Convergence of the Adiabatic Nuclear Potential. II

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A conjecture made in a previous paper concerning the non-convergence of the series of adiabatic nuclear potentials for meson pair theory obtained by means of perturbation methods is shown to be incorrect. The correct series is derived and summed and is in agreement with a result given previously by Wentzel. The same methods suffice for the derivation and summation of two additional series of potentials of the pseudoscalar theory with pseudoscalar coupling. One of these has as its leading term the one-pair potential of fourth order, and the other begins with the leading term of sixth order. Each series has the same radius of convergence which is determined by the condition $xe^x > 2\alpha$, where x is the separation of the nucleons in units of the meson Compton wavelength and $\alpha = (g^2/4\pi)(\mu/2M)$. With $(g^2/4\pi) = 15$, perturbation theory converges for $x > 0.85$; with $(g^2/4\pi) = 10$, for $x > 0.57$. The convergence for $x \lesssim 1$ is in any case very slow for these values of the coupling constant. The possibility remains that for substantially smaller values of the coupling constant, as are suggested by the inclusion of radiative corrections, perturbation calculations of adiabatic potentials may yield a meaningful first approximation when used in conjunction with a suitable cut-off.

I. INTRODUCTION

In a previous paper,[1] a qualitative discussion of the behavior of the series of adiabatic potentials of the *ps-ps* theory was presented on the basis of the calculation of the leading terms through eighth order in the coupling constant. There appeared to be a definite indication of non-convergence of the series for $x \lesssim 1$, where $x = \mu r$ is the nucleon separation measured in units of the meson Compton wavelength, μ^{-1}. The attempt was then made to infer a general result for the leading pair term of the potential of order $4n$. The result put forward, without full proof, was that the perturbation series is catastrophically divergent. This result is, in fact, incorrect. The purpose of the present work is to demonstrate that it is possible to obtain, in the adiabatic limit, the general term in a few well-defined series of potentials, to investigate the convergence of these series, and to sum them.

The potentials to be investigated are the leading pair terms of order $4n$, considered in I; the potential of order $4n$ with one pair fewer, prototypes of which (one pair term of fourth order and three pair term of eighth order) were given in I; the leading terms of order $4n+2$. The latter can be characterized diagrammatically by open meson-line perimeters with end points at each of the nucleon positions. Thus the leading term is a sixth-order potential in which each nucleon undergoes one pair and one gradient interaction. The contribution of the sixth-order term has been previously computed[2] by means other than are contemplated here. The calculations are carried out in Secs. IIA, B, and C, respectively.

For the purposes of this presentation it is simplest to use the form of the theory which results from the Dyson[3] or Foldy[4] transformations,

$$\mathcal{H}'(x) = g\bar{\psi}(x)\gamma_5\tau_i\psi(x)\phi_i(x) \rightarrow (g^2/2M)\bar{\psi}\psi\phi^2 + (g/2M)\bar{\psi}\boldsymbol{\sigma}\tau_i\psi\cdot\nabla\phi_i + \text{higher order terms.} \quad (1)$$

By computing the irreducible interactions that Eq. (1) contributes to the kernel of the relativistic two-body equation,[5-7] it is a straightforward matter to verify that the method proposed in I for the computation of the leading pair terms in the adiabatic limit is indeed applicable. The argument given there was that the restriction, in obtaining the leading contributions, to matrix elements with at most one nucleon pair at a time, as suggested by the original pseudoscalar coupling,[2,8] could be lifted; it was sufficient merely to demand that the requisite number of nucleon pairs be associated with the motion of each nucleon. This argument is indeed tantamount to carrying out the transformation of Eq. (1) term by term in the interaction function.

The potentials will be computed from an expression which is a trivial modification of Eq. (10) of I:

$$V(\mathbf{r},\mathbf{r}') = -it^{-1}\int dt_1dt_2dt_1'dt_2'd\mathbf{R}' \times \exp[iM(t_1+t_2-t_1'-t_2')]I(x_1,x_2;x_1'x_2'). \quad (2)$$

Aside from the use of individual time coordinates, we have added, for reasons of symmetry, an additional time integration and correspondingly divided by a "large" time interval t.

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II. COMPUTATION OF POTENTIALS

A. Pair Potential of Order 4n

For the case under consideration, the interaction function which enters Eq. (2) has the form[9]

$$I(x_1, x_2; x_{2n-1}, x_{2n}) = \sum 3(2)^{2n}(-i\lambda)^{2n} \int dx_3 \cdots dx_{2n-2}$$
$$\times [G(x_1-x_3)\cdots G(x_{2n-3}-x_{2n-1})]^{(1)}$$
$$\times [G(x_2-x_4)\cdots G(x_{2n-2}-x_{2n})]^{(2)}$$
$$\times \Delta(x_1-x_{i_1})\Delta(x_1-x_{i_2})\cdots$$
$$\times \Delta(x_{2n-1}-x_{i_{n-1}})\Delta(x_{2n-1}-x_{i_n}). \quad (3)$$

Here $i_1, i_2, \cdots, i_n$ is one of the permutations of 2, 4, $\cdots 2n$, the coordinates of the second nucleon, which yields a closed meson-line perimeter; the summation is over the $n!(n-1)!/2$ such terms; $\lambda = g^2/2M$. Equation (3) can be determined either directly from the interaction Hamiltonian of Eq. (1) or can be inferred term by term from the form of the theory used in I. The same statement can be made for the other interactions to be used in this paper.

We insert Eq. (3) into Eq. (2) and immediately carry out the reduction to the adiabatic limit by means of the equation

$$G(x) \cong i\delta(\mathbf{r})e^{-iMt}, \quad t>0$$
$$\cong 0, \quad t<0. \quad (4)$$

As in I, there results immediately a local interaction which is given by

$$V_{4n}(r) = it^{-1}3(2\lambda)^{2n}\sum \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_3 \cdots$$
$$\times \int_{-\infty}^{t_{2n-3}} dt_{2n-1} \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_4 \cdots$$
$$\times \int_{-\infty}^{t_{2n-2}} dt_{2n}\Delta(\mathbf{r}, t_1-t_{i_1})\Delta(\mathbf{r}, t_1-t_{i_2})\cdots$$
$$\times \Delta(\mathbf{r}, t_{2n-1}-t_{i_{n-1}})\Delta(\mathbf{r}, t_{2n-1}-t_{i_n}). \quad (5)$$

Equation (5) was also obtained by the methods of I. The essential symmetry property required for its evaluation was overlooked, however.

Suppose we permute the time coordinates of one of the nucleons,

$$t_1 \to t_{\alpha 1} \cdots t_{2n-1} \to t_{\alpha 2n-1}. \quad (6)$$

Then in virtue of the definition of the set of perimeters, the integrand of each term of the sum in Eq. (5) is transformed into the integrand of another term of the sum. In short the integrand as a whole is invariant. On the other hand, the time integrations, previously subject to the condition $t_1 > t_3 > \cdots > t_{2n-1}$ are now subject to the condition $t_{\alpha 1} > t_{\alpha 3} > \cdots > t_{\alpha 2n-1}$. If we carry out all possible permutations of the time coordinates of the two particles independently and then average over the resulting $(n!)^2$ expressions, we obtain the following form for the potential:

$$V_{4n}(r) = it^{-1}3(2\lambda)^{2n}(n!)^{-2}\sum \int dt_1 \cdots$$
$$\times dt_{2n}\Delta(\mathbf{r}, t_1-t_{i_1})\Delta(\mathbf{r}, t_1-t_{i_2})\cdots$$
$$\times \Delta(\mathbf{r}, t_{2n-1}-t_{i_{n-1}})\Delta(\mathbf{r}, t_{2n-1}-t_{i_n}), \quad (6)$$

in which the integrations are now carried out over the *entire* $2n$ dimensional space $t_1, t_2, \cdots t_{2n}$.

The remainder of the calculation is straightforward. We Fourier-analyze the meson propagation functions,

$$\Delta(\mathbf{r}, t) = (2\pi)^{-4}\int d^4k \exp[i\mathbf{k}\cdot\mathbf{r} - ik_0t][\omega^2 - k_0^2 + i\eta]^{-1}, \quad (7)$$

interchange the order of momentum and temporal integrations, and perform the latter. For each term of the sum in Eq. (6) we then obtain $2n-1$ delta functions of linear combinations of the meson energies $k_{0,1}\cdots k_{0,2n}$ and one remaining time integral which cancels the factor t^{-1}. It is then possible to perform trivially $2n-1$ integrals with respect to the meson energies, leaving one such integral which we label with the variable k_0. At this stage it is seen that all $n!(n-1)!/2$ terms of Eq. (6) contribute equally to the sum, so that the latter is most simply rewritten as[10]

$$V_{4n}(r) = 3i(2\lambda)^{2n}(8\pi^3)^{-2n}(4\pi n)^{-1}$$
$$\times \int d\mathbf{k}_1 \cdots d\mathbf{k}_{2n} \exp[i(\mathbf{k}_1 + \cdots + \mathbf{k}_n)\cdot\mathbf{r}]$$
$$\times \int dk_0[(k_0^2-\omega_1^2)\cdots(k_0^2-\omega_{2n}^2)]^{-1}$$
$$= -3\mu\alpha^{2n}(2^{2(n-1)}/n)(2/\pi)K_1(2nx)/x^{2n}, \quad (8)$$

where $\alpha = (g^2/4\pi)(\mu/2M)$. Equation (8) can also be inferred by expanding the exact solution given by Wentzel[11,12] for meson pair theory with stationary sources. From the results of the latter author, one can

[9] It is perhaps worthwhile to emphasize that the propagation functions employed are those defined, for example, in reference 5. We have been unconventional here and in references 1 and 2 in omitting identifying subscripts on these functions.

[10] See the appendix of reference 1 for the evaluation of the integral.

[11] G. Wentzel, Helv. Phys. Acta **15**, 111 (1942).

[12] The agreement of the result with that of reference 11 demonstrates the correctness of the assumption made in reference 1 that higher-order pair diagrams which do not consist of single closed meson perimeters are cancelled in the adiabatic limit by the iterates of lower-order diagrams.

obtain a closed expression for the sum

$$V(x)=\sum_{n=1}^{\infty} V_{4n}(x)$$
$$=-\frac{3\mu}{4\pi i}\int_0^\infty \frac{kdk}{(1+k^2)^{\frac{1}{2}}}\log\left\{\frac{1-(2\alpha e^{ikx}/x)^2}{1-(2\alpha e^{-ikx}/x)^2}\right\}$$
$$=-\frac{3\mu}{2\pi}\int_0^\infty \frac{kdk}{(1+k^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{4\alpha^2 \sin 2kx/x^2}{1-4\alpha^2\cos 2kx/x^2}\right\}. \quad (9)$$

The integral of Eq. (9) is well-defined for values of x which are larger than x_c, defined by

$$x_c=2\alpha. \quad (10)$$

It is, however, possible to investigate the convergence of the series *per se*, most simply by introducing the asymptotic form of the function

$$(2/\pi)K_1(2nx)\sim e^{-2nx}/(\pi nx)^{\frac{1}{2}}. \quad (11)$$

A straightforward application of the ratio test then informs us that the series converges provided that $x>x_c$, with x_c here defined by

$$x_c\exp(x_c)=2\alpha, \quad (12)$$

a less stringent condition than Eq. (10). We shall prefer for purposes of discussion the less accurate Eq. (12), since it will be seen that the same condition of convergence will obtain rigorously for the other series of potentials to be derived below. We therefore defer numerical discussion of Eq. (12) until this has been done.

B. Potential of Order 4*n* with One Pair Fewer

A typical diagram is shown in Fig. 3(a) of I. The number of distinct diagrams in the general order can be computed as the product of the number of ways of choosing two vertices of one of the nucleons for the gradient interaction by the number of ways of drawing continuous meson perimeters beginning with these points. The result is $n!(n+1)!/2$. There is actually twice this number of interactions, since either nucleon could have been selected to bear the gradient coupling. This will be taken account of in the final result. By arguments previously mentioned, the interaction can be shown to have the form

$$I(x_1x_2;x_{2n+1},x_{2n})=-\sum 3(2)^{2n}(-i\lambda)^{2n}(2M)^{-1}$$
$$\times\int dx_3\cdots dx_{2n-1}[G(x_1-x_3)\cdots G(x_{2n-1}-x_{2n+1})]^{(1)}$$
$$\times[G(x_2-x_4)\cdots G(x_{2n-2}-x_{2n})]^{(2)}(\boldsymbol{\sigma}_1\cdot\nabla_{2j+1})$$
$$\times(\boldsymbol{\sigma}_1\cdot\nabla_{2l+1})\Delta(x_1-x_{i1})\cdots\Delta(x_{2j+1}-x_{i,2j+1})\cdots$$
$$\times\Delta(x_{2l+1}-x_{i,2l})\cdots\Delta(x_{2n+1}-x_{in}), \quad (13)$$

where ∇_{2j+1} operates on the variable $\mathbf{r}_{2j+1}$ for the first nucleon, etc., and the summation is over $n!(n+1)!/2$ terms.

The argument now proceeds precisely as in Sec. A. The adiabatic limit is first taken. The resulting static potential has an integrand which is invariant as a whole under independent permutations of the time coordinates of either particle. Averaging over all possible permutations which is, in this case, $n!(n+1)!$, we obtain the potential

$$V_{4n}'(r)=\mathrm{Lim}(r_i\rightarrow r)\Big\{3(2\lambda)^{2n}[n!(n+1)!]^{-1}t^{-1}$$
$$\times\sum\int dt_1\cdots dt_{2n+1}(\boldsymbol{\sigma}_1\cdot\nabla_j)(\boldsymbol{\sigma}_1\cdot\nabla_l)$$
$$\times\Delta(r_1,t_1-t_{i1})\cdots\Delta(r_{2n},t_{2n+1}-t_{in})\Big\}. \quad (14)$$

The integrals over the time variables now require that $2n$ linear homogeneous equations be satisfied by the $2n$ meson energy variables. Summing over the $n!(n+1)!/2$ terms, we are left with

$$V_{4n}'(r)=-3(2)^{2n-1}\lambda^{2n}(8\pi^3)^{-2n}\int d\mathbf{k}_1\cdots$$
$$\times d\mathbf{k}_{2n}\exp[i(\mathbf{k}_1+\cdots+\mathbf{k}_n)\cdot\mathbf{r}]$$
$$\times\mathbf{k}_j\cdot\mathbf{k}_l[\omega_1^2\cdots\omega_j^2\cdots\omega_l^2\cdots\omega_{2n}^2]^{-1}$$
$$=3\mu\alpha^{2n}(\mu/2M)2^{2n-1}(1+x^{-1})^2e^{-2nx}/x^{2n}. \quad (15)$$

The sum of the series with Eq. (15) as general term is obtained trivially:

$$V'(x)=\sum_{n=1}^{\infty}V_{4n}'(x)=6\mu\alpha^2(\mu/2M)(1+x^{-1})^2e^{-2x}/x^2$$
$$\times[1-4\alpha^2e^{-2x}/x^2]^{-1}$$
$$=V_4'(x)[1-4\alpha^2e^{-2x}/x^2]^{-1}. \quad (16)$$

It is thus seen that Eq. (12) determines the radius of convergence of Eq. (16).

C. Potential of Order 4*n*+2

Here one chooses a single gradient interaction for each particle. The total number of diagrams in the general order is easily found to be $[(n+1)!]^2$, and the interaction is given by

$$I(x_1,x_2;x_{2n+1},x_{2n+2})$$
$$=2^{2n}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2(-i\lambda)^{2n+1}(2M)^{-1}\sum\int dx_3\cdots dx_{2n}$$
$$\times[G(x_1-x_3)\cdots G(x_{2n-1}-x_{2n+1})]^{(1)}$$
$$\times[G(x_2-x_4)\cdots G(x_{2n}-x_{2n+2})]^{(2)}$$
$$\times(\boldsymbol{\sigma}_1\cdot\nabla_{2j+1})(\boldsymbol{\sigma}_2\cdot\nabla_{il})\Delta(x_1-x_{i1})\cdots$$
$$\times\Delta(x_{2n+1}-x_{in+1}), \quad (17)$$

where the summation is over $[(n+1)!]^2$ terms and the subscripts on the gradient operators are meant to imply that they operate on appropriate members of the set of coordinates for the first and second nucleon respectively. The remainder of the calculation differs in no wise from that described in Sec. B. It suffices therefore to state the result,

$$V_{4n+2}(r)=\mu\tfrac{1}{3}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 2^{2n}\alpha^{2n+1}(\mu/2M)\times[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+S_{12}]e^{-(2n+1)x}/x^{2n+1}. \quad (18)$$

The series of which Eq. (18) is the general term has the sum

$$\begin{aligned}V''(x)&=\sum_{n=1}^{\infty}V_{4n+2}(x)\\&=\frac{4}{3}\mu\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\alpha^3(\mu/2M)(1+x^{-1})^2e^{-3x}/x^3\\&\quad\times[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+S_{12}][1-4\alpha^2e^{-2x}/x^2]^{-1}\\&=V_6(x)[1-4\alpha^2e^{-2x}/x^2]^{-1}.\end{aligned} \quad (19)$$

The same remarks about convergence as in Sec. B are therefore applicable.

III. DISCUSSION OF RESULTS

Conclusions concerning the validity of perturbation theory can be drawn immediately from the application of Eq. (12). Thus for $(g^2/4\pi)(\mu/2M)=1$, corresponding to $g^2/4\pi=15$, Eq. (12) requires $x>0.85$ for convergence, for $(g^2/4\pi)=10$, $x>0.57$. For such values of the coupling constant, it is clear that no plausible account of nuclear jorces can be based on the leading terms of the series. Moreover, as already indicated in I, the repulsive potential $V'(x)$ of Eq. (17) predominates numerically over the other pair terms to such extent that were the coupling constant as large as the above values, the possibility of obtaining agreement with the low-energy two-nucleon data from the *ps-ps* theory would effectively be ruled out.

However, all treatments of the potential problem which have taken into account radiative corrections[8,11,13] have indicated that as a consequence of self-interactions the pair coupling is strongly damped. If we take $(g^2/4\pi)_{\rm eff}(\mu/2M)\sim 0.1$, which is probably as much of a reduction as self-energy effects are likely to produce, then our series converge for $x>0.17$. Under these circumstances, perturbation theory is applicable for distances as small as $x=0.5$. Assuming that the gradient interaction of Eq. (1) is undamped compared to the pair coupling, one has from this result a good indication of the domain of applicability of perturbation theory to the former interaction for which the effective expansion parameter is $(g^2/4\pi)(\mu/2M)^2\sim 0.1$.

A more direct attack on the applicability of perturbation theory to the gradient interaction will be presented in a subsequent publication. The author also hopes to discuss in later publications the relationship of the potentials computed in this paper to a possible consistent model for nuclear forces.

[13] M. Ruderman, Phys. Rev. **90**, 183 (1953); Brueckner, Gell-Mann, and Goldberger, Phys. Rev. **90**, 476 (1953).