On the Interpretation and Generalization of Dirac's Theory of the Electron

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(Received December 18, 1952)

The independence of momentum and velocity in Dirac's theory of the electron may be understood classically as a consequence of the radiation reaction force. In earlier work one of the authors has tried to interpret the whole spin phenomenon on this basis, considering the new degrees of freedom as representatives of the higher time derivatives in the equation of motion of a particle extended in space. A consequent treatment of this question by Bopp on the basis of his linear electrodynamics revealed that this program is possible only for integral spin; also, it seemed to be necessary to ignore the non-conservative part of the radiation reaction force. We show here that without alteration of the new formalism the spin may be introduced as an intrinsic feature of the particles, as in older theories, so that half-integral values are included. The Poisson brackets are derived in full generality without reference to a specific model by extension of an idea of Anderson. It is then shown by a contact transformation first discovered in quantum mechanics that the motion under the (third order) radiation reaction force is contained as a particular integral in the (fourth order) equations of motion, which we have in common with Bopp. This holds exactly in absence of the Lorentz force and suggests a simple scheme for the interpretation in quantum mechanics of the radiation reaction. In order to maintain nearly the particular motion in presence of the Lorentz force, a slight alteration of the Hamiltonian is necessary, which has empirical support from the evaluation of the corresponding wave equation.

I

IT has often been observed that Dirac's wave equation of the electron may be interpreted classically as an energy theorem of the form

$$E=\mathbf{v}(\mathbf{p}+e\mathbf{A}/c)-eV+m_0c^2(1-\beta^2)^{\frac{1}{2}}, \tag{1.1}$$

where the velocity $\mathbf{v}$ and the momentum $\mathbf{p}$ are treated as independent quantities. ($\mathbf{A}$ and V mean the vector and scalar potential, and $\beta=v/c$.) On the other hand, it seems to have escaped the attention of most physicists that Eq. (1.1) with its characteristic independence of $\mathbf{v}$ and $\mathbf{p}$ may be interpreted classically in a generalized form in the following way: Let the electron be under the influence of the Lorentz force and the radiation reaction force,

$$mcu_j'=-(e/c)F_{jk}u^k+(2e^2/3c)(u_j''-u_j\cdot u_k'u^{k'}), \tag{1.2}$$

and let its mass m be variable according to

$$m'=(2e^2/3c^2)u_k'u^{k'}. \tag{1.3}$$

Here u_jc means the four velocity ($u_ku^k=-1$), F_{ik} is the field tensor, and a prime denotes derivation along the world line. Then (1.1) is an intermediate integral of this motion, if m_0 is replaced by m. As a proof[1] rewrite Eq. (1.2) after substituting (1.3) in the form

$$[mcu_j-(2e^2/3c)u_j']'=-(e/c)F_{jk}u^k. \tag{1.4}$$

With

$$F_{ik}=\partial A_k/\partial x^i-\partial A_i/\partial x^k, \tag{1.5}$$

and

$$p_j=mcu_j-(e/c)A_j-(2e^2/3c)u_j', \tag{1.6}$$

this may, using a contraction well known from ordinary relativistic mechanics, be written in the form

$$p_j'=-(e/c)\partial A_ku^k/\partial x^j. \tag{1.7}$$

In the following we use as a standard abbreviation

$$g_j=p_j+(e/c)A_j. \tag{1.8}$$

Solving (1.6) for u_j' so that

$$u_j'=(3c/2e^2)(mcu_j-g_j), \tag{1.9}$$

we have two independent equations of motion for p_j and u_j. Only if the small constant $2e^2/3c$ is taken to be zero, one has the ordinary dependence $mcu_j=g_j$, with constant m. Now it is easy to see that

$$H=u_kg^k+mc=0 \tag{1.10}$$

is an integral of the foregoing equations of motion. Note first that (1.4) may be written, by using (1.6) and (1.8), as

$$g_j'=-(e/c)F_{jk}u^k, \tag{1.11}$$

so that $u_kg'^k=0$ because of the skew-symmetry of F_{ik}. It follows upon noting (1.9) that

$$\begin{aligned}H'&=u_k'g^k+m'c\\&=(3c/2e^2)(mcu_kg^k-g_kg^k)+m'c.\end{aligned} \tag{1.12}$$

Upon substituting Eq. (1.9) into (1.3), we get

$$m'c=3c/2e^2\cdot(m^2c^2u_ku^k-2mcu_kg^k+g_kg^k), \tag{1.13}$$

and after using $u_kg^k=-mc$, from (1.10), and the identity $u_ku^k=-1$, it follows immediately that

$$H'=0, \tag{1.14}$$

which proves that Eq. (1.10) is an intermediate integral of the motion. Formula (1.1), of course, is merely

[1] W. Wessel, Ann. Physik (5) **43**, 565 (1943). Proof is reproduced here in a condensed form, because the distribution of the original paper has been greatly hampered by the war events.

formula (1.10), solved for $E=cp^4$, with m instead of m_0, in nonrelativistic notation.

II

The quantity on the right side of Eq. (1.3) is the energy equivalent radiated by the moving charge per unit time in its rest system. Thus, it may be said that the analog—or more precisely one analog—of the motion described by the Dirac equation is one in which the radiated energy is permanently added to the rest mass. It is of interest to keep to this interpretation in contrast to an alternative one developed by Bopp,[2] which we shall discuss below. Besides other arguments in its favor, which are the main subject of this paper, we should like to mention in passing that it provides at least in principle the possibility of constructing a quantum mechanics of the radiation process, which has not yet been integrated into Bopp's theory. The identity of (1.1) and (1.10) does not imply that the energy is constant. Formula (1.10) is only an intermediate integral, which states that the variable $E=cp^4$ is expressible by the other variables as in formula (1.1). Let now these variables be represented by operators, to be defined by a set of commutation rules. Also, the rest mass will be an operator.[3] Both energy E and rest mass m will then have eigenvalues, and the variability of the rest mass, given by the classical formula (1.3), infers quantum theoretically that it does not commute with the energy. One has, then, the choice of keeping either to a state of constant energy or to a state of constant rest mass. In the first case one has a "stationary state" in the ordinary sense, involving states of different rest mass; in the other case one has a particle of well-defined mass which changes its energy. The details of this procedure should be analyzed more thoroughly, but it seems satisfactory to have at least a simple program for the treatment of this basic question.

III

We are now coming to the main problem of this paper, which is the following: The classical equation of motion (1.2) is of third order in the coordinates. The usual, although somewhat heuristic procedure of its quantization requires three steps: (a) decomposition into a set of equations of first order; (b) derivation of the first-order equations from a Hamiltonian by Poisson brackets (p.b.); and (c) interpretation of the p.b.'s by commutation rules. Note that only the existence of a Hamiltonian, not of a Lagrangian function, is essential. Of course step (a) is always possible; it is accomplished in the foregoing by formulas (1.7) and (1.9) together with the definition $x_j'=u_j$. Also, step (c) is possible. But, in order to do step (b) it has been found necessary to introduce additional variables.[4] The corresponding classical equation of motion is of higher order and accordingly more complicated. Now our question is: Cannot the simpler motion characterized by (1.2) and (1.3) be shown to be a special type of the more general case? We shall show that this is true, if there are no external forces ($F_{ik}=0$), and that in this case H given by (1.10) is the Hamiltonian; but in the presence of such forces ($F_{ik}\neq 0$) the tendency of keeping near to this motion requires additional terms in the Hamiltonian depending on the F_{ik}, which are also suggested by experimental evidence in a rather unexpected way.

The p.b.'s which we shall use in the following were first set up[4] in a heuristic way anticipating a magnetic moment of the particle as an additional variable. There was a decided tendency to consider it as a substitute for the higher derivatives in the equation of motion of the electron which were to be expected as a consequence of its finite extension in space. A very similar program was simultaneously carried through by Bopp.[2] The difference is that Bopp obtained a conservative character for the motion not by our artifice (1.3) but by complete omission of the nonconservative part of the radiation reaction force. In the formulation of Wheeler and Feynman[5] the term $\frac{1}{2}(F_{ik}^{\text{ret}}-F_{ik}^{\text{adv}})$ in which we are mostly interested is omitted and only $\frac{1}{2}(F_{ik}^{\text{ret}}+F_{ik}^{\text{adv}})$ is retained. With this omission Bopp was able to found the present "field mechanics" on a variation principle and to derive the p.b.'s unambiguously.

Now, of course, a consistent interpretation cannot simply skip the energy consuming part of the radiation reaction force. Moreover, there is another drawback for this conception in that it has not yet been possible to explain the half-integral spin of the electron by this procedure in spite of considerable effort to do so.[6] Thus one might be forced to abandon this scheme in favor of the older concept of the spin as a true additional feature of the particles. The question then arises as to how to introduce its p.b.'s in a logical way.[7] This has been at least partly achieved by Anderson[8] by the

[2] F. Bopp, Z. Naturforsch. **1**, 196 (1946); **3a**, 564 (1948); Z. Physik **125**, 615 (1949). A report on this theory has recently been given by H. Hönl, Ergeb. exakt. Naturwiss. **26**, 291 (1952).

[3] The following considerations have been influenced by interesting remarks of G. Falk, Z. Physik **130**, 51 (1951); **132**, 44 (1952).

[4] W. Wessel, Z. Naturforsch. **1**, 622 (1946); FIAT Report No. 1131.

[5] J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **17**, 157 (1945).

[6] F. Bopp and R. Haag, Z. Naturforsch. **5a**, 644 (1950).

[7] We are referring here to Poisson brackets—to be indicated by { }—because the following considerations are mainly of classical character. It must be kept in mind, that p.b.'s, in contrast to commutation rules, always refer to a system of basic conjugate variables. It is convenient to be reminded of this fact by the factor $\hbar$ in front of the p.b.'s. The existence of such a system is here silently admitted, because it is known to exist from previous work. If the p.b. of two quantities is zero, we say that they commute.

[8] Leon Tasso Anderson, Ph.D. thesis, Ohio State University, Columbus, Ohio, 1952 (unpublished). The reasoning is, in short: the kinetic energy of the linear Hamiltonian does not commute with the orbital angular momentum. Adding a spin angular momentum M_{ik} so that the sum becomes constant, and determining M_{ik}' in the sense of the text, one finds formula (3.1) by separation of the three independent momentum components.

simple remark that one has only to postulate conservation of angular momentum in order to get the p.b.'s of the spin angular momentum components with the four velocity. They follow as a consequence of the rule $M_{ik}'=\{H, M_{ik}\}$, with (1.10) as the Hamiltonian and under the assumption that m commutes with M_{ik}. There results

$$\hbar\{u_jM_{ik}\}=u_i\delta_{kj}-u_k\delta_{ij}, \tag{3.1}$$

if the spin angular momentum components are supposed to be of the form $-\hbar M_{23}$, $-\hbar M_{31}$, $-\hbar M_{12}$. This formula may be relativistically generalized, if the M_{ik}, $i, k=1, 2, 3$, are considered as part of a six-vector M_{ik}, $i, k=1, 2, 3, 4$, which may be interpreted as an electromagnetic moment (magnetic components opposite to spin). The p.b.'s of the M_{ik} among themselves may then be introduced from the viewpoint of their rotational and translational invariance:

$$\{M_{ik}M^{rs}\}=\delta_i{}^rM_k{}^s-\delta_k{}^rM_i{}^s-\delta_i{}^sM_k{}^r+\delta_k{}^sM_i{}^r. \tag{3.2}$$

This process may be continued. The method of Anderson is restricted by some simplifying assumptions which are not necessary. Removing them, we are able to derive all p.b.'s without reference to any specific model. The essential point is that we do not assume the magnitude of the spin to be constant. Indeed the rest mass will depend on its amplitude, and the rest mass is supposed to be variable. As a consequence we will not assume, as does Anderson, that $u^iM_{ji}=0$, but rather that

$$u^iM_{ji}=KU_i, \tag{3.3}$$

where U_i is a unit vector and K is an invariant. As a consequence of the skew-symmetry of M_{ik} and the time-like character of u^i, the vector U_i is orthogonal to u^i, i.e., $u^iU_i=0$, and space-like: $U^iU_i=+1$. Moreover, the vector $u^iM_{ji}{}^*$, where $M_{ji}{}^*$ is the dual to M_{ji}, need not be zero. Let us only make the simplifying assumption that it is parallel to U_i. Using the notation $M_{12}{}^*=M^{34}$, etc.,[9] we may then write with another invariant I,

$$u^iM_{ji}{}^*=-IU_i. \tag{3.4}$$

By a few algebraic manipulations it may be shown (see Appendix), that I and K are the invariants of the momentum tensor in the form

$$I^2-K^2=\tfrac{1}{2}M_{ik}M^{ik}, \tag{3.5}$$

$$IK=\tfrac{1}{4}M_{ik}{}^*M^{ik}. \tag{3.6}$$

Knowing the p.b.'s of u_j and M_{ik}, formula (3.1), it is easy to derive from (3.5) and (3.6) the relations

$$\hbar\{K, u_j\}=U_j, \tag{3.7}$$

$$\hbar\{I, u_j\}=0. \tag{3.8}$$

Now it may be seen from (3.2) that the two invariants in (3.5) and (3.6) commute with all tensor components and among themselves. This leads to the disappearance of the corresponding p.b.'s:

$$\{K, M_{ik}\}=0, \tag{3.9}$$

$$\{I, M_{ik}\}=0. \tag{3.10}$$

As a consequence of (3.8) and (3.7) the invariant I commutes also with U_i, which makes it the center of the p.b. algebra.

IV

With the extension of formula (3.1) we have introduced all six components of the momentum tensor as independent quantities, which is unphysical. One should expect the electrical components to be expressible by the magnetic components and the four-velocity. Now only six of the eight Eqs. (3.3) and (3.4) are independent; as a consequence the six-vector M_{ik} is expressible by the vectors u_j and U_j. The most general form will be a linear combination of $u_iU_k-u_kU_i$ and its dual. Inserting such a linear combination into (3.2) and (3.3) one finds easily, using the fact that the dual is orthogonal to both u_i and U_i, that

$$M_{ik}=K(U_iu_k-U_ku_i)+I(U_iu_k-U_ku_i)^*. \tag{4.1}$$

With this, our apparatus is reduced to seven quantities, namely, the two four-vectors U_i, u_i, which due to orthogonality and normalization represent five independent variables, and the two invariants I and K. It can be shown by some vector calculus that one may choose the seven quantities u_1, u_2, u_3, M_{23}/I, M_{31}/I, M_{12}/I and K/I in order to express the U_k and the rest of the M_{ik}/I, which is essentially the desired result. We omit the explicit formulas, mentioning only that the electrical components vanish with $\mathbf{v}$ only for $K=0$. Formula (4.1) may be inverted and used to derive the p.b.'s of the u_i among themselves. The somewhat lengthy calculations are given in the Appendix. There results

$$\hbar\{u_iu_k\}=I(u_iU_k-u_kU_i)^*/(I^2+K^2). \tag{4.2}$$

The rest is mere algebraic manipulation since the U_i may be expressed by the u_j, M_{ik} and K with the help of (3.2) or (3.3). One finds that

$$\hbar\{U_iu_k\}=K(\delta_{ik}+u_iu_k-U_iU_k)/(I^2+K^2)=-\hbar\{u_iU_k\}, \tag{4.3}$$

$$\{U_iU_k\}=-\{u_iu_k\}. \tag{4.4}$$

With this we have derived the formulas first suggested by the considerations of reference 4 in a straightforward way. Besides being relieved of some unnecessary dimensional constants, in accordance with a previous note,[10] formula (4.2) appears in a simplified form due to the use of (4.1). This last relation was first derived in another paper[11] as a consequence of a matrix representation of the p.b. algebra; hence, there is no doubt

[9] We omit here a superfluous factor i used in earlier publications. The $-$ sign in (3.4) maintains the $+$ sign in (3.6).

[10] W. Wessel, Z. Naturforsch. **6a**, 478 (1951).

[11] W. Wessel, Z. Naturforsch. **7a**, 583 (1952).

that it is compatible with the remaining apparatus in quantum mechanics too.

Bopp's classical theory is characterized by the commutativity of the velocity components. According to (4.2) this leads to

$$I=0, \tag{4.5}$$

and it follows from the representation theory of the Lorentz group that only integral quantization is possible. For finite representations the explicit formulas are given by van der Waerden.[12] One finds after a few rearrangements[13] that

$$iIK=\frac{1}{4}iM_{ik}{}^{*}M^{ik}=j(j+1)-j'(j'+1), \tag{4.6}$$

where j ($\geqslant 0$) and j' ($\geqslant 0$) are integral of half-integral numbers characterizing the representation. The invariant I can disappear only if $j'=j$. Now the eigenvalues of the momentum components are of the form $m+m'$ (reference 12), where $-j\leqslant m\leqslant j$ and $-j'\leqslant m'\leqslant j'$; hence, for equal j, j' the sum $m+m'$ can only be integral. For infinite representations see a previous paper[14] and the literature quoted therein.

V

With the help of the p.b.'s developed in the foregoing two sections we are now able to set up the equations of motion for every quantity, say u_j, in the form $u_j'=\{H, u_j\}$, if a Hamiltonian H is given. It is the subject of this paper to use the H of Eq. (1.10), which is primarily only an intermediate integral of (1.2) and (1.3), as such a Hamiltonian and to see if it leads back to these equations. To do so we have to know the function m. As an invariant it can only depend on I and K, and because I commutes with all quantities, it is its K dependence which matters. Generalizations of Dirac's theory in the sense of (1.10) have often been tried, and there has been much guessing about the presumable form of $m(K)$. As one of us has recently shown, both quantum[15] and classical[10] arguments favor the choice of the simple function

$$m(K)=m_0\cosh(\Gamma K), \tag{5.1}$$

where m_0 is a parameter and Γ the large constant

$$\Gamma=3\hbar c/2e^2. \tag{5.2}$$

The classical argument was that in this way the mass variation (1.3) can be realized. If we wish to see if Eq. (1.2) is also fulfilled, we first have to set up the equations of motion for u_j, U_j, K and p_k, x_k, and then eliminate all additional variables, expressing them by x_k', x_k'', x_k''' or by u_k, u_k', etc., respectively. This can easily be done for the case $I=0$, for we are then in complete accord with Bopp. In this case the motion is derivable from a variation principle, and the desired higher-order equation appears immediately in the form of an Euler-Lagrange equation:

$$\frac{d}{ds}\left(\frac{\partial(L+\lambda u_k u^k)}{\partial u^j}-\frac{d}{ds}\frac{\partial L}{\partial u'^j}\right)=\frac{\partial L}{\partial x^j}. \tag{5.3}$$

The derivative d/ds refers to the world line; L is a Lagrangian function of the form (we follow here Hönl[2])

$$L=-m_0cF(Q)-(e/c)A_k u^k; \tag{5.4}$$

Q is an abbreviation for

$$Q=l^2u_k'u'^k, \tag{5.5}$$

where l is a suitable length, to make Q dimensionless; F is the "structure function" which characterizes a certain linear electrodynamics of Bopp; and λ is a Lagrangian multiplier which takes care of the condition $u_k u^k=-1$. If one performs the derivations prescribed in (5.3), one finds that the equation is self-consistent only if

$$\lambda=m_0c(\tfrac{1}{2}F-2QdF/dQ). \tag{5.6}$$

As Bopp shows,[16] the structure function determines the rest mass by the two equations [in our notation: $g_5=K$, $m_0G/g_5=m(K)$]

$$Q(dF/dQ)^2=K^2, \tag{5.7}$$

and

$$m(K)/m_0=F(Q)-2QdF/dQ. \tag{5.8}$$

Inversely, with $m(K)$ given by (5.1), we may solve them for the structure function. Differentiating (5.8) with respect to Q, one finds quite generally, after repeated use of (5.7) and cancellation of a common factor dK/dQ, that

$$\frac{d}{dK}\frac{m(K)}{m_0}=2Q^{\frac{1}{2}}. \tag{5.9}$$

With our special choice of $m(K)$, namely formula (5.1), one has simply

$$Q=[\tfrac{1}{2}\Gamma\sinh(\Gamma K)]^2. \tag{5.10}$$

Thereafter it is easy to find

$$\begin{aligned}F(Q)&=\cosh(\Gamma K)-\Gamma K\sinh(\Gamma K)\\&=(1+4Q/\Gamma^2)^{\frac{1}{2}}-2(Q^{\frac{1}{2}}/\Gamma)\sinh^{-1}(2Q^{\frac{1}{2}}/\Gamma).\end{aligned} \tag{5.11}$$

Evidently this structure function is very "well behaved"; its development, consisting of powers of Q/Γ^2, is convergent for a wide range of Q, owing to the large numerical value of $\Gamma(\simeq\frac{3}{2}\cdot 137)$. The function has no poles, and its behavior for $Q\to\infty$ lies well within the limits of the functions discussed heretofore. This should be emphasized, for it shows that some "misbehavior"

[12] B. L. van der Waerden, *Die gruppentheoretische Methode in der Quantenmechanik* (Verlag Julius Springer, Berlin, Germany, 1932).
[13] K. J. LeCouteur, Proc. Roy. Soc. (London) **A202**, 394 (1950).
[14] W. Wessel, Z. Naturforsch. **4a**, 645 (1949).
[15] W. Wessel, Phys. Rev. **83**, 1031 (1951).
[16] F. Bopp and L. Bauer, Z. Naturforsch. **4a**, 611 (1949).

of $m(K)$, which we will have to discuss below is not a consequence of an absurd choice of this function.

Substituting (5.6) into (5.3) and introducing m from (5.8), one may put (5.3) into the form

$$\frac{dmcu_j}{ds}=-\frac{e}{c}F_{jk}u^k+2m_0c\frac{d}{ds}\left(Q\frac{dF}{dQ}u_j-l^2\frac{d}{ds}\frac{dF}{dQ}u_j'\right). \quad (5.12)$$

This is the equation of motion of an electron with an acceleration dependent rest mass under the influence of the Lorentz force augmented by force-like terms depending further on the acceleration term of the Lagrangian function. Our program seems to require some sort of coincidence between this formula of the Bopp theory and our Eq. (1.2), which may be written, as in (1.4), in the form

$$dmcu_j/ds=-(e/c)F_{jk}u^k+(2e^2/3c)u_j'', \quad (5.13)$$

but a comparison does not reveal the slightest similarity of the additional terms on the right side; even the order of differentiation is not the same. Nevertheless, the Hamiltonian of Bopp is identical with our function (1.10), which has been shown to be an integral of Eq. (5.13) also. Indeed we are going to show in the next section that for vanishing F_{ik} the motion described by (5.13) is a special case of the motion described by (5.12) if the mass function (5.1) and the structure function (5.11) are used. This does not imply that the additional terms of (5.12) and (5.13) are identical in this case; even the one time integrated form of (5.12) is not an immediate consequence of (5.13). This is the reason why a direct attempt at writing (1.2) and (1.3) in p.b.'s failed.[1] The assertion is that for $F_{ik}=0$ the general integral of the third order equation (5.13), with m' given by (1.3), is a particular integral of the fourth order equation (5.12).

The direct proof of this theorem is rather involved, but it is simple and very instructive to derive Eq. (1.2), for $F_{ik}=0$, from the general equations of motion directly, including nonvanishing I. We will find a very characteristic type of motion and be able to investigate the general motion in the vicinity of this restricted one, which will lead to an entirely new interpretation of this whole mechanics.

VI

The equations of motion to be derived by the p.b.'s of Secs. III and IV from (1.10) as the Hamiltonian, first given[4] and in a purified form,[10] are here further simplified by the use of (4.1), which eliminates the M_{ik} entirely, and are specified by the use of (5.1) for the mass. The value zero of the Hamiltonian (1.10) implies

$$u_kg^k=-mc. \quad (6.1)$$

As before we introduce for symmetry of writing

$$U_kg^k=-Mc. \quad (6.2)$$

This is primarily only an abbreviation and does not mean that M is a given function of K, as is m. Furthermore we write

$$\hbar/m_0c=\lambda\!\!\!^{-}, \quad (6.3)$$

which is the Compton wavelength, divided by 2π, if m_0 is the electron mass. With this notation we have now

$$u_j'=[I/\hbar(I^2+K^2)](u_kU_j-u_jU_k)^*g^k+(\Gamma/\lambda\!\!\!^{-})\sinh(\Gamma K)U_j, \quad (6.4)$$

$$U_j'=[K/\hbar(I^2+K^2)](mcu_j-McU_j-g_j)+(\Gamma/\lambda\!\!\!^{-})\sinh(\Gamma K)u_j, \quad (6.5)$$

$$K'=Mc/\hbar, \quad (6.6)$$

$$I'=0. \quad (6.7)$$

These are 10 equations for the u_j, U_j, K and I, reduced to 7 independent equations by the conditions $u_ku^k=-1$, $U_kU^k=+1$, $u_kU^k=0$, which are particular integrals of (6.4) and (6.5). In addition we have as a consequence of $\{p_i, x_k\}=\delta_{ik}$:

$$g_j'=\{g_kg_j\}u^k=-(e/c)F_{jk}u^k \quad (6.8)$$

and of course

$$x_j'=\{g_kx_j\}u^k=u_j. \quad (6.9)$$

We are now going to show that for vanishing F_{ik}, where

$$g_j=\text{const}=p_j, \quad (6.10)$$

there is another particular integral

$$M^2-m^2+m_0^2=0, \quad (6.11)$$

provided the invariant of the energy-momentum four vector is

$$p_kp^k=-(m_0c)^2. \quad (6.12)$$

Indeed, we have from (6.1), taking into account the skew-symmetry of the starred tensor in (6.4), for $g^k=p^k=\text{const}$:

$$m'c=-u_k'g^k=-(\Gamma/\lambda\!\!\!^{-})\sinh(\Gamma K)U_kp^k=(\Gamma/\lambda\!\!\!^{-})\sinh(\Gamma K)Mc, \quad (6.13)$$

and from (6.2) and (6.5)

$$M'c=-U_k'g^k=[K/(I^2+K^2)\hbar](m^2c^2-M^2c^2+p_kp^k)+(\Gamma/\lambda\!\!\!^{-})\sinh(\Gamma K)mc. \quad (6.14)$$

Now, if (6.11) holds at one time, the first term on the right of (6.14) disappears as a consequence of (6.12). It follows immediately that $MM'-mm'=0$, i.e., (6.11) holds at all times. We infer from (6.11) and (5.1) that for this type of motion there is

$$M=m_0\sinh(\Gamma K). \quad (6.15)$$

Let us now make the transformation

$$\bar{u}_j=u_j\cosh(\Gamma K)-U_j\sinh(\Gamma K), \quad (6.16)$$

$$\bar{U}_j=-u_j\sinh(\Gamma K)+U_j\cosh(\Gamma K). \quad (6.17)$$

It corresponds to a unitary transformation (n.b. with $\exp i\Gamma K^2/2$), which has been found to play a key role in quantum mechanics,[11] and may directly be shown to be canonical. With this transformation the equations of motion (6.4) and (6.5) go over, for the particular case (6.11), into

$$\hbar\bar{u}_j'(I^2+K^2)=I(\bar{u}_k\bar{U}_j-\bar{u}_j\bar{U}_k)^*p^k\cosh\Gamma K +K(p_j-m_0c\bar{u}_j)\sinh(\Gamma K), \quad (6.18)$$

$$\hbar\bar{U}_j'(I^2+K^2)=I(\bar{u}_k\bar{U}_j-\bar{u}_j\bar{U}_k)^*p^k\sinh\Gamma K -K(p_j-m_0c\bar{u}_j)\cosh(\Gamma K). \quad (6.19)$$

It is now immediately seen that

$$\bar{u}_j=p_j/m_0c=\text{const}, \quad (6.20)$$

which makes the last terms vanish, is a particular integral; for the starred tensor, after insertion of (6.20), becomes orthogonal to p^k, so that $\bar{u}_j'=0$. At the same time there follows $\bar{U}_j'=0$; hence

$$\bar{U}_j=\text{const}, \quad (6.21)$$

to be chosen perpendicular to $\bar{u}_j$ and so that $\bar{U}_k\bar{U}^k=1$. Accordingly the original u_j, U_j are expressible, by inversion of (6.16), (6.17), in terms of the variable K alone:

$$u_j=\bar{u}_j\cosh(\Gamma K)+\bar{U}_j\sinh(\Gamma K), \quad (6.22)$$

$$U_j=\bar{u}_j\sinh(\Gamma K)+\bar{U}_j\cosh(\Gamma K). \quad (6.23)$$

In order to obtain the differential equation of u_j, we have now only to eliminate the two constant vectors by two differentiations. This is done very easily with the use of K as an auxiliary variable, obeying

$$K'=(1/\lambda)\sinh\Gamma K, \quad (6.24)$$

as a consequence of (6.6) and (6.15). We have first

$$u_j'=(\Gamma/\lambda)U_j\sinh\Gamma K, \quad U_j'=(\Gamma/\lambda)u_j\sinh\Gamma K. \quad (6.25)$$

There follows

$$u_k'u^{k'}=(\Gamma^2/\lambda^2)\sinh^2(\Gamma K)=-U_k'U^{k'} \quad (6.26)$$

and finally

$$u_j''=(\Gamma^2/\lambda^2)[u_j\sinh^2(\Gamma K)+U_j\sinh(\Gamma K)\cosh(\Gamma K)] =u_j(u_k'u^{k'})+(\Gamma/\lambda)u_j'\cosh(\Gamma K). \quad (6.27)$$

Solving for u_j' and introducing m from (5.1) we have

$$mu_j'=(m_0\lambda/\Gamma)(u_j''-u_j\cdot u_k'u^{k'}), \quad (6.28)$$

which due to the values (5.2) and (6.3) of Γ and λ is just Eq. (1.2) for vanishing F_{ik}. Finally we have

$$m'=m_0\Gamma\sinh(\Gamma K)K' =(m_0\Gamma/\lambda)\sinh^2(\Gamma K) =(m_0\lambda/\Gamma)u_k'u^{k'}, \quad (6.29)$$

which is Eq. (1.3). Hence the motion described by (1.2) and (1.3) is contained in the general motion, if $F_{ik}=0$, whereas I is arbitrary. That the integral given by (6.22) and (5.1), with arbitrary m_0, is the general integral of (1.2), (1.3), may be inferred from the number of constants. Equation (1.2) represents three equations of second order for the three independent components of the four velocity, and (1.3) is one equation of first order. Hence, the number of integration constants must be seven. They are present in the five independent components of u_j, U_j in (6.22), the value of m_0, and the time constant.

VII

The integration of the force free motion which hitherto has been carried to the determination of $u_j=x_j'$ may be accomplished either through (6.24), which gives K as a function of the world line, or by direct treatment of (1.2) and (1.3). As in similar cases[17] one finds a strong variability of all ingoing quantities including m. This must not be considered as absurd, for we have here the clear distinction between momentum and velocity. The momentum is constant for force free motion, according to (6.20), and the variable velocity together with the variable mass describe an internal motion, which can only be checked indirectly. To check the behavior of m one has to check the mass operator in the wave equation, and we have already shown[11] that its behavior is reasonable. Also, as we have seen in Sec. V, the corresponding structure function is absolutely reasonable. There is only one calamity for the interpretation, already discussed,[11] namely that Eq. (6.24) does not always lead to real values of K for real s. Instead by using an imaginary world line, as we did,[11] one solves this difficulty more easily by exchanging the roles of (6.1) and (6.2), i.e., by considering U_kg^k+Mc, with M given by (6.15) as the Hamiltonian and (6.1) as a consequence. This modifies K' in the right sense and leads immediately to the wave equation used;[11] besides it shows directly, from $x_j'=\{H, x_j\}=U_j$, that the velocity of the internal motion is greater than the velocity of light, which causes[11] the imaginary character of the world line and of all normally real vectors. In the present paper we will keep to the Hamiltonian (1.10), for the question, whether u_j or U_j should be considered as the "real" velocity is unessential for the following considerations.

Rather our next step must be to introduce the Lorentz force, $F_{ik}\neq0$. If we do this in the usual way by replacing p_k in the Hamiltonian by g_k, formula (1.8), we obtain for g_k the equation of motion (6.8); and if nothing else is modified, the equation of motion for u_j is given by (5.12), that is for $I=0$. This equation looks formally very satisfactory, but the motion it describes is very far from the classical motion of an electron, since it refers to the internal velocity u_j. What one should expect is rather, in first approximation,

$$m_0c\bar{u}_j=-(e/c)F_{ik}\bar{u}^k, \quad (7.1)$$

[17] C. J. Eliezer, Revs. Modern Phys. **19**, 147 (1947).

where $\bar{u}_j$ reduces to p_j/m_0c, formula (6.20), in the force free case, and m_0 is strictly constant. With the present Hamiltonian (1.10) this equation does not hold. Neither u_j' nor U_j', Eqs. (6.4) and (6.5), contain the Lorentz force, and accordingly $\bar{u}_j'$ contains only the g^k. The Lorentz force in (5.12) appears first at the next stage of the elimination process described in Sec. V and is intimately connected with the appearance of terms with u_j''' in (5.12). From the point of view of an independent spin this is unnatural. If the spin is treated as an autonomous quantity, it is purely formal to eliminate its variables in favor of the higher derivatives of u_j. A direct interpretation of the Euler-Lagrange equation (5.12) is no longer available, and for the essential case $I\neq 0$ there is even no Lagrangian. In any case, even apart from these reflections, a constant momentum in the force free case should be modified in the sense of Eq. (7.1) without the detour of higher derivations. This is true for the vector

$$p_j=mcu_j-2m_0c(QF_Qu_j-l^2dF_Qu_j'/ds) \tag{7.2}$$

($F_Q=dF/dQ$), according to Eq. (5.12), at least for u_j instead of $\bar{u}_j$ on the right side of (7.1), and this vector is coincident with our p_j, Eq. (6.20), if the u_j of Eq. (6.22) and the $F(Q)$ of Eq. (5.11) are used (n.b. for $l=\lambda\!\!\!^{-}/2$). But this holds only in the force free case; the equation of motion of our p_j or $\bar{u}_j$ is different from the Eq. (5.12) obeyed by the p_j just mentioned.

Let us compute

$$\bar{u}_j'=\{H, \bar{u}_j\}, \tag{7.3}$$

with (1.10) as Hamiltonian. We obtain similar terms as in (6.18), with p^k on the right side replaced by g^k. But this is not the only change, because in the derivation of (6.18) we have used the form (6.15) for M as a consequence of $M^2-m^2+m_0^2=0$, which in presence of external forces is no longer a particular integral. Rather the equation of motion of $\bar{u}_j$ reads now, after using (6.16) and by direct use of (6.4), (6.5), and (6.6),

$$\begin{aligned}h\bar{u}_j'(I^2+K^2)=&I(\bar{u}_k\bar{U}_j-\bar{u}_j\bar{U}_k)^*g^k\cosh\Gamma K\\&+K(g_j+\bar{u}_j\cdot\bar{u}_kg^k-\bar{U}_j\cdot\bar{U}_kg^k)\sinh\Gamma K\\&+m_0c\Gamma\bar{U}_j(g_kU^k/m_0c+\sinh\Gamma K)\cdot(I^2+K^2).\end{aligned} \tag{7.4}$$

Let us see what happens, if we try to generalize (6.20) in the form

$$\bar{u}_j=g_j/m_0c. \tag{7.5}$$

The term with the starred bracket disappears for the same reason as in (6.18). In the second bracket we have $\bar{u}_kg^k=\bar{u}_k\bar{u}^k\cdot m_0c=-m_0c$, $\bar{U}_kg^k=\bar{U}_k\bar{u}^k\cdot m_0c=0$, i.e., it disappears too, and likewise the last one is zero, because there is $g_kU^k/m_0c=\bar{u}_kU^k=-\sinh(\Gamma K)$. Hence, there follows $\bar{u}_j'=0$, in contradiction to Eq. (7.5), which would lead to

$$\bar{u}_j'=-(e/m_0c^2)F_{jk}u^k. \tag{7.6}$$

Now, there seems only a little required to make (7.5) and (7.6) compatible. Let us join to the Hamiltonian an additional term H_1 and keep to (7.5). As before, the right side of (7.4) will disappear, but we will have

$$\bar{u}_j'=\{H_1, \bar{u}_j\}. \tag{7.7}$$

On the other hand, we have still in the first approximation, if H_1 is small compared with H,

$$g_j'=\{H+H_1, g_j\}\simeq\{H, g_j\}=-(e/c)F_{jk}u^k. \tag{7.8}$$

Hence, to maintain (7.5), we would have to find an H_1 so that

$$\{H_1, \bar{u}_j\}=-(e/m_0c^2)F_{jk}u^k. \tag{7.9}$$

It does not seem possible to fulfill this relation exactly in a simple way. Indeed we did not expect to find Eq. (7.6), but rather Eq. (7.1), with $\bar{u}^k$ instead of u^k on both sides. There exists an H_1 which accomplishes this task very simply, namely,

$$H_1=-e\hbar/2m_0c^2\cdot M_{ik}F^{ik}. \tag{7.10}$$

It follows first from (3.1) in view of the commutativity of M_{ik} and K that

$$\{H_1, u_j\}=-(e/m_0c^2)F_{jk}u^k, \tag{7.11}$$

and due to the canonical character of the transformation (6.16) (see the comment to this formula) one has also

$$\{H_1, \bar{u}_j\}=-(e/m_0c^2)F_{jk}\bar{u}^k. \tag{7.12}$$

With this choice of H_1, (7.5) is no longer true; accordingly we will have additional terms in Eq. (7.1), coming from the right side of (7.4), which no longer vanishes; but such terms are as natural as in (5.12), where they arise, if one tries to replace u_j by p_j [Eq. (7.2)] on both sides.

VIII

From the foregoing considerations it seems very suggestive to generalize our particular motion by introduction of the term H_1, Eq. (7.10), into the Hamiltonian. Evidently, it represents the magnetic energy of one Bohr magneton (the factor $1/c$ comes from the reference to the world line instead of to the proper time), and the question arises: Is there any experimental evidence for the addition of such a term? In Dirac's theory it would be impossible. Before we discuss this question, we should like to discuss another apparent difficulty which is even more obvious. The Lorentz force in (7.12) appears as the consequence of a magnetic moment of the particle and not of its charge. Apparently, by this mechanism, an uncharged particle with a magnetic moment, like the neutron, would move in much the same way as an electron. In fact the phenomenon is very surprising, but the present consequence is wrong. If (7.1) is supposed to hold to a fair approximation, the perturbation coming from the term on the right side of (7.4) must be small, i.e., $m_0c\bar{u}_j=g_j$ must hold approximately, at least in some time average. This is possible only when the particle has the right charge-moment ratio. Without charge the variation of g_j would be much smaller [namely, $\{H_1, g_j\}$ instead of

$\{H, g_j\}$] than that of $m_0c\bar{u}_j$. The particle would move more or less like a dipole, and the use of $\bar{u}_j$ would simply lose its meaning.

In Dirac's original Hamiltonian, as is well known, a spin moment does not appear explicitly, and a term like H_1 would destroy the accord with experiment, e.g., the fine structure of the H atom. More recently the much discussed inconsistency discovered by Lamb and Retherford (LR shift) has modified the picture, but in order to describe it phenomenologically in the terms of the Dirac theory one would have to add only $(1/2\pi)(1/137)$ of the present H_1.

This is very much different in the present approach. Our Hamiltonian is coincident with Dirac's only if the variability of the rest mass is disregarded, and our p.b.'s can be interpreted quantum mechanically only in the sense of "minus" commutation rules. The classical interpretation demands the use of Hermitean matrices for their representation, and this leads to infinite matrices[18] for the velocity components, similar to those of momenta and coordinates, but similar to the original Dirac matrices only in the first rows and columns. The theory of these infinite representations has been worked out during the last years by one of the authors with permanent regard to the integration into Dirac's theory of the finer reactions of the electron upon itself. It was in strong support of this tendency that the use of these representations together with formula (5.1) for the mass could be shown to account for the LR shift at least qualitatively.[11] An approximate evaluation gave a term of the right form, but with too great a factor $[(16/3)(1/2\pi)(1/137)]$. This deviation is certainly at least in part a consequence of the rather rough approximation. We evaluated the LR shift classically, using for the rest of the Hamiltonian a finite instead of an infinite matrix representation. This was done, since the interest was primarily centered on the shift term and since the technique of the infinite matrices was not yet sufficiently developed. The wave equation obtained in this way was just Dirac's except for the correction term and accordingly included the correct relativistic (Sommerfeld) fine structure. In the meantime a way has been found to treat the rigorous equation by a perturbation method.[19] It had to be applied first to check the normal relativistic fine structure, and the somewhat unexpected result was that this term was too small by a factor of 1/9. In this case the approximation method seems trustworthy, and the deviation has a physical reason. In the infinite representations also the matrix of the spin is infinite. There exists a minimum spin which, incidentally, is given by the center I of the p.b. algebra, and "excited states" with $I+1$, $I+2$, $\cdots$. The perturbation caused by the spin $\frac{3}{2}$ in its first order is of similar magnitude as the second-order perturbation caused by the spin $\frac{1}{2}$. To correct for the observed value of the fine structure one may try to introduce additional terms into H, and we have found that it may be done by an appropriate combination of terms with $(e\hbar/m_0c^2)U_ip_kF^{ik}$ and $(e\hbar/m_0c^2)u_ip^kF^{ik*}$; but without the support of the foregoing classical considerations this would be extremely unsatisfactory. Now we consider this result as much in favor of the whole attempt. The additional terms found by trying are not simply of the form (7.10); but the demand, that the classical motion in the presence of external forces should be "near to" $m_0c\bar{u}_j=g_j$ is too vague, to make the form (7.10) compulsory. It would even be more in the sense of the original formula (7.9) to replace the M_{ik}, formula (4.1), by mixed terms proportional to $\bar{u}_iU_k-\bar{u}_kU_i$ or even $p_iU_k-p_kU_i$, etc., which due to the skew-symmetry of F_{ik} would just give terms of the form found quantum theoretically. In any case the correspondence is evident, and one has no reason to hesitate with the introduction of such additional terms into the wave equation. We give no more details here, because we have not yet surmounted a half-empirical stage. In a satisfactory theory the correction terms should follow from a clear principle, which would also deduce the necessary numerical factors from the invariants I and K. Such principle would also have to explain, why just the particular motion studied in Sec. VI should be realized in quantum mechanics; but, of course, quantum mechanics in general is in favor of particular motions.

IX

The reader will probably be more willing to accept an independent spin than a basic change in Dirac's equation. Our opinion is the opposite. The idea of the spin as an independent quantity has been a little overemphasized in the foregoing to make things clearer; actually we think that the program of Bopp's linear electrodynamics is too attractive to be abandoned so readily. On the other hand, the necessity for a thoroughgoing modification of Dirac's theory seems very probable to the authors. In its present form it is still pioneer work. Its elegant simplicity is largely due to the "plus" character of its commutation relations, and that frustrates every classical understanding. Its successes may be due to the same fortunate chance as was Sommerfeld's derivation of the fine structure formula which he did without knowledge of the spin. If a classical interpretation becomes possible at no greater expense than a small additional term in the Hamiltonian it should be taken very seriously.

APPENDIX

The proof of Eqs. (3.5), (3.6) rests on two identities for skew-symmetric tensors

$$M_{ji}{}^*M^{ik}=\tfrac{1}{4}\delta_i{}^kM_{mn}{}^*M^{mn}, \qquad \text{(A1)}$$

$$M_{ji}M^{ik}-M_{ji}{}^*M^{ik*}=\tfrac{1}{2}\delta_i{}^kM_{mn}M^{mn}. \qquad \text{(A2)}$$

[18] W. Wessel, Phys. Rev. **76**, 1512 (1949).

[19] Unpublished; the procedure is based on the Lorentz transformation formula (3.7) (see reference 11). Its application to the potential gives the additional terms.

One proves them, following Minkowski,[20] by direct computation, using matrix notation for the tensor multiplication. With them one obtains (3.5) and (3.6) from (3.3) and (3.4) by squaring and subtracting these relations or multiplying them, respectively, remembering that $U_kU^k=1$.

For the proof of (4.2) we first invert formula (4.1) by combination with its dual. There follows, e.g.,

$$u_iU_k-u_kU_i=(-KM_{ik}+IM_{ik}^*)/(I^2+K^2). \quad \text{(A3)}$$

Let us denote the tensor on the right by R_{ik}. The p.b. with u_j is

$$\{u_ju_i\}U_k+u_i\{u_jU_k\}-\{u_ju_k\}U_i -u_k\{u_jU_i\}=\{u_jR_{ik}\}. \quad \text{(A4)}$$

The right side may be computed with the help of (3.1), (3.7), and (3.8):

$$\hbar\{u_jR_{ik}\}=[(I^2-K^2)U_jM_{ik}+2IKU_jM_{ik}^*]/(I^2+K^2)^2 +[K(\delta_{ij}u_k-\delta_{kj}u_i)-I(\delta_{ij}u_k-\delta_{kj}u_i)^*]/(I^2+K^2). \quad \text{(A5)}$$

We now multiply (A4) by U^k and contract. Due to (3.3), there is $U^ku_k=0$; and as a consequence of $U^kU_k=1$, we have $U^k\{u_jU_k\}=0$. With this there results

$$\{u_ju_i\}-\{u_ju_k\}U^kU_i=U^k\{u_jR_{ik}\}. \quad \text{(A6)}$$

This is a system of six inhomogeneous linear equations for $\{u_ju_i\}$. The determinant is unity, that is, if a solution is found somehow, it is unique. Try now

$$\{u_ju_i\}=U^k\{u_jR_{ik}\}. \quad \text{(A7)}$$

Due to the skew-symmetry of R_{ik} there holds then $U^i\{u_ju_i\}=U^iU^k\{u_jR_{ik}\}\equiv 0$. Accordingly, the second term in (A6) vanishes, and the equation is fulfilled. Furthermore formulas (3.3) and (3.4) may be inverted with the help of the tensor identities (A1) and (A2), giving

$$U^iM_{ji}=Ku_i, \quad \text{(A8)}$$

$$U^iM_{ji}^*=-Iu_i, \quad \text{(A9)}$$

and finally there holds

$$U^k(\delta_{ij}u_k-\delta_{kj}u_i)^*\equiv(u_iU_j-u_jU_i)^*. \quad \text{(A10)}$$

With these formulas the computation of $\{u_j, u_i\}$ from (A7) and (A5) is straightforward and leads to Eq. (4.2) of the text.

[20] H. Minkowski, Math. Ann. **68**, 472 (1910).

PHYSICAL REVIEW VOLUME 91, NUMBER 4 AUGUST 15, 1953

Repulsive Core and Charge Independence

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(Received April 30, 1953)

In an earlier paper Schwinger derived expressions for the effective strengths of the neutron-proton and proton-proton interactions in the singlet S state. He showed their difference is small and can be accounted for by magnetic forces if a long-tailed potential (Yukawa) is assumed but not for a short-tailed potential. In this paper an equivalent analysis is carried out for nuclear potentials which have a repulsive core. It is shown that for core radii of more than about 0.3×10^{-13} cm the effect of the magnetic interaction is decreased and the difference between the n-p and p-p interactions is increased. Numerical values of the discrepancy are given for different core radii.

1. INTRODUCTION

ONE test for the hypothesis of charge independence of nuclear forces consists in comparing the zero-energy scattering lengths for the singlet S states of the neutron-proton and proton-proton systems, a_{np} and a_{pp}, respectively. The experiments from which these two quantities are derived are very accurate, but the value obtained for a_{pp} depends to an appreciable extent on the assumed shape of the nuclear potential. It was further pointed out by Schwinger[1] that the magnetic interaction between nucleons gives different contributions to the effective potential strengths for the n-p and the p-p systems. Formulas for these magnetic contributions, also shape-dependent, and hence for a_{np}' and a_{pp}', the effective scattering lengths resulting from the purely nuclear potentials alone, were derived by Sc. He found that a_{np}' and a_{pp}' are practically equal if a very long-tailed potential shape (Yukawa or Hulthén) is assumed, but that there is a definite discrepancy between them for more short-tailed potential shapes.

The presence of large repulsive nuclear forces at short internuclear distances for the singlet n-p and p-p states is now considered very likely.[2] It is the purpose of the present paper to point out that a sizable discrepancy between a_{np}' and a_{pp}' (and hence between the effective strengths of the two potentials) is again obtained if a repulsive core is assumed, even if the attractive part of

[1] J. Schwinger, Phys. Rev. **78**, 135 (1950). This paper will be referred to as Sc and the same notation will be used throughout.

[2] R. Jastrow, Phys. Rev. **81**, 165 (1951); M. M. Lévy, Phys. Rev. **88**, 725 (1952).