One proves them, following Minkowski,[20] by direct computation, using matrix notation for the tensor multiplication. With them one obtains (3.5) and (3.6) from (3.3) and (3.4) by squaring and subtracting these relations or multiplying them, respectively, remembering that $U_kU^k=1$.

For the proof of (4.2) we first invert formula (4.1) by combination with its dual. There follows, e.g.,

$$u_iU_k-u_kU_i=(-KM_{ik}+IM_{ik}^*)/(I^2+K^2). \quad \text{(A3)}$$

Let us denote the tensor on the right by $R_{ik}$. The p.b. with $u_j$ is

$$\{u_ju_i\}U_k+u_i\{u_jU_k\}-\{u_ju_k\}U_i-u_k\{u_jU_i\}=\{u_jR_{ik}\}. \quad \text{(A4)}$$

The right side may be computed with the help of (3.1), (3.7), and (3.8):

$$\hbar\{u_jR_{ik}\}=[(I^2-K^2)U_jM_{ik}+2IKU_jM_{ik}^*]/(I^2+K^2)^2 +[K(\delta_{ij}u_k-\delta_{kj}u_i)-I(\delta_{ij}u_k-\delta_{kj}u_i)^*]/(I^2+K^2). \quad \text{(A5)}$$

We now multiply (A4) by $U^k$ and contract. Due to (3.3), there is $U^ku_k=0$; and as a consequence of $U^kU_k=1$, we have $U^k\{u_jU_k\}=0$. With this there results

$$\{u_ju_i\}-\{u_ju_k\}U^kU_i=U^k\{u_jR_{ik}\}. \quad \text{(A6)}$$

This is a system of six inhomogeneous linear equations for $\{u_ju_i\}$. The determinant is unity, that is, if a solution is found somehow, it is unique. Try now

$$\{u_ju_i\}=U^k\{u_jR_{ik}\}. \quad \text{(A7)}$$

Due to the skew-symmetry of $R_{ik}$ there holds then $U^i\{u_ju_i\}=U^iU^k\{u_jR_{ik}\}\equiv 0$. Accordingly, the second term in (A6) vanishes, and the equation is fulfilled. Furthermore formulas (3.3) and (3.4) may be inverted with the help of the tensor identities (A1) and (A2), giving

$$U^iM_{ji}=Ku_i, \quad \text{(A8)}$$

$$U^iM_{ji}^*=-Iu_i, \quad \text{(A9)}$$

and finally there holds

$$U^k(\delta_{ij}u_k-\delta_{kj}u_i)^*\equiv(u_iU_j-u_jU_i)^*. \quad \text{(A10)}$$

With these formulas the computation of $\{u_j, u_i\}$ from (A7) and (A5) is straightforward and leads to Eq. (4.2) of the text.

[20] H. Minkowski, Math. Ann. 68, 472 (1910).

PHYSICAL REVIEW VOLUME 91, NUMBER 4 AUGUST 15, 1953

# Repulsive Core and Charge Independence

E. E. Salpeter
*Laboratory of Nuclear Studies, Cornell University, Ithaca, New York*
(Received April 30, 1953)

In an earlier paper Schwinger derived expressions for the effective strengths of the neutron-proton and proton-proton interactions in the singlet $S$ state. He showed their difference is small and can be accounted for by magnetic forces if a long-tailed potential (Yukawa) is assumed but not for a short-tailed potential. In this paper an equivalent analysis is carried out for nuclear potentials which have a repulsive core. It is shown that for core radii of more than about $0.3\times10^{-13}$ cm the effect of the magnetic interaction is decreased and the difference between the $n$-$p$ and $p$-$p$ interactions is increased. Numerical values of the discrepancy are given for different core radii.

## 1. INTRODUCTION

One test for the hypothesis of charge independence of nuclear forces consists in comparing the zero-energy scattering lengths for the singlet $S$ states of the neutron-proton and proton-proton systems, $a_{np}$ and $a_{pp}$, respectively. The experiments from which these two quantities are derived are very accurate, but the value obtained for $a_{pp}$ depends to an appreciable extent on the assumed shape of the nuclear potential. It was further pointed out by Schwinger[1] that the magnetic interaction between nucleons gives different contributions to the effective potential strengths for the $n$-$p$ and the $p$-$p$ systems. Formulas for these magnetic contributions, also shape-dependent, and hence for $a_{np}'$ and $a_{pp}'$, the effective scattering lengths resulting from the purely nuclear potentials alone, were derived by Sc. He found that $a_{np}'$ and $a_{pp}'$ are practically equal if a very long-tailed potential shape (Yukawa or Hulthén) is assumed, but that there is a definite discrepancy between them for more short-tailed potential shapes.

The presence of large repulsive nuclear forces at short internuclear distances for the singlet $n$-$p$ and $p$-$p$ states is now considered very likely.[2] It is the purpose of the present paper to point out that a sizable discrepancy between $a_{np}'$ and $a_{pp}'$ (and hence between the effective strengths of the two potentials) is again obtained if a repulsive core is assumed, even if the attractive part of

[1] J. Schwinger, Phys. Rev. 78, 135 (1950). This paper will be referred to as Sc and the same notation will be used throughout.

[2] R. Jastrow, Phys. Rev. 81, 165 (1951); M. M. Lévy, Phys. Rev. 88, 725 (1952).

the potential is of Yukawa or Hulthén shape. Following Sc, nonrelativistic theory will be used throughout. In Sec. 2 we evaluate $a_{pp}$ from the experimental data for potentials containing an infinite repulsive core, for various values of the core radius $r_C$. In Sec. 3 we estimate the effect of a "mild" repulsive core on the magnetic interaction energy.

## 2. DETERMINATION OF $a_{pp}$

We consider in this section a central potential for the proton-proton singlet $S$ state, consisting of an infinite rectangular repulsive potential of core radius $r_C$ plus an attractive potential outside the core of approximately Yukawa (Hulthén) shape. We use as $u_0$, the radial wave function for zero kinetic energy,

$$\begin{aligned} u_0 &= 0, && r<r_C; \\ u_0 &= (1+r/a_{pp})-Ae^{-\beta r}, && r>r_C, \end{aligned} \tag{1}$$

where $a_{pp}$ is the zero-energy scattering length, $A$ is a constant fitted to make $u_0$ continuous at $r_C$, and $\beta$ is a parameter fitted so as to give the correct effective range[3] for zero energy $r_e$. As the core radius $r_C$ increases, the parameter $\beta$ increases rapidly, becoming infinite for a limiting core radius, $r_C \approx 1.22\times 10^{+13}$ cm. Thus, as $r_C$ is increased, the potential in effect becomes more and more short tailed.[4,5]

Using the shape-independent approximation of the effective range theory, an analysis[6] of low-energy $p$-$p$ scattering gives

$$r_e=(2.65\pm 0.07)\times 10^{+13}\text{ cm},\quad b_0/a=3.755\pm 0.025, \tag{2}$$

where $b_0=\hbar^2/Me^2$ and $a$ is a parameter connected with the scattering length $a_{pp}$. Since $r_e$ and $a$ are most accurately determined from experimental data for energies of the order of 1 Mev or more, the values obtained for these parameters depend slightly on the shape of the nuclear potential.[6,7] Values[5,6] for $r_e$, corresponding to various core radii $r_C$, are given in Table I. The variation with shape of $b_0/a$ is slightly less than the experimental error.

The derivation of the scattering length $a_{pp}$ from the parameter $a$ involves another parameter $\bar{r}$, defined by Sc. This parameter $\bar{r}$ is connected with the effective range $r_e$ but is strongly shape-dependent. Using the wave functions, Eq. (1), and formulas (10) and (11) of Sc, both $\bar{r}$ and $b_0/a_{pp}$ were calculated for various values of the core radius $r_C$. These values, together with those for an attractive square well potential without a repulsive core, are given in Table I.

TABLE I. Values of various parameters for square well ($S$) and Hulthén ($H$) potentials, and for three Hulthén potentials with repulsive cores of different radii $r_C$ (in $10^{-13}$ cm).

| | $S$ | $H$ | (0.3) | (0.6) | (1.2) |
|---|---|---|---|---|---|
| $r_e$ (in $10^{-13}$ cm) | 2.61 | 2.74 | 2.65 | 2.61 | 2.60 |
| $\bar{r}$ (in $10^{-13}$ cm) | 1.21 | 1.06 | 1.22 | 1.32 | 1.42 |
| $b_0/a_{pp}$ | 1.75 | 1.59 | 1.75 | 1.83 | 1.90 |
| $\delta b_0/a_{np}$ | $-0.04$ | $-0.14$ | $-0.07$ | $-0.02_5$ | $-0.01_5$ |
| $\delta b_0/a_{pp}$ | 0.06 | 0.20 | 0.10 | 0.05 | 0.02 |

As the core radius $r_C$ is increased, the results of this section become less and less sensitive to the shape of the attractive part of the potential. For a value of $r_C=0.6\times 10^{+13}$ cm, the parameter $\bar{r}$ was also calculated for two different shapes of the attractive potential. For square well shape, $\bar{r}$ is about 1.37, for a shape approximating that of the attractive part of Lévy's[2] potential (which has no singularity at $r=r_C$) $\bar{r}$ is about 1.34, as compared with 1.32 for the Hulthén potential, which has a singularity at $r=r_C$ ($\bar{r}$ in $10^{+13}$ cm).

## 3. THE EFFECT OF MAGNETIC INTERACTION

An analysis[3] of experimental $n$-$p$ scattering data at very low energies gives a value for $a_{np}$, the zero-energy scattering length for the singlet $S$ state of the neutron-proton system, which does not depend on the potential shape,

$$b_0/a_{np}=1.216\pm 0.003. \tag{3}$$

The values of $a_{np}$, Eq. (3), and of $a_{pp}$, Table I, are measures of the effective strengths in the two systems of the sum of the purely nuclear potential and of the magnetic interaction energy. Formulas for this magnetic interaction energy were derived by Sc, using non-relativistic theory and assuming no spread of the nucleonic magnetic moment, and two parameters $a_{np}'$ and $a_{pp}'$ were defined which are measures of the purely nuclear potential alone.

Schwinger's[1] expressions for the magnetic interaction energy operator [Sc, Eqs. (26) and (27)] are

$$V_{np}^{(\text{mag})}=8\pi\mu_n(\mu_p-\tfrac{1}{2})\mu_0^2\delta(\mathbf{r}), \tag{4}$$

$$V_{pp}^{(\text{mag})}=8\pi(\mu_p^2-\mu_p+\tfrac{1}{2})\mu_0^2\delta(\mathbf{r})+4\mu_0^2\frac{1}{\hbar^2}\mathbf{p}\cdot\frac{1}{r}\mathbf{p}. \tag{5}$$

In Eq. (5), the term involving the delta-function gives a numerically much larger contribution to the interaction energy than the term involving the momentum operator. The value for the magnetic interaction energy [expectation value of Eqs. (4) and (5)] therefore depends strongly on the value of $|\psi_0(0)|^2$, where $\psi_0(r)=u_0(r)/r$ is the spatial wave function for zero energy. Since $\mu_n$ is negative, Eqs. (4) and (5) give energies of opposite sign.

[3] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949); H. A. Bethe, Phys. Rev. **76**, 38 (1949); E. E. Salpeter, Phys. Rev. **82**, 60 (1951); G. Snow, Phys. Rev. **87**, 21 (1952).

[4] Hafner, Horynak, Falk, Snow, and Coor, Phys. Rev. **89**, 204 (1953).

[5] The well-shape parameter (references 3 and 4) $P_S$ decreases with increasing $r_C$ from about $+0.05$ for a pure Hulthén potential to about $-0.04$ for the limiting core radius. $P_S$ is zero for $r_C$ of the order of magnitude of $0.3\times 10^{-13}$ cm.

[6] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950).

[7] J. Rouvina, Phys. Rev. **81**, 593 (1951).

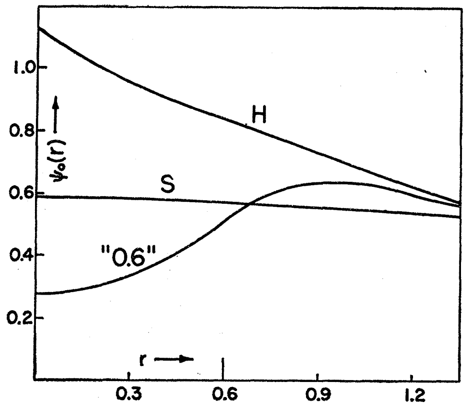


FIG. 1. The spatial wave function $\psi_0(r)$ *versus* $r$ for a pure Hulthén ($H$), pure square well ($S$), and a Hulthén potential ("0.6") plus a repulsive core of radius 0.6. $r$ is in units of $10^{-13}$ cm, $\psi_0(r)$ in arbitrary units (same asymptotic form for all three curves).

To estimate the effect of a repulsive core on the magnetic interaction in a nonrelativistic manner, we evaluated $\psi_0(r)$ for a potential containing a "soft repulsive core" only; i.e., a finite repulsive rectangular potential of radius $r_C$ and of depth about $\frac{1}{2}Mc^2$ plus an attractive potential outside of $r_C$ of approximately Yukawa (Hulthén) shape. The range of the Hulthén potential was adjusted to give the correct effective range $r_e$. A plot of $\psi_0(r)$ for such a potential with $r_C=0.6\times10^{+13}$ cm, as well as for a Hulthén and for a square well potential without repulsive core, is given in Fig. 1. It will be seen that $\psi_0(0)$ for the pure Hulthén potential (and similarly for other long-tailed potentials with an attractive singularity at the origin) is much larger than $\psi_0(0)$ for the other potentials and that $\psi_0(r)$ decreases more rapidly with increasing $r$ for the pure Hulthén potential. For these reasons the magnetic interaction energy is considerably smaller than Schwinger's[1] result for a Hulthén potential if (a) a repulsive core is added to the potential, or if (b) the nucleonic magnetic moment is considered spread over a finite distance (making contributions from larger values of $r$ more important).

If a repulsive core of radius $r_C$ is assumed, it seems reasonable to assume the nucleonic magnetic moments to be spread out over distances of the order of magnitude of $r_C$. We therefore estimated the magnetic contributions to $b_0/a_{np}$ and $b_0/a_{pp}$, $\delta b_0/a_{np}$ and $\delta b_0/a_{pp}$, respectively, by using the above-mentioned wave functions $\psi_0(r)$ and replacing the delta-functions in Eqs. (4) and (5) by

$$\left.\begin{array}{ll} 3/4\pi r_C^3 & \text{for} \quad r<r_C, \\ 0 & \text{for} \quad r>r_C, \end{array}\right\} \tag{6}$$

corresponding to a spread of the moments over a sphere of radius $r_C$. Values for $\delta b_0/a_{np}$ and $\delta b_0/a_{pp}$ are given in Table I. These values are, of course, only rough estimates, but for $r_C \lesssim 0.3\times10^{+13}$ cm they are small compared with the variation of $b_0/a_{pp}$ with $r_C$. It should be noted that the values of $\delta b_0/a_{np}$ and $\delta b_0/a_{pp}$ are much smaller still if a repulsive core but no spread of magnetic moment is assumed.

We finally obtain values for $b_0/a_{np}'$ and $b_0/a_{pp}'$, the parameters comparing the purely nuclear potentials, for a square well ($S$) and Hulthén ($H$) potential and for Hulthén potentials plus repulsive cores of radii $r_C$ equal to 0.3, 0.6, and 1.2 (in $10^{-13}$ cm), respectively,

| | ($S$) | ($H$) | (0.3) | (0.6) | (1.2) |
|---|---|---|---|---|---|
| $b_0/a_{np}'$: | 1.26 | 1.36 | 1.28 | 1.24 | 1.23 |
| $b_0/a_{pp}'$: | 1.69 | 1.39 | 1.65 | 1.78 | 1.88 |

The uncertainty due to experimental error alone is only about $\pm0.04$ for $b_0/a_{pp}'$ and much less still for $b_0/a_{np}'$. It will be seen that the difference between $b_0/a_{np}'$ and $b_0/a_{pp}'$ is considerably greater than the experimental error, for $r_C \lesssim 0.3\times10^{-13}$ cm (a difference of about 0.54 for $r_C=0.6$, the value suggested by Jastrow and Lévy[2]). For $r_C \lesssim 0.3$ this difference does not depend very critically on the exact value of $r_C$, nor on the exact shape of the attractive part of the potential.

We therefore conclude that there is a definite discrepancy between the effective strengths of the $n$-$p$ and $p$-$p$ singlet potentials, if a reasonable repulsive core is assumed and nonrelativistic theory is used. It should, however, be pointed out that this discrepancy corresponds to a difference of only a few percent in the strengths of the attractive part of the potential and that there might be appreciable relativistic corrections to the magnetic interaction of two nucleons which were not considered in this paper. A $C$ meson field[8] of range (0.5 to 1)$\times10^{-13}$ cm (or a spread of the charge of the proton over similar distances) would contribute to the $p$-$p$ interaction energy an amount of the same order of magnitude but of wrong sign for removing the discrepancy.

[8] O. Hara and M. Tatsuoka, Progr. Theoret. Phys. (Japan) **3**, 369 (1948), and private communication.