu$ $=11.8$ Mev.

VI. NONADIABATIC CORRECTIONS TO THE SECOND-ORDER POTENTIAL

It was originally pointed out by Lèvy[15] that a certain type of velocity-dependent corrections to the second order potential can be expressed as static corrections to the fourth-order potential. In the case of the pseudoscalar theory, if recoil is taken into account in evaluating the energies of the intermediate states, one obtains for the second-order potential in momentum space:

$$V^{(2)}(p,k)=-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\frac{g^2}{4M^2}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}}{w_k[w_k+E_p+E_{p-k}-E]}. \quad (62)$$

Here E is the energy eigenvalue (in this case the deuteron binding energy) and E_p and E_{p-k} are the nucleon kinetic energies for momenta $\mathbf{p}$ and $\mathbf{p}-\mathbf{k}$. The nonadiabatic terms in this potential, which appear through the explicit dependence on the nucleon momenta, can be separated if we write

$$V^{(2)}(p,k)=V_s^{(2)}(k)+\delta V(p,k), \quad (63)$$

where the static potential is

$$V_s^{(2)}(k)=-(g^2/2M)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}[1/w_k^2], \quad (64)$$

and the velocity dependent correction is

$$\delta V(p,k)=\frac{g^2}{4M^2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left[\frac{\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\sigma}_2\cdot\mathbf{k}}{w_k^2}\right]\frac{E_p+E_{p-k}-E}{w_k+E_p+E_{p-k}-E}. \quad (65)$$

To express this correction in terms of a static potential, using the method of Lévy, Klein[11] makes use of the nonrelativistic Schrödinger equation in momentum space:

$$\left(\frac{p^2}{M}-E\right)\phi(p)=\int[V_s^{(2)}(k)+\delta V(p,k)]\varphi(p-k)d^3k, \quad (66)$$

which determines the wave function $\varphi(p)$ correct to second order. If δV is treated as a perturbation, the relation

$$[E_p+E_{p-k}-E]\phi(p-k)\simeq\int V_s^{(2)}(k')\varphi(\mathbf{p}-\mathbf{k}-\mathbf{k}')d^3k' \quad (67)$$

can be used to eliminate the momentum dependence from δV. One then finds

$$\left[\frac{p^2}{M}-E\right]\phi(p)\simeq\int V_s^{(2)}(k)\phi^{(2)}(\mathbf{p}-\mathbf{k})d^3k-\int V_s^{(2)}(k')\left[\frac{1}{w_k}\right]V_s^{(2)}(k)\phi(\mathbf{p}-\mathbf{k}-\mathbf{k}')d^3kd^3k' \quad (68)$$

where the second term, arising from the velocity dependent δV, is now expressible as a static potential. Further investigation of this term shows that it gives rise to a very strong repulsive central force in the triplet even states, in fact changing the sign of the potential, and leads to rather unacceptable results for the deuteron ground state.

This method of approximating the nonadiabatic terms is, of course, valid only if they are in fact a small correction since only then is the above iterative procedure valid. The correction is, however, not small; its importance can be traced to the predominance of high-momentum components in the wave function $\phi(p)$ determined by Eq. (66). This is a consequence of the singularity of $V_s^{(2)}$ of Eq. (64) which in coordinate space has a singularity of $1/r^3$ with the result that $\phi(p)$ is a collapsed state with an infinite expectation value for the nucleon kinetic energies. An estimate of the velocity dependent corrections based on Eq. (66) is therefore probably erroneous.[17]

As an alternative procedure which does not make this possibly incorrect treatment of δV of Eq. (65), we have evaluated the velocity dependent terms making use of the entire static potential (second and fourth order and phenomenological repulsive core) to determine the wave function. This wave function is well behaved; the expectation value of the nucleon kinetic energies is quite low so that velocity dependent corrections should be small. The procedure we have followed therefore has been to neglect the velocity-dependent terms in v_2 and v_4, to evaluate the problem using the static potentials, and finally to determine the velocity-dependent corrections using the correct wave functions.

We have found empirically that an excellent fit to the tabulated wave function resulting from the numerical solution of the deuteron problem is (for the S-state

[17] A similar conclusion has been reached independently by Fukuda, Sawada, and Taketani (private communication).

alone)

$$\psi(r)=\frac{12}{7}[4\pi r_D]^{-1}[e^{-(r-r_0)/r_D}-e^{-8(r-r_0)/r_D}]\left(\frac{1}{r}\right) \quad \text{for} \quad r>r_0$$

$$=0 \quad \text{for} \quad r<r_0, \tag{69}$$

where r_0 is the core radius and $r_D=\hbar[ME]^{-\frac{1}{2}}$ is the deuteron radius. In evaluating the velocity dependent correction δV of Eq. (65), we have not considered the small D-state admixture. For the ground state of the deuteron, explicit evaluation then gives

$$\langle\delta V\rangle_{\text{Av}}\simeq -0.11 \text{ Mev}, \tag{70}$$

which is to be compared with a 20-Mev expectation value for the static potential. The velocity-dependent corrections therefore would modify the potential by less than 1 percent, and presumably have little effect on the solutions.

VII. CONCLUSIONS

We have seen that it is possible to derive a nucleon-nucleon potential, working entirely with a nonrelativistic approximation to the pseudoscalar meson theory, which gives a quantitative description of the low-energy properties of the two-nucleon system. The principal difficulties of the analysis concern the closely related questions of convergence of the potential expansions used (S matrix, non-covariant perturbation theory, or the method used in this paper) and the treatment of radiative corrections. We have depended rather strongly on the suppression of nucleon pair formation by radiative effects; the contributions to the potential from pair formation in high order calculated without taking into account such effects otherwise tend to be so large as to invalidate the power series expansions usually used.[11] Radiative effects associated with the low-momentum components of the meson coupling are, however, small so that they do not modify appreciably the potential (which arises from the low-momentum components) in the nonrelativistic region. In this region $[r>0.5\hbar/\mu c]$ our investigations have also shown that the expansion in powers of the coupling constant converges fairly well in that multiple-scattering corrections which start as g^6 are not important corrections to the g^2+g^4 potential. We also have found that nonadiabatic effects are small, principally because the mean kinetic energies of the nucleons in the deuteron are low.

We have also concluded, however, that the potential expansion breaks down quite rapidly as r becomes less than $0.5\hbar/\mu c$ as strong multiple scattering of the virtual mesons sets in, the precise value of r depending on the effect that radiative corrections, for example, have on the high-momentum components of the coupling. The treatment of this region is probably best left phenomenological; the uncertainties are most simply represented by the insertion of the adjustable core radius.

In most of these conclusions we are in qualitative agreement with the comments of Lévy[15] who also considered pseudoscalar theory, although the potential which he derived omitted several terms of importance at least equal to those which he retained. Our work is also closely related to that of Taketani *et al.*,[10] which differs principally in the treatment of the nonadiabatic corrections to the potential (which they found to be very important), but also in that they considered pseudovector coupling. Our conclusions are somewhat more optimistic than those of Klein[11] who also considered pseudoscalar coupling, principally because of the difference again of his treatment of the nonadiabatic terms and because of his estimate of the predominance of the contributions to the potential associated with nucleon-pair formation in high order. This estimate is very sensitive to the extent to which radiative corrections are taken into account; these were, however, not considered by Klein.

Finally, we would like to remark that it is at least possible to conclude that the nuclear potentials given by pseudoscalar meson theory, with the only freely adjustable parameter being the core radius (the coupling constant being fixed to within a small range by other experiments), give a remarkably adequate determination of the six parameters which characterize interactions of nucleons at low energy.

The authors are indebted to Drs. S. Drell, E. Henley, M. Ruderman, and A. Klein for very stimulating and useful discussions of this and related problems.