Stability of Orbits in a Strong-Focusing Synchrotron

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The influence of gaps for injection, acceleration, etc., on the stability of orbits in a strong-focusing synchrotron is examined and found not to be negligible. The calculation also gives some information about the resonances resulting from irregularities.

I. INTRODUCTION

In earlier constructions of the synchrotron, the oscillations of the particles are governed by the equation

$$(d^2w/d\theta^2)+nw=0, \tag{1}$$

with n a constant. It has been pointed out by Courant, Livingston, and Snyder[1] that the engineering construction may be considerably simplified by making n a function of θ. These authors have considered the case where n shows a rectangular ripple. The result seems promising, allowing a wide variation in the absolute values of n. It has also been suggested that the conditions for stability will not be remarkably influenced by the presence of small intermediate intervals where $n=0$. In practice it is necessary to have evenly spaced intervals with $n=0$, which are not small. As the motion is more complicated in this case, one is inclined to believe that the stable regions will be diminished. This suggestion is shown to be correct by the following analysis. Furthermore, it is shown that the stable regions become "perforated" in the presence of irregularities in the engineering construction.

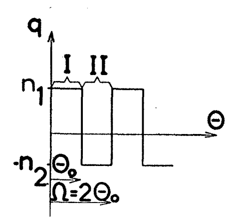

FIG. 1. Field gradient q as function of the angle θ for sections arranged without gaps.

II. REMARKS ON HILL'S EQUATION

The following mathematical theorems concerning the equation

$$(d^2w/d\theta^2)+q(\theta)w=0 \tag{2}$$

will be useful in treating the stability problem. Here $q(\theta)$ is a periodic function of θ with the period Ω. Two different solutions w_1, w_2 of this equation are linearly independent if, and only if,

$$\Delta(\theta)=\begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix}\neq 0. \tag{3}$$

[1] Courant, Livingston, and Snyder, Phys. Rev. **88**, 1190 (1952).

For the solutions of Eq. (2),

$$d\Delta/d\theta=0; \tag{4}$$

i.e., Δ is constant for all values of θ. From the theory of Floquet[2] it is known that Hill's equation has normal solutions W_1, W_2 satisfying

$$\begin{aligned} W_1(\theta+\Omega)&=\sigma_1 W_1(\theta),\\ W_2(\theta+\Omega)&=\sigma_2 W_2(\theta). \end{aligned} \tag{5}$$

This fact is due to the periodicity of $q(\theta)$.

From an arbitrary set of fundamental solutions, w_1, w_2, satisfying Eq. (3), the normal solutions W_1, W_2 can be constructed in the following way. A normal solution, which can always be written

$$W=aw_1+bw_2, \tag{6}$$

satisfies

$$W(\Omega)=\sigma W(0), \tag{7}$$

$$W'(\Omega)=\sigma W'(0). \tag{8}$$

Hence,

$$\begin{aligned} a\{w_1(\Omega)-\sigma w_1(0)\}+b\{\omega_2(\Omega)-\sigma w_2(0)\}&=0,\\ a\{w_1'(\Omega)-\sigma w_1'(0)\}+b\{w_2'(\Omega)-\sigma w_2'(0)\}&=0. \end{aligned} \tag{9}$$

A solution exists if

$$\begin{vmatrix} w_1(\Omega)-\sigma w_1(0) & w_2(\Omega)-\sigma w_2(0) \\ w_1'(\Omega)-\sigma w_1'(0) & w_2'(\Omega)-\sigma w_2'(0) \end{vmatrix}=0, \tag{10}$$

or

$$\sigma^2-(\sigma/\Delta)\{w_1(0)w_2'(\Omega)-w_1'(0)w_2(\Omega) +w_2'(0)w_1(\Omega)-w_1(0)w_1'(\Omega)\}+1=0. \tag{11}$$

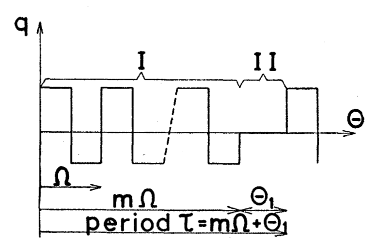

FIG. 2. Field gradient q for the case of m double sections followed by a gap.

[2] See, e.g., E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Macmillan Company, New York, 1946), p. 412.

Here Eq. (4) has been used. Denoting the roots of Eq. (11) by σ_1 and σ_2, the normal solutions are

$$\begin{aligned} W_1 &= \{w_2(\Omega)-\sigma_1 w_2(0)\}w_1-\{w_1(\Omega)-\sigma_1 w_1(0)\}w_2,\\ W_2 &= \{w_2(\Omega)-\sigma_2 w_2(0)\}w_1-\{w_1(\Omega)-\sigma_2 w_1(0)\}w_2. \end{aligned} \tag{12}$$

In order that all solutions of Hill's equation be bounded, it is necessary that

$$|\sigma_1|\leqslant 1 \quad \text{and} \quad |\sigma_2|\leqslant 1. \tag{13}$$

Since, according to Eq. (11), $\sigma_1\sigma_2=1$, the condition for stability is that the roots of Eq. (11) be complex, and thus

$$|\sigma_1|=|\sigma_2|=1,$$

or

$$|[w_1(0)w_2'(\Omega)-w_1'(0)w_2(\Omega)+w_1(\Omega)w_2'(0) -w_1'(\Omega)w_2(0)]/2\Delta|<1. \tag{14}$$

III. RECTANGULAR RIPPLE WITHOUT GAPS

Let us consider the case where the function $q(\theta)$ is defined as shown in Fig. 1. Elementary calculations give the following set of solutions (continuous and with continuous derivatives of first order): In region I:

$$w_1=\cos(n_1^{\frac{1}{2}}\theta), \quad w_2=n_1^{-\frac{1}{2}}\sin(n_1^{\frac{1}{2}}\theta). \tag{15}$$

In region II:

$$w_1=A\cosh(n_2^{\frac{1}{2}}\theta)+B\sinh(n_2^{\frac{1}{2}}\theta),$$
$$w_2=Cn_2^{-\frac{1}{2}}\cosh(n_2^{\frac{1}{2}}\theta)=Dn_2^{-\frac{1}{2}}\sinh(n_2^{\frac{1}{2}}\theta).$$

Here

$$\begin{aligned} A&=\cos\alpha_1\cosh\alpha_2+\gamma\sin\alpha_1\sinh\alpha_2,\\ B&=-\gamma\sin\alpha_1\cosh\alpha_2-\cos\alpha_1\sinh\alpha_2,\\ C&=\gamma^{-1}\sin\alpha_1\cosh\alpha_2-\cos\alpha_1\sinh\alpha_2,\\ D&=\cos\alpha_1\cosh\alpha_2-\gamma^{-1}\sin\alpha_1\sinh\alpha_2, \end{aligned} \tag{16}$$

where $\gamma=(n_1/n_2)^{\frac{1}{2}}$, $\alpha_1=n_1^{\frac{1}{2}}\theta_0$, and $\alpha_2=n_2^{\frac{1}{2}}\theta_0$. This is a set of fundamental solutions, and the functional determinant has the value 1. The equation for σ is

$$\sigma^2-2\sigma\cos\varphi+1=0, \tag{17}$$

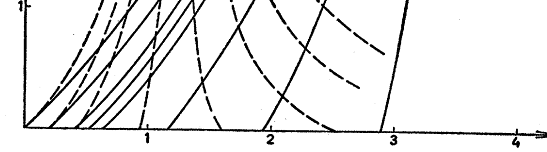

FIG. 3. $\cos\varphi$ and β as functions of α_1^2 and α_2^2. The quantity $\cos\varphi$ characterizes a double section, and β describes the arrangement of the gaps.

where

$$\cos\varphi=\cos\alpha_1\cosh\alpha_2-\frac{1}{2}\frac{\alpha_1^2-\alpha_2^2}{\alpha_1\alpha_2}\sin\alpha_1\sinh\alpha_2. \tag{18}$$

The condition for stability,

$$|\cos\varphi|<1, \tag{19}$$

is identical with that given by Courant *et al.*

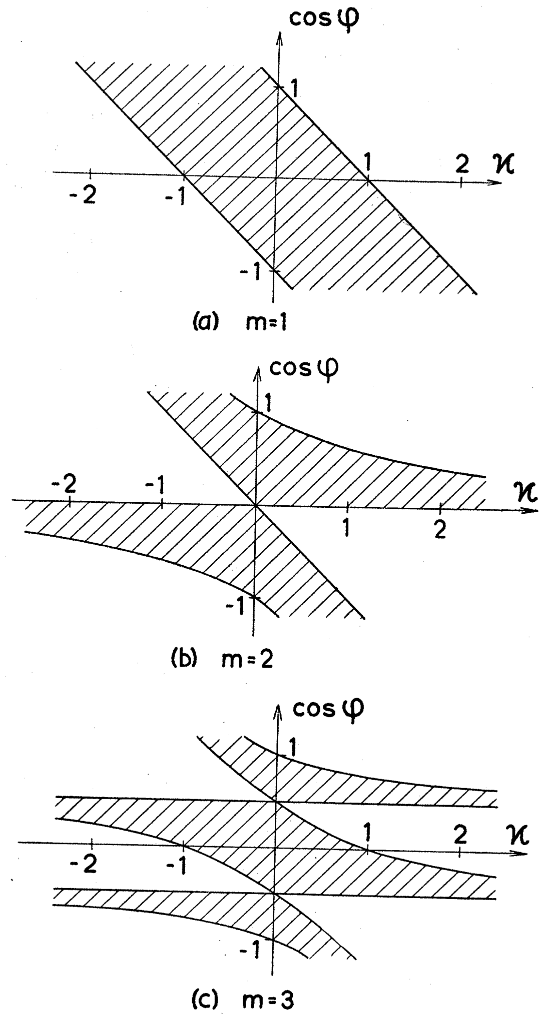

FIG. 4. The stability of the motion depends on the properties of the magnetic sections and the gaps, which are characterized by $\cos\varphi$ and κ. The number of double sections between the gaps is m. Regions of stability for one type of oscillation are shown for 3 different values of m.

It is easy to verify the following relations:

$$\sigma_1=e^{i\varphi}, \quad \sigma_2=e^{-i\varphi}, \tag{20}$$

$$\sigma_1^m+\sigma_2^m=2\cos(m\varphi), \tag{21}$$

$$\sigma_1^m-\sigma_2^m=2i\sin(m\varphi), \tag{22}$$

$$\begin{aligned} W_1(0)/W_1'(0)&=-w_2(\Omega)/[w_1(\Omega)-\sigma_1],\\ W_2(0)/W_2'(0)&=-w_2(\Omega)/[w_1(\Omega)-\sigma_2], \end{aligned} \tag{23}$$

where the W_i are the normal solutions defined by Eq. (12).

IV. RIPPLE WITH GAPS

In the actual case, $q(\theta)$ is defined according to Fig. 2. A fundamental set of solutions u_1, u_2 is obtained by starting with the normal solutions W in the region I.

$$\begin{aligned} u_1(0)&=W_1(0), & u_1(\tau)&=\sigma_1{}^m\{W_1(0)+\theta_1W_1'(0)\},\\ u_2(0)&=W_2(0), & u_2(\tau)&=\sigma_2{}^m\{W_2(0)+\theta_1W_2'(0)\},\\ u_1'(0)&=W_1'(0), & u_1'(\tau)&=\sigma_1{}^mW_1'(0),\\ u_2'(0)&=W_2'(0), & u_2'(\tau)&=\sigma_2{}^mW_2'(0). \end{aligned} \tag{24}$$

The condition of stability (14) corresponding to this problem is

$$\left|\frac{\sigma_2{}^m+\sigma_1{}^m}{2}-\frac{\theta_1}{2}\frac{\sigma_2{}^m-\sigma_1{}^m}{W_1(0)/W_1'(0)-W_2(0)/W_2'(0)}\right|<1, \tag{25}$$

which, by using Eqs. (21)–(23), may be reduced to

$$|\cos(m\varphi)+\kappa\sin(m\varphi)/\sin\varphi|<1. \tag{26}$$

Here $\cos\varphi$ is given by Eq. (18) and

$$\kappa=\beta\theta_1/\theta_0,$$
$$\beta=\tfrac{1}{2}\alpha_2\cos\alpha_1\sinh\alpha_2-\tfrac{1}{2}\alpha_1\sin\alpha_1\cosh\alpha_2. \tag{27}$$

Figure 3 shows $\cos\varphi$ and β as functions of α_1^2 and α_2^2. Figure 4 gives the stable regions in the plane κ, $\cos\varphi$.

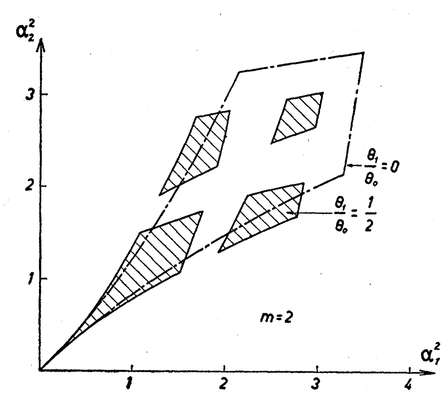

FIG. 5. Stability regions for an arrangement of 2 double sections followed by a gap. This figure is constructed from Figs 3 and 4(b).

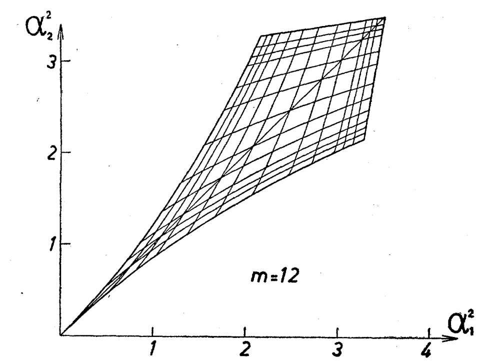

FIG. 6. An irregularity in the engieeering construction represented by a small gap repeated after 12 double sections gives rise to 22 unstable stripes, shown as fine lines.

The stable regions of Fig. 4 may be mapped into the α^2 plane for different values of θ_1/θ_0. In order that the stability of both radial and axial oscillations may be examined, the stable regions should be transformed according to $\alpha_1^2=\alpha_2^2+\theta_0^2$. If $\alpha\gg\theta_0$, this is equivalent to a reflection in the line $\alpha_1^2=\alpha_2^2$. The final region of stability is then found as the area common to both regions.

This construction has been worked out for $m=2$, $\theta_1/\theta_0=0$, $\frac{1}{2}$, and the result is shown in Fig. 5. By means of Figs. 3 and 4 the process is easily repeated for other combinations. It is seen that the stable regions are split up by the presence of the gaps. The effect is larger the larger the ratio θ_1/θ_0, and if this ratio is not very small the effect can probably not be neglected.

It is possible to investigate the influence of a small irregularity in the engineering construction by applying these results in a slightly different manner. Let us assume a rectangular ripple interrupted after a distance $k\Omega$ by a small error represented by a gap. This corresponds to case III with $m=k\gg1$ and $\theta_1/\theta_0\ll1$. One finds (see Fig. 6) that the stable regions will be "perforated" by small unstable regions crossing the line $\alpha_1^2=\alpha_2^2$, the number of such regions being $2(k-1)$. Though of small area the perforation may seriously influence the operation. By letting the oscillation be of a nonlinear character, a stable operation may be obtained. This question, however, will not be discussed here.