The Hyperfine Structure of Hydrogen*

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The two-body formalism of Schwinger is modified to consider the case of the hydrogen atom. The proton's anomalous moment is treated by adding a Pauli-type term to the Lagrangian. A perturbation theory based upon the Green's function is developed and the first-order correction to the Fermi hyperfine splitting of the ground state is calculated. The method of calculation used is that of Karplus and Klein in their positronium work. Aside from the usual renormalizations encountered, an extra infinity appears in the calculation associated with the assumption of a point anomalous magnetic moment for the proton. On the hypothesis that the proton's moment is actually distributed, cutoffs are inserted. The modified hyperfine formula leads to a new value of $\alpha: 1/\alpha = 137.0378$ for a cutoff at the meson length and $1/\alpha = 137.0374$ for a cutoff at the proton length.

I. INTRODUCTION

DURING the past two years, several derivations of relativistic two-body Green's function and wave-function equations have appeared in the literature.[1] The derivation of Bethe and Salpeter followed the methods of Feynman while that of Gell-Mann and Low followed those of Dyson. The approach that we shall use here is the one given by Schwinger, where use is made of the technique of variational derivatives with respect to external sources and the formalism developed in Schwinger's "Theory of Quantized Fields. I."[2]

The specific problem in which we shall be interested is the reduced mass corrections to the Fermi hyperfine splitting of the ground state of hydrogen. To obtain an adequate two-body equation for hydrogen, certain modifications must be made in Schwinger's derivation. First, we are here dealing with two distinguishable particles rather than with two particles of the same field. Second, account must be taken of the proton's meson anomalous moment. The latter is accomplished phenomenologically by adding a Pauli type term, $\frac{1}{2}\mu'\sigma_{\mu\nu}F_{\mu\nu}$ (where μ' is the anomalous magnetic moment), to the Lagrangian. The net effect is to allow the proton to interact with the electromagnetic field via its normal current and its spin current.

Aside from the usual infinities of quantum electrodynamics, two others appear in these calculations. The first is an ultraviolet divergence in certain of the Pauli moment terms. This is due to the fact that the Pauli moment introduces a point dipole. On the assumption that the anomalous moment is due to a meson-nucleon interaction, one may postulate that the moment actually has the extension of the order of the meson Compton wavelength and cut off all divergent integrals accordingly. Fortunately, the results depend upon the cutoff only logarithmically and hence the calculation is not sensitive to its precise value. The second difficulty involves an infrared divergence and is due to the fact that we have assumed free particle intermediate state Green's functions in the perturbation theory. The appropriate method of treating this difficulty for a hyperfine splitting has been shown by Karplus and Klein.[3] Since the spin-spin interactions are all high frequency, the infrared divergences will cancel if one makes the free particle approximation consistently throughout the entire calculation. As the actual calculations are similar to the ones done by Karplus and Klein, they will only be sketched here.

II. THE ELECTRON-PROTON TWO-BODY GREEN'S FUNCTION

For the system of coupled electron, proton, and photon fields, the Lagrange function may be taken as[4]

$$\begin{aligned}L=&-\tfrac{1}{4}[\bar{\psi},\gamma_{1\mu}(-i\partial_\mu-e_1A_\mu)\psi+m_1\psi]+\tfrac{1}{2}[\bar{\psi},\eta]\\&-\tfrac{1}{4}[\bar{\phi},\gamma_{2\mu}(-i\partial_\mu-e_2A_\mu)\phi\\&-\tfrac{1}{2}\mu'\sigma_{2\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)\phi+m_2\phi]\\&+\tfrac{1}{2}[\bar{\phi},\zeta]+\text{Herm. conj.}+\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}\\&-\tfrac{1}{4}\{F_{\mu\nu},\partial_\mu A_\nu-\partial_\nu A_\mu\}+J_\mu A_\mu,\end{aligned}\quad(2.1)$$

where ψ and ϕ are the electron and proton field variables respectively. Subscripts "1" and "2" refer to electron and proton quantities. η and ζ are the prescribed external source for the electron and proton fields and J_μ for the photon fields. η and ζ (and their variations) anti-commute with all fermion field variables. The equations of motion obtained from (2.1) are

$$\begin{aligned}&[\gamma_{1\mu}(-i\partial_\mu-e_1A_\mu)+m_1]\psi=\eta,\\&[\gamma_{2\mu}(-i\partial_\mu-e_2A_\mu)\\&\qquad-\tfrac{1}{2}\mu'\sigma_{2\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+m_2]\phi=\zeta,\\&\partial_\nu F_{\mu\nu}=J_\mu+j_{1\mu}+j_{2\mu},\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\end{aligned}\quad(2.2)$$

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[1] J. Schwinger, Proc. Nat. Acad. Sci. U. S. **37**, 452, 455 (1951); E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951); M. Neuman, Phys. Rev. **85**, 129 (1952); H. Kita, Prog. Theor. Phys. (Japan) **7**, 217 (1952).

[2] J. Schwinger, Phys. Rev. **82**, 914 (1951).

[3] R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952).

[4] Natural units are employed throughout: $\hbar=c=1$.

where

$$j_{1\mu}=\tfrac{1}{2}e_1[\bar{\psi},\gamma_{1\mu}\psi],$$
$$j_{2\mu}=\tfrac{1}{2}e_2[\bar{\phi},\gamma_{2\mu}\phi]-\tfrac{1}{2}\mu'\partial_\lambda[\bar{\phi},\sigma_{2\lambda\mu}\phi]. \tag{2.3}$$

For any operator $F(x)$, one may define the quantity

$$\langle F(x)\rangle=(0\sigma_1|F(x)|0\sigma_2)/(0\sigma_1|0\sigma_2), \tag{2.4}$$

where σ_1 and σ_2 are two space-like surfaces and 0 signifies the vacuum state. The one-particle electron and proton Green's functions are defined by the equations[5]

$$G_1(x_1,x_1')=(\delta/\delta\eta(x_1'))\langle\psi(x_1)\rangle]_{\eta,\zeta\to0},$$
$$G_2(x_2,x_2')=(\delta/\delta\zeta(x_2'))\langle\phi(x_2)\rangle]_{\zeta,\eta\to0}. \tag{2.5}$$

By use of the fundamental dynamical principle,[2] the one-particle Green's functions may be expressed by

$$G_1(x_1,x_1')=i\langle(\psi(x_1)\bar{\psi}(x_1'))_+\rangle\epsilon(x_1,x_1'),$$
$$G_2(x_2,x_2')=i\langle(\phi(x_2)\bar{\phi}(x_2'))_+\rangle\epsilon(x_2,x_2'). \tag{2.6}$$

From the equations of motion (2.2) and the defining equations (2.5), it can be seen that

$$[\gamma_{1\mu}(-i\partial_{1\mu}-e_1\langle A_\mu(x_1)\rangle)+\mathfrak{M}_1]G_1(x_1,x_1')=\delta(x_1-x_1'),$$
$$[\gamma_{1\mu}(-i\partial_{2\mu}-e_2\langle A_\mu(x_2)\rangle)-\mu'\sigma_{2\mu\nu}\partial_{2\mu}\langle A_\nu(x_2)\rangle+\mathfrak{M}_2]\times G_2(x_2,x_2')=\delta(x_2-x_2'), \tag{2.7}$$

where

$$\mathfrak{M}_1=m_1+ie_1\gamma_1\frac{\delta}{\delta J(x_1)},$$
$$\mathfrak{M}_2=m_2+ie_2\gamma_2\frac{\delta}{\delta J(x_2)}+i\mu'\sigma_{2\mu\nu}\partial_{\mu'}\frac{\delta}{\delta J_\nu(x')}\Big]_{x'\to x}. \tag{2.8}$$

For the photon field, the equations of motion yield

$$-\partial_\nu{}^2\langle A_\mu(x)\rangle+\partial_\mu\partial_\nu\langle A_\nu(x)\rangle=J_\mu(x)+\langle j_{1\mu}(x)\rangle+\langle j_{2\mu}(x)\rangle, \tag{2.9}$$

where

$$\langle j_{1\mu}(x)\rangle=ie_1\,\mathrm{tr}\gamma_{1\mu}G_1(x,x),$$
$$\langle j_{2\mu}(x)\rangle=ie_2\,\mathrm{tr}\gamma_{2\mu}G_2(x,x)-i\mu'\partial_\lambda\,\mathrm{tr}\sigma_{2\lambda\mu}G_2(x,x), \tag{2.10}$$

the symmetric limit being understood on the right of (2.10). In analogous fashion, the one-particle Maxwell Green's function may be defined by

$$\mathcal{G}_{\mu\nu}(x,x')=(\delta/\delta J_\nu(x'))\langle A_\mu(x)\rangle=i[\langle(A_\mu(x)A_\nu(x'))_+\rangle-\langle A_\mu(x)\rangle\langle A_\nu(x')\rangle]. \tag{2.11}$$

[5] The variational derivatives with respect to η and ζ here (and elsewhere) are right variational derivatives, i.e., if $\delta_\eta F=\int F(x)\delta\eta(x)dx$, then $\delta F/\delta\eta=F(x)$. A convention is necessary as $F(x)$ and $\delta\eta(x)$ need not commute.

In the Lorentz gauge, this obeys the equation

$$-\partial^2\mathcal{G}_{\mu\nu}(x,x')=\delta_{\mu\nu}\delta(x-x')+ie_1\,\mathrm{tr}\gamma_{1\mu}(\delta/\delta J_\nu(x'))G_1(x,x)+ie_2\,\mathrm{tr}\gamma_{2\mu}(\delta/\delta J_\nu(x'))G_2(x,x)-i\mu'\partial_\lambda\,\mathrm{tr}\sigma_{2\lambda\mu}(\delta/\delta J_\nu(x'))G_2(x,x). \tag{2.12}$$

The two-particle electron-proton Green's function is given by

$$G_{12}(x_1x_2,x_1'x_2')=(\delta/\delta\eta(x_1'))(\delta/\delta\zeta(x_2'))\langle(\psi(x_1)\phi(x_2))_+\rangle]_{\eta,\zeta\to0}\epsilon(x_1,x_2)=-\langle(\psi(x_1)\phi(x_2)\bar{\psi}(x_1')\bar{\phi}(x_2'))_+\rangle\epsilon(x_1,x_1')\epsilon(x_2,x_2')\times\epsilon(x_1,x_2)\epsilon(x_2',x_1')\epsilon(x_1,x_2')\epsilon(x_2,x_1'), \tag{2.13}$$

and obeys the equations

$$\mathfrak{F}_1G_{12}=\delta(x_1-x_1')G(x_2,x_2'),$$
$$\mathfrak{F}_1\mathfrak{F}_2G_{12}=\delta(x_1-x_1')\delta(x_2-x_2'). \tag{2.14}$$

The $\mathfrak{F}$'s are the functional operators on the left of (2.7).

Due to the fact that we have been considering vacuum states on σ_1 and σ_2, the Green's functions defined above have all been outgoing wave Green's functions. It is convenient, then, to set the surfaces σ_1 and σ_2 at $+\infty$ and $-\infty$ and introduce the "matrix" notation of Schwinger,[1] x and p being used for particle operators, ξ and k for photon operators. We also introduce two quantities which are photon vectors and particle matrices, $\gamma_\mu(\xi)$ and $\sigma_{\mu\nu}(\xi)$, such that

$$(x|\gamma_\mu(\xi)|x')=\gamma_\mu\delta(x-x')\delta(\xi-x),$$
$$(x|\sigma_{\mu\nu}(\xi)|x')=\sigma_{\mu\nu}\delta(x-x')\delta(\xi-x). \tag{2.15}$$

By use of the vertex operators,

$$\Gamma_{1,2}(\xi)=-[\delta/\delta(e_{1,2}A(\xi))]G_{1,2}{}^{-1}, \tag{2.16}$$

the variational derivatives in $\mathfrak{M}$ may be re-expressed as an integral operator M:

$$\mathfrak{M}_1G_1=(m_1+ie_1{}^2\,\mathrm{T}\rho\gamma_1G_1\Gamma_1\mathcal{G})G_1=M_1G_1,$$
$$\mathfrak{M}_2G_2=(m_2+ie_2{}^2\,\mathrm{T}\rho\bar{\gamma}_2G_2\Gamma_2\mathcal{G})G_2=M_2G_2, \tag{2.17}$$
$$\bar{\gamma}_2(\xi)=\gamma_2(\xi)-i(\mu'/e_2)(k\cdot\sigma_2)(\xi).$$

In general, the one-particle proton operators are identical to the electron in form with γ replaced by $\bar{\gamma}$. The variational derivatives in the Maxwell Green's function may also be eliminated in favor of the integral operator P:

$$(k^2+P_1+P_2)\mathcal{G}=1,$$
$$P_{1\mu\nu}(\xi,\xi')=-ie_1{}^2\,\mathrm{Tr}\gamma_{1\mu}(\xi)G_1\Gamma_{1\nu}(\xi')G_1. \tag{2.18}$$

Turning to the two-body equation, one wishes to recast it into the form

$$\{[\gamma_2(p_2-e_2\langle A_2\rangle)-\mu'\sigma_{2\mu\nu}\partial_{2\mu}\langle A_{2\nu}\rangle+M_2]$$
$$\times[\gamma_1(p_1-e_1\langle A_1\rangle)+M_1]-I_{12}\}G_{12}=1_1 1_2. \quad (2.19)$$

On comparison with (2.14), one sees that the interaction operator I_{12} is given by

$$I_{12}G_{12}=G_2^{-1}(M_1-\mathfrak{M}_1)G_{12}, \quad (2.20)$$

which may be rearranged to

$$I_{12}G_{12}=-ie_1e_2\,\mathrm{T}\rho\gamma_1\Gamma_2\mathcal{G}G_{12}$$
$$-ie_1\,\mathrm{T}\rho\gamma_1 G_1(\delta/\delta J)(I_{12}G_{12}). \quad (2.21)$$

The integral operators discussed in this section may all be expanded in a power series in e^2. The lowest-order interaction is

$$I^{(1)}=ie^2\,\mathrm{T}\rho\gamma_1\bar{\gamma}_2\mathcal{G}^0$$
$$=ie^2\int d\xi d\xi'\gamma_{1\mu}(\xi)\bar{\gamma}_{2\mu}(\xi')D_+(\xi',\xi). \quad (2.22)$$

In general, the interaction operator is symmetric in the proton and electron indices. Thus, to second order it may be shown that

$$I_{12}=(ie^2\,\mathrm{T}\rho\Gamma_1\Gamma_2\mathcal{G})_{\text{second order}}$$
$$+(ie^2)^2\,\mathrm{T}\rho\gamma_1 G_1{}^0(\mathrm{T}\rho'\gamma_1'\bar{\gamma}_2'\mathcal{G}^{0\prime})G_2{}^0\bar{\gamma}_2\mathcal{G}^0. \quad (2.23)$$

The superscript 0 means zeroth-order function. In these perturbation expansions for I_{12}, etc., the free particle intermediate state assumption has been made.

III. PERTURBATION THEORY

Salpeter[6] has presented a perturbation theory based on the two-body wave equation. We present here an alternate form based upon the Green's function. From Eq. (2.13), for the case $t_1, t_2>t_1', t_2'$, the Green's function may be written as

$$G_{12}=-\sum_\zeta \frac{(0\sigma_1|(\psi(x_1)\phi(x_2))_+\epsilon(x_1,x_2)|\zeta\sigma)}{(0\sigma_1|0\sigma)}$$
$$\times\frac{(\zeta\sigma|(\bar{\psi}(x_1')\bar{\phi}(x_2'))_+\epsilon(x_2',x_1')|0\sigma_2)}{(0\sigma|0\sigma_2)}$$
$$=-\sum_\zeta\psi_\zeta(x_1,x_2)\bar{\psi}_\zeta(x_1',x_2'), \quad (3.1)$$

where ζ is some complete set on surface σ including the total field energy. Introducing the center-of-mass and relative coordinates

$$X=(m_1x_1)/(m_1+m_2)+(m_2x_2)/(m_1+m_2)$$
$$=\mu_1x_1+\mu_2x_2, \quad (3.2)$$
$$x=x_1-x_2,$$

the factor $\exp(iP_{\mu\zeta}X_\mu)$ may be separated from ψ_ζ, where $P_{\mu\zeta}$ is the field energy-momentum eigenvalue. In general, we may set the momentum part to zero. For the equal-times situation, $t_1=t_2=t>t_1'=t_2'=t'$, the Green's function becomes

$$G_{12}(\mathbf{r}_1\mathbf{r}_2t,\mathbf{r}_1'\mathbf{r}_2't')$$
$$=-\sum_\zeta\psi_\zeta(\mathbf{r}_1\mathbf{r}_2)\exp(-iP_{0\zeta}T_0)\bar{\psi}_\zeta(\mathbf{r}_1'\mathbf{r}_2'), \quad (3.3)$$

where $T_0=t-t'$.

The eigenvalues, $P_{0\zeta}$, are infinitely degenerate. Such a set of eigenvalues (and eigenfunctions) may be replaced by a discrete set (for the bound states), P_{0n}. The P_{0n} are complex, the real part representing the energy of the two-body system, the complex part representing the possibility of decay (via the emission of photons and pairs). Thus, Eq. (3.3) may be replaced by

$$G_{12}(\mathbf{r}_1\mathbf{r}_2t,\mathbf{r}_1'\mathbf{r}_2't')$$
$$=-\sum_n\psi_n(\mathbf{r}_1\mathbf{r}_2)\exp(-iP_{0n}T_0)\bar{\psi}_n(\mathbf{r}_1'\mathbf{r}_2'), \quad (3.4)$$

where $\psi_n(x_1,x_2)$ presumably obeys the two-body wave equation. The situation is very similar to that of radioactive α decay. There, the continuum of positive energy eigenstates is replaced by a discrete set with complex eigenvalues, the complex part of the energy representing the possibility of decay of the α particle out of the nuclear well (in analogy to the decay of the two-body system via emission processes).

Equation (3.4) may be used as the basis of a perturbation theory for the eigenvalue P_0. Before doing this, however, it is best to extract out the various infinite constants implicit in electrodynamics. The Green's function equation, in the absence of an external field is

$$[(\gamma_1p_1+M_1)(\gamma_2p_2+M_2)-I_{12}]G_{12}=1_1 1_2. \quad (3.5)$$

It may be shown, to order α, that

$$\gamma_1p_1+M_1=(1+A_1)(\gamma_1p_1+m_1'+M_1{}^{(r)}), \quad (3.6)$$

where m_1' is the renormalized mass, A_1 is an infinite constant and $M_1{}^{(r)}$ is a finite residue. Similarly, Γ_1 may be shown to equal

$$\Gamma_1=(1+A_1)(\gamma_1+\Gamma_1{}^{(r)}), \quad (3.7)$$

where $\Gamma_1{}^{(r)}$ is finite. Referring to Eq. (2.23), one sees that to order α, the factor $1+A_1$ may be extracted from all terms of the bracket in (3.5) and be absorbed into the amplitude of G_{12}. A similar result holds for the proton quantities,[7] completing the mass and Green's function renormalizations. The remaining renormalization is that of the charge appearing in the interaction

[6] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).

[7] The actual calculations of the proton's mass and vertex operators appears to run into difficulties due to the extra dipole divergences mentioned earlier. However, one may show from general invariance considerations that equations analogous to (3.6) and (3.7) hold for proton quantities. $M_2{}^{(r)}$ and $\Gamma_2{}^{(r)}$ are infinite but we may assume such integrals have cutoffs. These difficulties produce no net effects on the finite parts of the calculations.

function. It may be shown that the Maxwell Green's function, to first order, can be written as

$$\mathcal{G}=(1+B)(\mathcal{G}^0+\mathcal{G}^{(r)}), \tag{3.8}$$

where B is infinite and of order α, and $\mathcal{G}^{(r)}$ is finite (to within the dipole infinities). From (2.23), one may then write

$$I_{12}=e^2(1+B)[i\,\mathrm{T}\rho\Gamma_1\Gamma_2(\mathcal{G}^0+\mathcal{G}^{(r)})-e^2I^{(2)}], \tag{3.9}$$

where $I^{(2)}$ is the second order interaction of (2.23). Defining the renormalized charge as

$$e'^2=e^2(1+B), \tag{3.10}$$

all the charges appearing in I_{12} may be written in terms of e', to the desired order. Having completed the renormalizations, we will drop the prime notation and assume that in all future equations, the quantities appearing are the renormalized ones. As will be shown in the Appendix, $M^{(r)}$ and $\mathcal{G}^{(r)}$ produce no energy shifts to the order desired and hence may be neglected. $\Gamma_1{}^{(r)}$ produces the correction to the hyperfine formula corresponding to the first order anomalous magnetic moment of the electron. $\Gamma_2{}^{(r)}$ would produce a similar effect for the proton. As the first case has been included in previous calculations,[8] and the second effect may be assumed to have been included in the experimental value to be inserted for the protons's magnetic moment, these terms also need not be considered further.

Having completed the renormalization, one may proceed with the perturbation theory. Separating out from the interaction function a static, non-spin dependent part I_0,

$$I_{12}=I_0+I', \tag{3.11}$$

one may define the Green's function, G_0, by the equation

$$[(\gamma_1p_1+m_1)(\gamma_2p_2+m_2)-I_0]G_0=1_11_2. \tag{3.12}$$

In the electrodynamical case, I_0 is proportional to the Coulomb energy. A first-order expression for G_{12} may be obtained by an iteration procedure:

$$G_{12}=G_0+G_0I'G_0. \tag{3.13}$$

Considering the equal times situation where the separation T_0 of (3.4) is a large positive number, an expansion such as (3.4) may always be made for G_0, yielding

$$\begin{aligned}G_{12}&(\mathbf{r}_1\mathbf{r}_2t,\ \mathbf{r}_1'\mathbf{r}_2't')\\&=-\sum_n\psi_n{}^0(\mathbf{r}_1\mathbf{r}_2)\exp(-iP_{0n}{}^0T_0)\bar\psi_n{}^0(\mathbf{r}_1'\mathbf{r}_2')\\&\quad+\sum_{n,n'}\psi_n{}^0(\mathbf{r}_1\mathbf{r}_2)\exp(-iP_{0n}{}^0t)\int\bar\psi_n{}^0(x_1'',\,x_2'')\\&\quad\times I'\psi_{n'}{}^0(x_1''',\,x_2''')\exp(iP_{0n'}{}^0t')\bar\psi_{n'}{}^0(\mathbf{r}_1'\mathbf{r}_2'),\end{aligned} \tag{3.14}$$

where the superscript "0" means quantities related to G_0. We wish to rearrange (3.14) into a form resembling (3.4). The phases then will be the energy eigenvalues, the coefficients of each phase the wave functions. To obtain the phases, one need only consider terms in the second sum where $n=n'$.[9] These terms are

$$\begin{aligned}\sum_n\psi_n{}^0(\mathbf{r}_1\mathbf{r}_2)\Big[&\exp(-iP_{0n}{}^0T_0)\int\psi_n{}^0(x_1'',\,x_2'')\\&\times I'\bar\psi_n{}^0(x_1''',\,x_2''')\Big]\bar\psi_n{}^0(\mathbf{r}_1'\mathbf{r}_2').\end{aligned} \tag{3.15}$$

In combination with the first sum of (3.14), the bracket may be viewed as the first order expansion of an exponential, and hence to the required accuracy, the right side of (3.14) can be written as

$$\begin{aligned}-\sum_n\psi_n{}^0(\mathbf{r}_1\mathbf{r}_2)&\exp(-iT_0)\\&\times\Big[P_{0n}{}^0-i/T_0\int\bar\psi_n{}^0I'\psi_n{}^0\Big]\bar\psi_n{}^0(\mathbf{r}_1'\mathbf{r}_2').\end{aligned} \tag{3.16}$$

Thus the first order energy shift becomes

$$\begin{aligned}\Delta E^{(1)}=-(i/T_0)\int&\bar\psi^0(x_1,\,x_2)\\&\times I'(x_1x_2,\,x_1'x_2')\psi^0(x_1',\,x_2').\end{aligned} \tag{3.17}$$

To obtain the second-order shift, one iterates twice, giving, for the diagonal terms,

$$\begin{aligned}-\sum_n\psi_n{}^0&\exp(-iP_{0n}{}^0T_0)\bar\psi_n{}^0+\sum_n\psi_n{}^0\exp(-iP_{0n}{}^0T_0)\\&\times\Big[\int\bar\psi_n{}^0I'\psi_n{}^0+\int\bar\psi_n{}^0I'G_0I'\psi_n{}^0\Big]\bar\psi_n{}^0.\end{aligned} \tag{3.18}$$

Thus the second-order energy shift becomes

$$\Delta E^{(2)}=-(i/T_0)\int\bar\psi^0I'G_1G_2I'\psi^0, \tag{3.19}$$

where, in concordance with our plan of treating the infrared difficulty, we have replaced the intermediate state Green's function G_0 by the free particle approximation, G_1G_2.[10]

The energy shifts may be expressed in terms of momentum integrals. In view of the center-of-mass dependence, one may write (using a "box normalization"):

$$\psi^0(x_1,\,x_2)=\exp(iP_\mu X_\mu)V^{-\frac{1}{2}}(2\pi)^{-2}\int e^{ipx}\psi^0(p)dp. \tag{3.20}$$

[8] R. Karplus and A. Klein, Phys. Rev. **85**, 972 (1952).

[9] The off-diagonal elements will give the first-order perturbed wave functions.

[10] In turning the expansion of (3.18) into an exponential one really requires the presence of the term $\frac{1}{2}(i\Delta E^{(1)}T)^2$ as the expansion must be adequate to second order. This term is actually present in the second term of the bracket of (3.18), as may be seen by breaking the G_0 there into its bilinear sum. Since n is a bound state, this term corresponds to a low-frequency intermediate state and hence, for the hyperfine problem, may be neglected. In general, it would seem essential that the rigorous G_0 be used in (3.19).

Defining

$$I'(p,p')=(2\pi)^{-4}\int e^{-iP(X-X')}e^{-ipx}e^{ip'x'}\times I'(x_1x_2,x_1'x_2')dxdx'd(X'-X), \quad (3.21)$$

one obtains

$$\Delta E^{(1)}=-i\int\bar{\psi}^0(p)I'(p,p')\psi^0(p')dpdp', \quad (3.22)$$

$$\Delta E^{(2)}=-i\int\bar{\psi}^0(p)I'(p,p')G_1(\mu_1P+p')\times G_2(\mu_2P-p')I'(p',p'')\psi^0(p'')dpdp'dp''. \quad (3.23)$$

The bound-state zeroth-order wave functions obey the homogeneous equation,

$$[(\gamma_1p_1+m_1)(\gamma_2p_2+m_2)-I_0]\psi^0=0. \quad (3.24)$$

IV. CALCULATIONS

As shown in the preceding section, the energy shifts may be obtained in terms of matrix elements of the interaction between the lowest-order wave functions. Due to the fact that I_0 is a static interaction, it is possible to obtain an adequate solution for the wave function based upon an iteration scheme stemming from the nonrelativistic Coulomb wave function.

In momentum space, the lowest-order interaction (2.22) becomes

$$I(p,p')=\frac{ie^2}{(2\pi)^4}\frac{\gamma_1\gamma_2}{(p-p')^2}+\frac{\lambda e^2}{(2\pi)^4}\frac{\gamma_{1\mu}\sigma_{2\alpha\mu}(p_\alpha-p_\alpha')}{(p-p')^2}, \qquad \lambda=\mu'/e. \quad (4.1)$$

Selecting out the non-spin-dependent static part for I_0, (3.24) becomes in momentum space,

$$[\mu_1P_0+p_0-(\boldsymbol{\alpha}_1\cdot\mathbf{p}+\beta_1m_1)]\times[\mu_2P_0-p_0-(-\boldsymbol{\alpha}_2\cdot\mathbf{p}+\beta_2m_2)]\psi(p)=-\frac{ie^2}{(2\pi)^4}\int\frac{dp'}{(\mathbf{p}-\mathbf{p}')^2}\psi(p'). \quad (4.2)$$

$\boldsymbol{\alpha}$ and β are the usual Dirac matrices. Defining an equal relative times wave function,

$$\phi(\mathbf{p})=(2\pi)^{-\frac{1}{2}}\int dp_0\psi(p), \quad (4.3)$$

Salpeter[6] has shown that an equation for ϕ may be derived, and that to a first approximation, ϕ is the Schrödinger Coulomb wave function. Putting this approximate function in on the right of (4.2), one obtains the wave function

$$\psi(p)=-\frac{i}{2\mu(2\pi)^{\frac{1}{2}}}\frac{(\mathbf{p}^2+\alpha^2\mu^2)\phi(\mathbf{p})}{[\mu_1P_s+p_0-(\boldsymbol{\alpha}_1\cdot\mathbf{p}+\beta_1m_1)]\times[\mu_2P_s-p_0-(-\boldsymbol{\alpha}_2\cdot\mathbf{p}+\beta_2m_2)]}, \quad (4.4)$$

where μ is the reduced mass of the electron and $P_s=m_1+m_2-\frac{1}{2}\alpha^2\mu$ is the energy of the ground state of hydrogen. It is convenient in treating the first order perturbation shift to return to coordinate space for the relative time variable via the equation

$$\psi(\mathbf{p},t)=(2\pi)^{-\frac{1}{2}}\int e^{-ip_0t}\psi(p)dp_0. \quad (4.5)$$

The calculation of $\psi(\mathbf{p},t)$ is very similar to that done by Karplus and Klein.[3] We will only state the result:

$$\psi(\mathbf{p},t)=(2\pi)^{-\frac{1}{2}}(m_1+m_2)^{-1}\alpha\phi(0)(\mathbf{p}^2+\alpha^2\mu^2)^{-2}K(\mathbf{p},t), \quad (4.6)$$

where

$$\begin{aligned}K(\mathbf{p},t)=&(4E_1E_2)^{-1}[\exp^+\{(m_1E_2+m_2E_1)\\&\times(2m_1+\boldsymbol{\alpha}_1\cdot\mathbf{p})(2m_2-\boldsymbol{\alpha}_2\cdot\mathbf{p})-\mathbf{p}^2[(E_2-E_1)\\&\times(2m_1+\boldsymbol{\alpha}_1\cdot\mathbf{p}-2m_2+\boldsymbol{\alpha}_2\cdot\mathbf{p})-(m_1E_2+m_2E_1)]\}\\&+\exp^-\{(2E_1E_2\epsilon(t)+m_1E_2-m_2E_1)\\&\times(2m_1+\boldsymbol{\alpha}_1\cdot\mathbf{p})(2m_2-\boldsymbol{\alpha}_2\cdot\mathbf{p})\\&-\mathbf{p}^2[(E_1+E_2)(2m_1+\boldsymbol{\alpha}_1\cdot\mathbf{p}-2m_2+\boldsymbol{\alpha}_2\cdot\mathbf{p})\\&+(2E_1E_2\epsilon(t)+m_2E_1-m_1E_2)]\}], \end{aligned} \quad (4.7)$$

and

$$\exp^\pm=e^{-i[E_1\epsilon(t)-m_1]t}\pm e^{-i[E_2\epsilon(t)+m_2]t}, \qquad E=(\mathbf{p}^2+m^2)^{\frac{1}{2}}, \quad \epsilon(t)=\begin{cases}+1 & t>0\\ -1 & t<0.\end{cases} \quad (4.8)$$

$\phi(0)$ is the value of the ground-state wave function at the origin in coordinate space.

We consider now the first-order energy shift with the first-order interaction. After subtracting out I_0, (4.1) may be divided into a "charge" and "dipole" part:

$$I^{(1)c}(p,p')=\frac{ie^2}{(2\pi)^4}\frac{\beta_1\beta_2}{(p-p')^2}\left[\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2-\frac{(p_0-p_0')^2}{(\mathbf{p}-\mathbf{p}')^2}\right],$$
$$I^{(1)d}(p,p')=\frac{\lambda e^2}{(2\pi)^4}\frac{\gamma_{1\mu}\sigma_{2\alpha\mu}(p_\alpha-p_\alpha')}{(p-p')^2}. \quad (4.9)$$

Substituting $I^{(1)c}$ into (3.22) yields

$$\Delta E^{(1)c}=-\frac{i}{2\pi}\frac{ie^2}{(2\pi)^4}\int\psi^*(\mathbf{p},t)e^{-ip_0t}\times\frac{\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2-(p_0-p_0')^2/(\mathbf{p}-\mathbf{p}')^2}{(\mathbf{p}-\mathbf{p}')^2-(p_0-p_0')^2}e^{ip_0't'}\psi(\mathbf{p}',t'). \quad (4.10)$$

The p_0 and p_0' integrations may be performed upon using the usual outgoing wave perscription:

$$\Delta E^{(1)c}=\frac{\pi ie^2}{(2\pi)^4}\int\psi^*(\mathbf{p},t)\exp\{-i[(\mathbf{p}-\mathbf{p}')^2]^{\frac{1}{2}}|t|\}\times\frac{\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2-1}{[(\mathbf{p}-\mathbf{p}')^2]^{\frac{1}{2}}}\psi(\mathbf{p}',t). \quad (4.11)$$

Inserting the wave function (4.6) into (4.11) gives[11]

$$\Delta E^{(1)c}=\frac{ie^2}{2(2\pi)^4}\frac{\alpha^2|\phi(0)|^2}{(m_1+m_2)^2}\times\int\frac{K(\mathbf{p},-t)}{(\mathbf{p}^2+\alpha^2\mu^2)^2}\frac{\exp\{-i[(\mathbf{p}-\mathbf{p}')^2]^{\frac{1}{2}}|t|\}}{[(\mathbf{p}-\mathbf{p}')^2]^{\frac{1}{2}}}\times[\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2-1]\frac{K(\mathbf{p}',t)}{(\mathbf{p}'^2+\alpha^2\mu^2)^2}. \quad (4.12)$$

The calculation now proceeds exactly as the corresponding one done by Karplus and Klein. We desire the α^{-2} and α^{-1} parts of the integral which can be obtained from the part of the integrand which is large when at most one of the momentum variable is large. We also desire the spin-spin interactions as we are interested only in the hyperfine shift. These two conditions restrict the portions of the integrand which are relevant. Further, where one momentum variable is large, we will neglect the $\alpha^2\mu^2$ in the corresponding denominator. This corresponds to the neglecting of binding effects in the intermediate states and will produce its quota of infrared divergences. Upon carrying through these simplifications, the integrations may be performed yielding

$$\Delta E^{(1)c}=\frac{2\pi}{3}\frac{\alpha}{m_1m_2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi(0)|^2\left[1-\frac{4\alpha}{\pi}-\frac{2\alpha}{\pi(m_1+m_2)}\times\left\{m_1\ln\frac{m_2}{2p}+m_2\ln\frac{m_1}{2p}\right\}\right], \quad (4.13)$$

where p is the infrared cutoff.

A similar calculation may be done for the dipole part of the interaction. The result here is[12]

$$\Delta E^{(1)D}=\frac{4\pi}{3}\frac{\alpha\lambda}{m_1}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi(0)|^2\left[1-\frac{4\alpha}{\pi}-\frac{2\alpha}{\pi(m_1+m_2)}\times\left\{m_1\ln\frac{m_2}{2p}+m_2\ln\frac{m_1}{2p}\right\}\right]. \quad (4.14)$$

We turn now to the effects of the second-order interaction [the second term of (2.23)] in the first order perturbation formula. This interaction contains the double photon processes and consists of three types of terms according to whether the proton interacts zero, once, or twice with the electromagnetic field via its spin current. In momentum space, the interaction is

$$I^{pp}=-\frac{e^4}{(2\pi)^8}\int dk\frac{N_1+N_2+N_3}{[(\mu_1P_s+p+k)^2+m_1^2][k-p'+p]^2\times[(\mu_2P_s+k-p')^2+m_2^2]k^2}, \quad (4.15)$$

where

$$N_1=\gamma_{1\mu}[m_1-\gamma_1(\mu_1P_s+p+k)]\times\gamma_1\gamma_2[m_2-\gamma_2(\mu_2P_s-p'+k)]\gamma_{2\mu},$$

$$N_2=i\lambda\gamma_{1\mu}[m_1-\gamma_1(\mu_1P_s+p+k)]\gamma_{1\lambda}\sigma_{2\alpha\lambda}\times(p'-p-k)_\alpha[m_2-\gamma_2(\mu_2P_s-p'+k)]\gamma_{2\mu}, \quad (4.16)$$

$$N_3=-\lambda^2\gamma_{1\mu}[m_1-\gamma_1(\mu_1P_s+p+k)]\gamma_{1\lambda}\sigma_{2\lambda\alpha}(p'-p-k)_\alpha\times[m_2-\gamma_2(\mu_2P_s-p'+k)]\sigma_{2\mu\beta}(k-p'+p)_\beta.$$

Since the interaction is already second order, we need not treat the integrals in the perturbation formula as accurately as before.

First, one requires only the low-momentum parts of the wave function and hence may set $\mathbf{p}$ and $\mathbf{p}'$ to zero in I^{pp}. Also, the p_0 poles, unlike the situation in the first order interaction, will occur at high-frequency values and give higher order contributions to the energy shift. Thus it is possible in general to set p and p' to zero in $I^{pp}(p,p')$. Under these conditions, (3.22) becomes:

$$\Delta E^{(1)pp}=-i(2\pi)^4|\phi(0)|^2I^{pp}(0,0). \quad (4.17)$$

This result is in direct agreement with the idea that for the hyperfine effects, only small distances are involved requiring only the wave function at the origin.

The calculation proceeds as follows: first, the spin-spin parts of the numerator are picked out. The three poles in the denominator may then be integrated. Following this, the remaining real variable integrations may be performed. The final answer is:

$$\Delta E^{(1)pp}=\frac{2\pi}{3}\frac{\alpha}{m_1m_2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi(0)|^2\left[\frac{4\alpha}{\pi}+\frac{3\alpha}{2\pi(m_2-m_1)}\times\left\{m_2\ln\frac{m_1}{2p}-m_1\ln\frac{m_2}{2p}\right\}\right]+\frac{4\pi}{3}\frac{\alpha\lambda}{m_1}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi(0)|^2\times\left[\frac{4\alpha}{\pi}+\frac{3\alpha}{\pi}\ln\frac{m_1}{2p}\right]+\lambda^2\alpha^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi(0)|^2\left[\frac{1}{6}-\frac{1}{m_2-m_1}\times\left\{4m_2\ln\frac{2\bar{p}}{m_2}-(3m_1+m_2)\ln\frac{2\bar{p}}{m_1}\right\}\right], \quad (4.18)$$

where $\bar{p}$ is a dipole ultraviolet cutoff. We have also assumed that the infrared cutoff is the same as in the previous calculations.

The final contribution to the energy shift that need be considered comes from the second-order perturbation formula. Here we consider only the lowest-order interaction:

$$\Delta E^{(2)}=-i\int\bar{\psi}(p)[I^{(1)c}(p,k)+I^{(1)d}(p,k)]\times G_1(\mu_1P_s+k)G_2(\mu_2P_s-k)[I^{(1)c}(k,p')+I^{(1)d}(k,p')]\psi(p'). \quad (4.19)$$

[11] $\psi^*(\mathbf{p},t)$ is not actually the complex conjugate of ψ. An analysis of its definition, (3.1), will show that for our function, (4.6), it is actually the time reversed function.

[12] I should like to thank Dr. W. A. Newcomb for finding an error in the calculation of this term.

As in the previous calculation, one may set p and p' to zero in the interaction terms, yielding

$$\Delta E^{(2)}=-i(2\pi)^4|\phi(0)|^2\int dk[I^{(1)c}(0,k)+I^{(1)d}(0,k)]G_1(\mu_1P_s+k)G_2(\mu_2P_s-k)\times[I^{(1)c}(k,0)+I^{(1)d}(k,0)]. \quad (4.20)$$

Again there are three types of terms corresponding to the three possible ways the proton may interact with the electromagnetic field. The calculation proceeds in an analogous fashion, giving for the energy shift:

$$\Delta E^{(2)}=\frac{2\pi}{3}\frac{\alpha}{m_1m_2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi(0)|^2\left[\frac{\alpha}{2\pi(m_1+m_2)}\times\left(m_2\ln\frac{m_1}{2p}+m_1\ln\frac{m_2}{2p}\right)\right]-\frac{4\pi}{3}\frac{\alpha\lambda}{m_1}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi(0)|^2\times\left\{\frac{\alpha}{\pi(m_1+m_2)}\left[(3m_1+m_2)\ln\frac{m_1}{2p}-2m_1\ln\frac{m_2}{2p}\right]\right\}+\lambda^2\alpha^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi(0)|^2\left\{\frac{1}{6}-\frac{1}{m_1+m_2}\times\left[(3m_1-m_2)\ln\frac{2\bar{p}}{m_1}+4m_2\ln\frac{2\bar{p}}{m_2}\right]\right\}. \quad (4.21)$$

V. RESULTS

Adding up the results of the preceding section, the infrared divergences cancel. Keeping terms only to order m_1/m_2, the total shift due to reduced mass corrections may be written as[13]

$$\Delta E=\frac{4\pi}{3}\frac{\alpha\lambda_p}{m_1}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\phi(0)|^2\times\left[1-\frac{\alpha}{\pi}\frac{m_1}{m_2\mu_p}\left\{[3-\tfrac{3}{4}(\mu_p-1)^2]\ln\frac{m_2}{m_1}+\frac{9}{4}(\mu_p-1)^2\ln\frac{2\bar{p}}{m_2}-\frac{1}{8}(\mu_p-1)^2\right\}\right], \quad (5.1)$$

where $\lambda_p=1/(2m_2)+\lambda$ and μ_p is the protons total magnetic moment expressed in units of the nuclear magneton. The coefficients in front of the bracket is the Fermi formula, modified by the usual reduced mass correction.

Neglecting the shift calculated here, Dumond and Cohen[14] have made a least-squares fitting of atomic constants and obtained a value for the fine structure constant of

$$1/\alpha=137.0377\pm0.0016. \quad (5.2)$$

Including the above corrections, one obtains

$$1/\alpha=\begin{cases}137.0378 & \text{for } \bar{p}=\text{meson mass},\\ 137.0374 & \text{for } \bar{p}=\text{proton mass},\end{cases} \quad (5.3)$$

the contributions from the Dirac and Pauli moments almost cancelling. The results are fairly insensitive to the cut-off value.

In evaluating the validity of the results, two points should be noted. Though the final formula is independent of the infrared cutoff we have assumed that all the cutoffs were equal. Since the divergences come from intrinsically different momentum integrations, this assumption need not be true. Recently, however, the two-body Green's function equation has been rearranged so that the infrared divergences can be eliminated from the beginning. It appears that the prescriptions used here are correct.[15] Difficulties also appear in dealing with the ultraviolet cutoff. In the dipole terms which are finite, we have assumed that the upper limits on the momentum are rigorously at infinity and have introduced cutoffs only in the divergent terms. Since the meson mass is not a particularly high frequency this procedure might disturb the values of the finite terms. The issue could be settled by inserting a form factor into the Pauli term rather than using cutoffs, but this would greatly complicate the calculations.

VI. ACKNOWLEDGMENTS

It is with great pleasure that I record my indebtedness to Professor J. Schwinger for suggesting this problem and for aid received throughout the course of my work. I also wish to thank Professor R. Karplus and Dr. A. Klein for suggesting that their method of calculation was applicable to this problem and for many helpful conversations explaining their technique.

APPENDIX

It will be shown here that both $M^{(r)}$ and $\mathcal{G}^{(r)}$ produce no contribution to the hyperfine splitting to the desired order. The interaction proportional to $M_1^{(r)}$ is

$$I(p,p')=M_1^{(r)}(\mu_1P_s+p)[\gamma_2(\mu_2P_s-p)+M_2^{(r)}(\mu_2P_s-p)]\delta(p-p'). \quad (A.1)$$

First-order perturbation theory leads to an energy shift of

$$\Delta E=-i\int\bar{\psi}(p)M_1^{(r)}(\mu_1P_s+p)[\gamma_2(\mu_2P_s-p)+M_2^{(r)}(\mu_2P_s-p)]\psi(p)dp. \quad (A.2)$$

Due to the $\delta(p-p')$ appearing in (A.1) only one

[13] This result is in agreement with that obtained by W. A. Newcomb (private communication).

[14] J. W. M. Dumond and E. R. Cohen, Special Technical Report No. 1, U. S. Atomic Energy Commission, Nov. 1952 (unpublished).

[15] Conversations with J. Schwinger and R. Karplus.

momentum integration remains in (A.2) and thus no spin-spin combinations can be formed. A similar phenomenon occurs in the second-order perturbation shift.

Upon evaluation of the first-order polarization operators, $\mathcal{G}^{(r)}$ may be written as

$$\mathcal{G}_{\mu\nu}{}^{(r)}(p)=-\delta_{\mu\nu}\frac{\alpha}{4\pi}\Big[4\lambda^2\int_0^\infty\frac{ds}{s}\exp(-sm^2)+\int_0^1 du\frac{f_1(u)}{m_1{}^2+u(1-u)p^2}+\int_0^1 du\frac{f_2(u)}{m_2{}^2+u(1-u)p^2}\Big],\quad(A.3)$$

the first term having an extra dipole infinity. These terms lead to a second-order interaction. Considering the constant term first, one obtains an interaction of

$$I(p,p')=\text{const}\times 4\alpha^2\lambda^2\gamma_1\gamma_2,\quad(A.4)$$

which leads to an energy shift of

$$\Delta E\sim\int\bar{\psi}(p)\gamma_1\gamma_2\psi(p')dpdp',\quad(A.5)$$

which has no spin-spin part. The evaluation of the energy shifts from the second and third terms would be analogous to the evaluation of $\Delta E^{(1)c}$ except that interaction is already second order and the low-frequency pole of D_+ has been replaced by a high-frequency pole. Thus these terms give no contribution to the desired order.

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A New Modification of Classical Electromagnetic Theory*

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A fundamental particle is treated as a unit charge whose rest mass and space time coordinates are variables of its motion. Classical electrodynamics, in its action at a distance formulation, is obtained from an action principle which is simpler than the usual one. In this new action principle the rest mass of a particle is varied as well as the coordinates. The rest masses of interacting particles, although not assumed constant *a priori*, become constants as a consequence of the equations of motion. Modifications of the old action principle can yield purely electromagnetic rest masses which are, however, the same for all particles. Similar modifications of the new action principle give purely electromagnetic rest masses to all charged fundamental particles. In this new modification of electrodynamics, particles interacting at small distances no longer have constant rest masses.

1. INTRODUCTION

OF the many fields which play an important role in quantum physics, the one whose classical counterpart is most familiar is the electromagnetic field. It has been known for a long time that classical electrodynamics can be formulated in two equivalent forms, as a field theory (Faraday-Maxwell-Lorentz) or as a theory of action at a distance between charged particles.[1] In the case of electrodynamics the two formulations are of the same order of simplicity. Other fields (such as meson fields) could also be described classically in an equivalent action at a distance formulation but, in general, the two descriptions would not be equally simple.

In modern physics it is the field-theoretic point of view which has been stressed. Ignoring quantum mechanical considerations such as statistics, each type of free fundamental particle (photon, electron, meson, nucleon, etc.) is described by a set of field variables whose behavior is characterized by a different Lagrangian function. Interaction is characterized by additional Lagrangians which are functions of the field variables of two or more different fundamental particles. Even if this kind of description gave good results, it can hardly be regarded as satisfactory at a time when the number of fundamental particles is of order 20 and still increasing.

It may be claimed, with only some measure of truth perhaps, that all simple field theories modeled on electrodynamics have been examined exhaustively, and that not one of them shows any indications of explaining all processes involving fundamental particles. It therefore seems worthwhile to investigate systematically all simple modifications of electrodynamics in its action at a distance formulation. The present field theories may well turn out to be asymptotic approximations of an even more complicated and nonlocal field theory which corresponds to a simple equivalent action at a

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[1] K. Schwarzschild, Nach. Akad. Wiss. Göttingen Math. physik. Kl. IIa **1903**, 128, 132, 245 (1903); H. Tetrode, Z. Physik **10**, 317 (1922); A. D. Fokker, Z. Physik **58**, 386 (1929); Physica **9**, 33 (1929); **12**, 145 (1932); J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **17**, 157 (1945); **21**, 425 (1949).