The Calculation of the Half-Widths of One-Body Resonances

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(Received March 13, 1953)

Half-widths of resonances for protons incident on a number of light nuclei with Z ranging between 4 and 14 are calculated in the one-body approximation. It is found that neglecting the tail of the wave function in the integral representing the probability of the proton being in the incident state can lead to errors of the order of 50 percent. Estimates of level shifts are also made using formulas derived by Breit. These represent approximately the difference between the energy at which the phase shift is 90° and the energies corresponding to maxima of the absolute value of the radial function $\mathfrak{F}$ or else of the absolute value of $\mathfrak{F}/G$, where G is the irregular function. Both $\mathfrak{F}$ and G are normalized to unit amplitude at a large distance.

INTRODUCTION

IN nuclear reaction theories there occurs a quantity usually denoted by Γ which is closely related to the half-value breadth of resonance levels. Differences in formulation and viewpoint lead to different forms of final expressions for the collision cross sections with correspondingly different definitions of the resonance half width Γ. In some discussions[1,2] the quantity Γ is defined in close analogy to the disintegration probability of a nearly stationary state. In other work[3] the connection with the semistable states is given less prominence. In the former type of consideration Γ is essentially proportional to

$$v\Big/\Big[\int|\Psi|^2 d\tau+\text{correction term}\Big],$$

where v is the relative velocity of the particles while the integral is extended over most of the region within which $|\Psi|^2$ is large. A correction term to the integral is introduced in order to reproduce the experimental half-value breadth. If the correction term is to be relatively small, the integration usually has to be extended through appreciable portions of the reaction channels.

In the latter type of development the definition of a level is made in terms of a preassigned nuclear radius and extensions of the region of integration to portions of the outer channels are not natural in these theories, the usual definitions of reduced widths involving integrals confined to the nuclear interior. In the former method the theoretical 2Γ is adjusted to be a first approximation to the experimental half-value resonance breadth. In the latter the adjustment to experiment is carried out by means of "level shift" calculations and changes such as the introduction of Γ' in the work of Thomas.[4]

In the present note the difference caused by including the tail of the wave function through a disintegration channel is partially studied in the special case of the one-body problem. It is realized that by neglecting the many-body character of the process an exaggerated importance may be given to this "tail" effect. On the other hand, the work of Thomas[4] shows that the interaction of an s particle with C^{12} or O^{16} is reasonably well represented by a one-particle model in a limited energy region and the exaggeration involved is therefore not always serious. Admittedly, however, the whole interpretation of C^{13} and N^{13} states still contains some contradictions regarding preference for one- *versus* many-body formulations. Table I contains a comparison of the values of different kinds of Γ computed in different conventions. The first of the three columns starting with the one marked Γ represents a reasonably conscientious approximation to the half-width. The second is calculated by extending the integral to the turning point in the radial motion corresponding to a classical dynamics treatment of the actual problem. The third gives the value computed by terminating the integration at the nuclear radius. The rather large differences between the values in the last two columns show that the level shift and Γ correction terms may be quite serious in the "black box" type of calculation. Special circumstances accounting for relative prominence of the effects in some cases and their relative absence in others will be discussed toward the end of this note. The values in the table have been calculated for hypothetical rather than actual resonances, the well depth and radius having been treated as in the work of Freeman and McHale,[5] who employed the one-body approximation of Ostrofsky, Breit, and Johnson.[6]

It was felt that this procedure provides a better picture of the relationships between quantities as a

* Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.

[1] G. Breit, Phys. Rev. **40**, 127 (1932); G. Breit and F. L. Yost, Phys. Rev. **48**, 203 (1935); G. Breit and E. P. Wigner, Phys. Rev. **49**, 519 (1936); H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937); H. A. Bethe, Revs. Modern Phys. **9**, 71 (1937); G. Breit, Phys. Rev. **58**, 1068 (1940); G. Breit, Phys. Rev. **69**, 472 (1946).

[2] G. Breit, Phys. Rev. **58**, 506 (1940).

[3] P. R. Kapur and R. Peierls, Proc. Roy. Soc. (London) **A166**, 277 (1938); E. P. Wigner, Phys. Rev. **70**, 15 (1946); E. P. Wigner, Phys. Rev. **70**, 606 (1946); E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947); T. Teichmann, Phys. Rev. **77**, 506 (1950); J. B. Ehrman, Phys. Rev. **81**, 412 (1951); R. G. Thomas, Phys. Rev. **81**, 148 (1951); **88**, 1109 (1952).

[4] See R. G. Thomas, footnote 3.

[5] B. E. Freeman and J. L. McHale, Phys. Rev. **89**, 223 (1953).

[6] Ostrofsky, Breit, and Johnson, Phys. Rev. **49**, 22 (1936).

function of the atomic number, orbital angular momentum, nuclear radius, and incident energy than could be accomplished by studying experimentally observed reactions, since for the latter the values of the parameters are restricted by practical circumstances. The conventions regarding nuclear radii are explained in the second of the two footnotes to Table I. During the course of the work some improvements in methods of computation have been developed which are also briefly reported on below.

Notation

F_L, $G_L\cdots$ regular and irregular Coulomb functions for angular momentum $L\hbar$ normalized to be asymptotic to the sine and cosine of the same phase at large r with $F>0$ at small r. When no ambiguity arises the subscript L is omitted in the list of notation and in the text.

r, $v\cdots$ relative distance and velocity.

$k\equiv\mu v/\hbar$.

$F'\equiv dF/d\rho$, $G'\equiv dG/d\rho$, $\rho\equiv kr$.

b, $r_{ctp}=r(ctp)\cdots$ values of r at nuclear boundary and at classical turning point.

$K=$ phase shift.

$\mathfrak{F}=e^{iK}[F\cos K+G\sin K]$ for $r>b$ and the continuation of this function into $r<b$ by means of the differential equation for r times the radial function.

$\Gamma=$ resonance half-width.

$U=$ depth of square well representing one-body potential.

$E_0=$ energy for which $K=90°$.

PROCEDURE AND RESULTS

For one-body resonances which are not too wide one may calculate the half-width[7] by means of the formula[8]

$$E/\Gamma\cong k\int_0^R|\mathfrak{F}|^2dr+[(G^2E/k)(\partial/\partial E)(kG'/G)]_{r=R}, \quad (1)$$

where $R\geq b$ and $E=E_0$. This formula was originally derived from the relation

$$\Gamma\cong-(F^2\delta)/(\partial/\partial E)(1-FG\delta), \quad (2)$$

which was employed by Freeman and McHale.[5] They determined the denominator of Eq. (2) graphically.

A difficulty in the numerical evaluation of the right side of Eq. (1) is the calculation of the derivative $(\partial/\partial E)(kG'/G)$. If kG'/G is obtained from the Coulomb tables,[9] the energy derivative can be obtained by differences with a corresponding loss of accuracy. This can often be improved by performing the calculation at a different radius, using the identity,[10]

$$\left[\frac{G^2E}{k}\frac{\partial}{\partial E}\left(\frac{kG'}{G}\right)\right]_{r=b}-\left[\frac{G^2E}{k}\frac{\partial}{\partial E}\left(\frac{kG'}{G}\right)\right]_{r=R}=k\int_b^R G^2dr. \quad (3)$$

In principle the quantity $[(G^2E/k)(\partial/\partial E)(kG'/G)]_{r=b}$ can be obtained by applying Eq. (3) in the limit $R\to0$.[11] Such a form is awkward for numerical work, especially for $L>0$.

The direct numerical evaluation of $(\partial/\partial E)(kG'/G)$ can be avoided by utilizing the equation[12]

$$(\partial/\partial E)(kG'/G)=(k/\rho)(\rho/FG)^2(\partial/\partial E)(FG/\rho)-(k^2/F^2E)\int_0^R F^2dr, \quad (4)$$

which follows from the Wronskian relation. Equation (4) is often more convenient because $(\partial/\partial E)(FG/\rho)$ is readily obtained by differences of the tabulated function[9] $\Phi\Theta$. Since in many cases F is relatively small the last term in Eq. (4) is often needed only with little relative accuracy and this form appears preferable, therefore, to others involving integrals over G^2, the danger of cancellation of terms of roughly equal absolute values being minimized.

As indicated previously the reactions and parameters L, b, U, and E_0 selected for the calculation of Γ are those used by Freeman and McHale[5] and do not correspond to real resonances. It is assumed in the present work that Γ is sufficiently well given by Eq. (1). Current ways of dealing with reduced widths suggest the desirability of comparison of Γ with an approximation to this quantity in which the last term on the right side of Eq. (1) is omitted and the upper limit of the integral is set at $r=b$. Accordingly, computations were made for the value of

$$\Gamma_b=E/k\int_0^b|\mathfrak{F}|^2dr, \quad E=E_0. \quad (5)$$

Since, on the other hand, the second term on the right side of Eq. (1) is likely to be relatively unimportant if the integration is extended to the classical turning point, an alternative approximation to Γ was also considered in the form

$$\Gamma_{ctp}=E/k\int_0^{r(ctp)}|\mathfrak{F}|^2dr, \quad E=E_0. \quad (6)$$

[7] The expression "half-width" is used here for half of the half-value breadth. The latter is the breadth measured between points of the resonance curve at half of the resonance maximum. The occurrence of the factor $\frac{1}{2}$ in resonance formulas is thus avoided.

[8] The second and third parts of reference 1.

[9] Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. 23, 147 (1951).

[10] This result is implicitly contained in BW, the third reference in footnote 1.

[11] R. G. Thomas (the last reference of footnote 3) obtained a formula for the $L=0$ case which corresponds to the calculation of the limit in Eq. (3), using the connection between the second term on the left of Eq. (3) and the j_0 function in the limit $R\to0$ with $K_0=90°$.

[12] Equations of this type are presented in BW. Compare formula appearing in their reference 17.

TABLE I. One-body half-widths and level shifts in Mev for protons incident on light nuclei.[a]

Isotope	b[b]	L	U(Mev)	E_0(Mev)	Γ	Γ_{ctp}	Γ_b	Π	$-(E'-E_0)$	$-(E''-E_0)$
Be^9	$\mathfrak{B}$	2	19.10	2.34	0.22	0.20	0.44	0.31	0.023	0.0061
B^{11}	$\mathfrak{B}$	2	19.10	0.76	4.9×10^{-3}	4.9×10^{-3}	8.7×10^{-3}	7.1×10^{-3}	3.7×10^{-5}	3.1×10^{-6}
C^{12}	$\mathfrak{A}$	2	28.63	2.54	0.11	0.11	0.22	0.17	0.0058	0.0011
N^{14}	$\mathfrak{A}$	2	28.63	0.63	3.3×10^{-4}	3.4×10^{-4}	5.3×10^{-4}	4.6×10^{-4}	2.0×10^{-7}	9.0×10^{-9}
F^{19}	$\mathfrak{B}$	3	19.10	2.57	0.034	0.033	0.054	0.041	0.00076	0.00014
Ne^{20}	$\mathfrak{B}$	0	0	2.10	1.0	0.69	0.93	1.1	0.52	0.33
Na^{23}	$\mathfrak{B}$	3	19.10	0.81	3.1×10^{-5}	3.0×10^{-5}	4.3×10^{-5}	3.9×10^{-5}	2.2×10^{-9}	1.2×10^{-10}
Mg^{24}	$\mathfrak{B}$	0	0	2.02	0.44	0.36	0.54	0.66	0.13	0.089
Al^{27}	$\mathfrak{A}$	3	28.63	1.83	0.81×10^{-3}	0.81×10^{-3}	1.2×10^{-3}	1.0×10^{-3}	6.6×10^{-7}	5.8×10^{-8}
	$\mathfrak{B}$	1	19.10	1.93	0.15	0.14	0.25	0.24	0.017	0.0084
	$\mathfrak{B}$	0	0	1.97	0.29	0.24	0.35	0.52	0.065	0.046
Si^{28}	$\mathfrak{A}$	3	28.63	1.40	1.0×10^{-4}	0.99×10^{-4}	1.4×10^{-4}	1.3×10^{-4}	1.4×10^{-8}	9.1×10^{-10}
	$\mathfrak{B}$	1	19.10	1.66	0.062	0.059	0.11	0.10	0.0032	0.0013
	$\mathfrak{B}$	0	0	2.03	0.26	0.21	0.33	0.45	0.052	0.035

[a] The reactions considered and the choice of parameters b, L, U, E_0 are the same as those in Table III of the reference in footnote 5.
[b] $\mathfrak{A}=1.6\times10^{-13}(A+1)^{\frac{1}{3}}$ cm; $\mathfrak{B}=2.1\times10^{-13}(A+1)^{\frac{1}{3}}$ cm. A is the mass number of the target nucleus.

In Table I comparisons are made of the half-width Γ with the approximations Γ_b and Γ_{ctp} as defined by Eqs. (5) and (6). The results indicate that Γ_{ctp} is a much better approximation than Γ_b in all cases except Ne^{20} for which the resonance is so broad that Eq. (1) should not be expected to hold. For the sharp resonances occurring for small E_0 and large L in the N^{14}, F^{19}, Na^{23}, Al^{27}, and Si^{28} cases, the agreement between Γ and Γ_{ctp} is particularly good, and the disparity between Γ and Γ_b is a minimum. In such cases the effect of the tail of the wave function on the half-width is adequately accounted for by evaluating the approximation Γ_R at a point slightly beyond the thin shell where G^2 is rapidly decreasing. For low flat barriers Γ_{ctp} remains a better approximation than Γ_b but the correction term to the integral is of increased importance. In the Be^9 case for example the difference between Γ and Γ_{ctp} is 8.5 percent, while Γ and Γ_b differ by almost a factor of 2. Similar results are obtained for C^{12}, and to a lesser extent for Mg^{24}, as well as the Al^{27} and Si^{28} reactions with $L\leq1$. The quantity[13]

$$\Pi=(2E/\rho^2)(\rho/G^2) \tag{7}$$

is also tabulated since it might be expected that $\Gamma\propto\rho/G^2$ for sharp resonances, the proportionality constant being a function of parameters like L or Z. This is in fact approximately the case. Thus, Γ/Π is roughly 0.8 for $L=3$, 0.7 for $L=2$, and 0.6 for $L=1$.

It is well known that $|\mathfrak{F}|^2$ is not a maximum when the phase shift is 90°. Breit[14] obtained the approximate formula,

$$E'-E_0\cong-\Gamma(F/G)+\Gamma^2(\partial/\partial E)\ln G, \tag{8}$$

for the energy shift of the minimum of $1/|\mathfrak{F}|^2$, and a corresponding relation

$$E''-E_0\cong-\Gamma(F/G) \tag{9}$$

for the minimum of $|G/\mathfrak{F}|^2$. For sharp resonances this is only a small fraction of Γ, the ratio $|(E'-E_0)|/\Gamma$ being about 10^{-4} when Γ is less than one kev and varying from 0.05 to about 0.1 when Γ is in the approximate range 50–200 kev. In Ne^{20}, where $\Gamma\sim1$ Mev, the ratio is about 0.5. Since the ratio $(E'-E_0)/(E''-E_0)$ contains the factor G/F, it is large (about 20) for narrow resonances and varies roughly from 1.5 to 5 for broad resonances.

It is thus seen that in a number of circumstances Eq. (1) and the approximation Γ_{ctp} are suitable for the calculation of resonance widths, the level shifts being relatively insignificant. The approximation Γ_b is seen to be far from safe, however, an error of 30 to 50 percent being reasonably representative. In some cases the error is much larger. It is expected nevertheless that for reactions of a typically many-body interaction type the errors will be less significant, the $\int|\Psi|^2d\tau$ owing its origin only in part to the radial integrals considered here.

We are greatly indebted to Professor G. Breit for initiating and guiding the course of this work and for several valuable suggestions in the preparation of this note. In particular we wish to thank him for pointing out the utility of transformations such as Eq. (4). We also wish to thank Miss Cecilie Smolen for assistance with the numerical work.

[13] This particular approximation is obtained from Eq. (1) by neglect of the correction term and the quantity $-\mathfrak{F}_{L-1}\mathfrak{F}_{L+1}/G_L^2$ which arises from the internal integration.
[14] Equations (5.8) and (5.9) of reference 2.