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Relation between Time Averages and Ensemble Averages in the Statistical Dynamics of Continuous Media

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The method of correlations, which has been widely used in the theory of turbulence, is applied to the case of a field that satisfies linear equations. It is found that most of the difficulties encountered in the theory of turbulence disappear when the equations are linear and that a number of general results can be obtained.

In particular, it is possible to use an analog of Gibbs' procedure of replacing a time average by an ensemble average. The mathematical foundations of this procedure are discussed and shown to be considerably different than for dynamical systems with a finite number of degrees of freedom, to which the method is usually applied. In particular, it is shown that it can be applied to dissipative fields excited by a randomly varying force.

Three examples are treated in some detail: statistically uniform but nonisotropic fields; statistically nonuniform, dissipative fields excited by a random force; and the scattering of monochromatic radiation by inhomogeneities of the refractive index.

The method of ensembles, as applied to these problems, appears to have a mathematical foundation that is rather simpler than in other cases that have been considered in the past, but several basic and unsolved problems are mentioned.

I. INTRODUCTION

THE methods of classical statistical mechanics were developed primarily to deal with the thermodynamic equilibrium of systems with a finite number of degrees of freedom. They have been extended to the case of continuous media, but only by the device of supposing a portion of the medium to be enclosed by ideal, perfectly reflecting boundaries. They have also been extended to include small departures from thermodynamic equilibrium, but in the case of continua, no methods similar to those of kinetic theory or general statistics have been developed. The existing methods are better adapted to calculating the constant properties of the medium than the statistical determination of its more complicated modes of motion.

Other methods have been introduced into the theory of turbulence, which do not require that the system be in thermodynamic equilibrium. These methods consist largely in the study of the correlation of two quantities when measured at different points in space and time.[1–4] Since the interest has centered on hydrodynamic problems, governed by nonlinear equations, progress has been difficult, and hampered by the fact that the equations for the double correlation $\langle uv\rangle$ also contain the triple correlation $\langle uvw\rangle$, etc.[1] Attention has been centered on this difficulty† and the search for valid hypotheses to enable it to be circumvented. As a result, only more or less conventional solutions have been obtained, and problems involving boundary conditions or driving forces have been ignored.

When the same methods are applied to electrodynamic[5] or acoustic[6] problems, the difficulty vanishes, and other problems claim the attention. These are of two kinds: the one has already been indicated, and involves boundary conditions and driving forces. In principle, these present no difficulty and it is merely a matter of arranging the calculations systematically. This is the primary topic of the following paper.

The second kind of problem is more fundamental. It has been customary in the theory of turbulence, to consider the averages to be temporal, rather than ensemble averages (see reference 1). This is an attractive approach, but can be expected to encounter all of the difficulties inherent in the ergodic theorem. However, these expected difficulties do not obtrude themselves at once, and the ensemble average appears in the calculation merely as a convenient device for the solution of the differential equations obeyed by the temporal averages (see Sec. III, below). It is only when one endeavors to generalize this method of solution that the expected difficulties arise (Sec. IX, below).

Let $p(\mathbf{x}t)$ be any field that obeys the equation

$$Lp=\partial p/\partial t, \tag{1}$$

L being a linear operator that commutes with t and $\partial/\partial t$, and such that the eigenvalue problem

$$Lf=-i\omega f \tag{2}$$

has no discrete eigenvalues. This implies that any boundary conditions are included in the operator L, but that they are not stringent, as are for example, those of a cavity with perfectly reflecting walls. If L contains no frictional terms, the boundary conditions

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[1] See references 2, 3, 4, or the review of J. E. Moyal, *Stochastic Processes and Statistical Physics*, J. Roy. Stat. Soc. **B11**, 150 (1949).

[2] O. Reynolds, Trans. Roy. Soc. (London) **186** (1895).

[3] G. I. Taylor, Proc. London Math. Soc. **20**, 196 (1922).

[4] T. v. Kármán and L. Howarth, Proc. Roy. Soc. (London) **A174**, 192 (1938).

† This difficulty is more obvious than serious, as will be shown in a subsequent paper.

[5] C. L. Pekeris, Proc. Intern. Congr. Math. **1**, 648 (1950).

[6] C. Eckart, J. Acoust. Soc. Am. **25**, 195 (1953).

must be such that energy can be dissipated by radiation to infinity, and may include the condition for "outgoing waves" (see Sec. VIII). In dealing with problems such as this, it has been customary to introduce additional, artificial boundary conditions, until the spectrum of Eq. (2) becomes a line spectrum and the system can be considered closed. At the end of all other calculations, these artificial boundary conditions are removed, which is a difficult limiting process.

There are also reasons for doubting the validity of these procedures. The statistical mechanics of closed systems has been developed with special reference to thermodynamic equilibrium, and its postulates are not certain to be applicable under other conditions. To emphasize the lack of equilibrium, a "driving force" will be added to Eq. (1), so that it becomes

$$Lp=\partial p/\partial t+a(\mathbf{x}t), \tag{1.1}$$

the function a being given as part of the data of the problem.

The usual method of solving such problems is to expand both p and a as Fourier integrals. This implies that the time averages of p^2 and a^2 are zero. This is not permissible if one wishes to deal with statistically steady states, for which these and other time averages are finite. The objective of the following paper will be to develop methods for calculating such averages without solving the more difficult problem of finding an analytic expression for the solution that is being averaged. This is essentially the basic idea of the Reynolds[2] and v. Kármán[4] approach to problems of turbulence.

II. CONVOLUTION AND CORRELATION

In the following, much use will be made of the two limiting processes,[7]

$$\{f(\mathbf{x},t)\}=\lim_{T\to\infty}\int_{-T}^{T}f(\mathbf{x},t)dt \tag{3}$$

and

$$\langle f(\mathbf{x},t)\rangle=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(\mathbf{x},t)dt. \tag{4}$$

With their aid, one may define the convolution of two fields $f(\mathbf{x},t)$ and $g(\mathbf{x},t)$ as

$$\{fg'\,|\,\tau\}=\{f(\mathbf{x},t)g(\mathbf{x}',t-\tau)\}, \tag{5}$$

and the correlation as

$$\langle fg'\,|\,\tau\rangle=\langle f(\mathbf{x},t)g(\mathbf{x}',t-\tau)\rangle. \tag{6}$$

It is important to note that both the convolution and the correlation depend on the two points $\mathbf{x}$ and $\mathbf{x}'$ in space, as well as on the delay time τ. If the fields f and g each have n components, the symbols here defined have n^2 components; they are fields in the seven-dimensional space $\mathbf{x}$, $\mathbf{x}'$, τ.

[7] In symbols such as $\{f(\mathbf{x},t)\}$, t is a bound variable, while $\mathbf{x}$ is free; $\{\cdots\}$ is a definite integral over t, and $\langle\cdots\rangle$ is very similar.

If p is a solution of Eq. (1.1) for a given a, and if both can be expressed as ordinary Fourier integrals in the time, then the convolutions $\{aa'\,|\,\tau\}$, $\{pa'\,|\,\tau\}$, $\{pp'\,|\,\tau\}$ will be finite and the corresponding correlations will be zero. Such solutions will be called Class I (I for integrable). If both p and a can be expressed as a Fourier (or similar but more general trigonmetric) series, the correlations will be similar series in the delay time τ, while the integrals defining the convolutions will diverge. Such solutions will be called Class AL (averageable, line spectrum). There will certainly be other solutions for which the correlations are finite, but which do not have pure line spectra; these will be called Class AC.

It is this Class AC that contains the solutions that are of interest in statistical mechanics. There are no simple general methods for representing them analytically, nor is it necessary to find such methods. It is typical of statistical mechanics that its objective is the calculation of averages (including correlations) rather than the determination of detailed dependence on time and space.

In the present case, this is relatively simple, for the correlations satisfy a set of linear equations that are readily deriveable from Eq. (1.1). It is also important to note that the convolutions satisfy exactly the same set of equations. Let

$$\begin{aligned}P&=\{pp'\,|\,\tau\}\quad\text{or}\quad\langle pp'\,|\,\tau\rangle,\\Q&=\{ap'\,|\,\tau\}\quad\text{or}\quad\langle ap'\,|\,\tau\rangle,\\Q'&=\{pa'\,|\,\tau\}\quad\text{or}\quad\langle pa'\,|\,\tau\rangle,\\A&=\{aa'\,|\,\tau\}\quad\text{or}\quad\langle aa'\,|\,\tau\rangle.\end{aligned} \tag{7}$$

Then, ignoring any difficulties that may arise in connection with such equations as

$$L\langle pp'\,|\,\tau\rangle=\langle (Lp)p'\,|\,\tau\rangle,$$

it is found that these fields are governed by the equations

$$\begin{aligned}LP&=+\partial P/\partial\tau+Q,\\L'P&=-\partial P/\partial\tau+Q',\\L'Q&=-\partial Q/\partial\tau+A,\\LQ'&=+\partial Q/\partial\tau+A,\end{aligned} \tag{8}$$

the operator L acting on $\mathbf{x}$, L' on $\mathbf{x}'$. Because the operator L includes boundary conditions, these equations are more elaborate than this formal notation indicates, and moreover they are not the only conditions to be satisfied. From Eqs. (5) and (6) it follows that

$$\{fg'\,|\,\tau\}=\{g'f\,|\,-\tau\},\quad \langle fg'\,|\,\tau\rangle=\langle g'f\,|\,-\tau\rangle, \tag{9}$$

so that the following symmetry conditions are required:

$$\begin{aligned}P(\mathbf{x},\mathbf{x}',\tau)&=P(\mathbf{x}',\mathbf{x},-\tau),\\Q(\mathbf{x},\mathbf{x}',\tau)&=Q'(\mathbf{x},\mathbf{x}',-\tau),\\A(\mathbf{x},\mathbf{x}',\tau)&=A(\mathbf{x}',\mathbf{x},-\tau).\end{aligned} \tag{8.1}$$

It is both natural and convenient to suppose that all these functions are real. Then it follows from Eqs. (3) and (4) that

$$\{[f+\theta g']^2\} \geqslant 0, \quad \langle [f+\theta g']^2 \rangle \geqslant 0, \tag{10}$$

for all real values of θ. From this, it is seen that the Schwarzian inequalities

$$\begin{aligned} &P(\mathbf{x}, \mathbf{x}, 0)+2\theta P(\mathbf{x}, \mathbf{x}', \tau)+\theta^2 P(\mathbf{x}', \mathbf{x}', 0) \geqslant 0, \\ &P(\mathbf{x}, \mathbf{x}, 0)+2\theta Q'(\mathbf{x}, \mathbf{x}', \tau)+\theta^2 A(\mathbf{x}', \mathbf{x}', 0) \geqslant 0, \\ &A(\mathbf{x}, \mathbf{x}, 0)+2\theta A(\mathbf{x}, \mathbf{x}', \tau)+\theta^2 A(\mathbf{x}', \mathbf{x}', 0) \geqslant 0, \end{aligned} \tag{8.2}$$

are also required of the P, Q, and A fields. If $\langle a \rangle \neq 0$, $\langle p \rangle \neq 0$, there are additional inequalities to be considered; however, it is no loss of generality to suppose that $\langle a \rangle = \langle p \rangle = 0$, because the Eq. (1.1) is linear, and this can always be brought about by subtracting a constant from the quantities concerned. This is a fundamental point in the following calculations, for the inequalities that involve $\langle p \rangle$ are not satisfied by $\{p\}$, in general. Thus the formal identity of the equations for convolutions and correlations depends on the elimination of the linear averages in this way.

While these many equations and inequalities are complicated, it will be seen that their solution is quite simple. Moreover, it will be possible to obtain solutions that cannot be either the convolution of a Class I solution of Eq. (1.1), nor yet the correlation of a Class AL solution. It does not follow that they are the correlation of a Class AC solution, but there is a strong presumption to this effect.

III. SOLUTION OF THE STATISTICAL EQUATIONS

Since the Eqs. (8) to (8.2) are all linear, their solutions will conform to a limited principle of superposition, the limitation arising from the fact that the Schwarzian inequalities are present. Let P_1, Q_1, Q_1', A_1, and P_2, Q_2, Q_2', A_2, be two solutions; then $P=c_1P_1+c_2P_2\cdots A=c_1A_1+c_2A_2$ will again be a solution if c_1 and c_2 are both real positive numbers. If both c_1 and c_2 are negative, $P\cdots A$ will certainly not be a solution; if one is positive, the other negative, $P\cdots A$ may or may not be a solution.

This may be generalized: if $P_q\cdots A_q$ is a family of solutions of Eqs. (8) to (8.2), q being a parameter or parameters, and $w(q)$ is any non-negative function of q, such that the integrals

$$P=\int w(q)P_q dq, \quad A=\int w(q)A_q dq, \tag{11}$$

converge well, these integrals will again be a solution. Now, let p_q, a_q be a family of Class I solutions of Eq. (1.1); their convolutions may then be substituted for $P_q\cdots A_q$ in Eq. (11), leading to

$$P=\int w(q)\{q_q p_q' | \tau\}dq, \quad A=\int w(q)\{a_q a_q' | \tau\}dq. \tag{12}$$

Since the techniques for constructing Class I solutions are well-known, the Eq. (12) provides a relatively simple and systematic method of solving the Eqs. (8) to (8.2).

In the next section, it will be shown that the functions defined by Eq. (12) cannot always be the convolution of any Class I solution, etc.; this is not, as has been remarked, proof that they are then the correlation of a Class AC solution p, but if this is the case (as one may be inclined to believe) then

$$\langle pp' | \tau \rangle = \int w(q)\{p_q p_q' | \tau\}dq, \text{ etc.} \tag{12.1}$$

The operation of forming the time average is thus replaced by the integration over q and t; this is analogous to Gibbs' ensemble average, and for that reason, a family of Class I solutions with an associated weight function will briefly be called an ensemble.

The weight function here appears as an arbitrary function introduced in solving the statistical equations. In general, it will be uniquely determined in specifying the function $A=\langle aa' | \tau \rangle$. This function is appropriately considered as part of the data of the statistical problem. In the case of homogeneous problems, $A=0$ and other data must be included in the formulation of the problem in order that it have a unique solution: i.e., in order that $w(q)$ be determinate.

IV. MATHEMATICAL PROPERTIES OF CORRELATIONS AND CONVOLUTIONS

If $a(\mathbf{x}, t)$ is a real function such that $\{aa' | \tau\}$ exists, it can be shown that a has a Fourier transform α,

$$a(\mathbf{x}, t)=\int \alpha(\mathbf{x}, \omega) \exp(-i\omega t)d\omega \tag{13}$$

and that

$$\alpha(\mathbf{x}, \omega)=\alpha^*(\mathbf{x}, -\omega), \tag{13.1}$$

while (by the Parseval theorem)

$$\{aa' | \tau\}=2\pi \int \alpha(\mathbf{x}, \omega)\alpha^*(\mathbf{x}', \omega) \exp(-i\omega\tau)d\omega. \tag{14}$$

The converse of this proposition is not as simple: let $A(\mathbf{x}, \mathbf{x}', \tau)$ be a real function of the seven variables indicated, having the symmetry

$$A(\mathbf{x}, \mathbf{x}', \tau)=A(\mathbf{x}', \mathbf{x}, -\tau) \tag{15}$$

and satisfying the Schwarzian inequality

$$A(\mathbf{x}, \mathbf{x}, 0)+2\theta A(\mathbf{x}, \mathbf{x}', \tau)+\theta^2 A(\mathbf{x}', \mathbf{x}', 0) \geqslant 0; \tag{16}$$

moreover, let

$$A(\mathbf{x}, \mathbf{x}', \tau)=\int B(\mathbf{x}, \mathbf{x}', \omega) \exp(-i\omega\tau)d\omega. \tag{17}$$

Then, because of the reality and symmetry of A,

$$B(\mathbf{x}, \mathbf{x}', \omega)=B^*(\mathbf{x}, \mathbf{x}', -\omega)=B^*(\mathbf{x}', \mathbf{x}, \omega); \tag{18}$$

because of the second of these equations, the integral equation,

$$B(\omega)f(\mathbf{x},\omega)=\int B(\mathbf{x},\mathbf{x}',\omega)f(\mathbf{x}',\omega)d\mathbf{x}',\qquad(19)$$

will have only real eigenvalues, $B(\omega)$.

If $A=\{aa'|\tau\}$, it will satisfy Eqs. (15) to (18) inclusive, and Eq. (19) will have exactly one eigenvalue

$$B(\omega)=2\pi\int|\alpha(\mathbf{x},\omega)|^2d\mathbf{x},$$

which is positive, and has the single eigenfunction $\alpha(\mathbf{x},\omega)$ associated to it. Conversely, if Eq. (19) has just one, positive and nondegenerate, eigenvalue, and if $f(\mathbf{x},\omega)$ is the normalized eigenfunction, then

$$B(\mathbf{x},\mathbf{x}',\omega)=B(\omega)f(\mathbf{x},\omega)f^*(\mathbf{x}',\omega)\qquad(20)$$

and the function

$$\alpha(\mathbf{x},\omega)=[B(\omega)/2\pi]^{\frac{1}{2}}f(\mathbf{x},\omega)\qquad(20.1)$$

will have the symmetry of Eq. (13.1). If, when this function is substituted into Eq. (13) the integral converges, the function A will be the convolution of the resulting function a.

Next, let the function A be defined in terms of an ensemble,

$$A=\int w(q)\{a_qa_q'|\tau\}dq;\qquad(21)$$

it will then satisfy Eqs. (15) to (18) inclusive, with

$$B(\mathbf{x},\mathbf{x}',\omega)=2\pi\int w(q)\alpha_q(\mathbf{x},\omega)\alpha_q(\mathbf{x}',\omega)dq,\qquad(22)$$

provided the integrals of Eqs. (21) and (22) converge well enough. A necessary condition that A be the convolution of a real function a can now be formulated: when the function B defined by Eq. (22) is substituted into Eq. (19), the latter must have exactly one, nondegenerate and positive, eigenvalue. This will certainly not always be the case.

Thus, it has been shown that those solutions of Eqs. (8) that are obtained by the method of ensembles are not always convolutions. That they are not always correlations of Class AL functions follows, because if they were, the function A of Eq. (21) could not be expanded as a Fourier integral, and this is usually the case.

V. MODIFIED FORMS OF THE STATISTICAL SOLUTION

The restriction to real ensembles is often inconvenient. It can be removed in various ways, one of which is the following. Let $p(\mathbf{x},t)$, $a(\mathbf{x},t)$ be a complex solution of Eq. (1.1) and construct the ensemble

$$p_q=pe^{iq}+p^*e^{-iq},\quad a_q=ae^{iq}+a^*e^{-iq},$$

with $w=1/4\pi$. Then Eq. (12) yields

$$P=\Re\{p^*p'|\tau\},\cdots,A=\Re\{a^*a'|\tau\}.$$

This can be used to generalize Eq. (12) into

$$P=\Re\int w(q)\{p_q^*p_q'|\tau\}dq,\text{ etc.}\qquad(23)$$

and thus enables one to make use of complex ensembles.

Another modification of Eq. (12) is obtained as follows: with the ensemble $p(q)$, $a(q)$, construct an ensemble $p(q_1)+p(q_2)$, $a(q_1)+a(q_2)$ with double the number of parameters, and the symmetric weight function $w_2(q_1,q_2)=w_2(q_2,q_1)$. Using the abbreviation,

$$w_1(q)=\int w_2(q,q_2)dq_2,$$

Eq. (12) yields

$$P=2P_1+2P_2,\text{ etc.},$$

where

$$P_1=\int w_1(q)\{p(q)p'(q)|\tau\}dq,\qquad(24.1)$$

$$P_2=\int\int w_2(q_1,q_2)\{p(q_1)p'(q_2)|\tau\}dq_1dq_2.\qquad(24.2)$$

One can also construct the ensemble $p(q_1)+\cdots+p(q_N)$, $a(q_1)+\cdots+a(q_N)$, with N times the number of parameters, and a symmetric weight function $w(q_1\cdots q_N)$. Then, if w_1 and w_2 are appropriately defined, Eq. (12) yields

$$P=NP_1+N(N-1)P_2.$$

Thus it follows that P_2 is itself a solution of the Eqs. (8), (8.1), (8.2), even though superposition with negative coefficients is not generally permissible. While solutions of the form of Eq. (24.2) are useful, the method of derivation shows that they need not be given a special place in the general theory.

VI. STATISTICALLY UNIFORM FREE FIELDS

As a first example of these methods, the homogeneous Eq. (1) will be considered, under the assumption that Eq. (2) has solutions of the form

$$f=\exp[ik(\omega)\mathbf{q}\cdot\mathbf{x}],\qquad(25)$$

as is frequently the case, ω taking on all real values, and k being a function of ω, while $\mathbf{q}$ is a unit vector that can be varied independently of ω. If the function $g(\mathbf{q},\omega)$ is appropriately chosen,

$$p(\mathbf{q})=\int g(\mathbf{q},\omega)\exp[i(k\mathbf{q}\cdot\mathbf{x}-\omega t)]d\omega\qquad(25.1)$$

will be an ensemble of Class I solutions. The element of solid angle $d\mathbf{q}$ associated with the unit vector $\mathbf{q}$ can be used as wdq. Then Eq. (23) yields

$$\langle pp'|\tau\rangle = 2\pi\int\int |g(\mathbf{q},\omega)|^2 \cos[k\mathbf{q}\cdot\boldsymbol{\xi}-\omega\tau]d\omega d\mathbf{q}, \quad (26)$$

where

$$\boldsymbol{\xi}=\mathbf{x}-\mathbf{x}'. \quad (26.1)$$

Since this correlation depends only on the difference $\mathbf{x}-\mathbf{x}'$, it represents a solution that is statistically uniform throughout space. The mean-square value of p at any point in space is

$$\langle p^2\rangle = \langle pp|0\rangle = 2\pi\int\int |g|^2 d\omega d\mathbf{q}$$

and is also independent of position. In most cases the correlation of $\partial p/\partial t$ with the gradient is associated with the flux of energy; by differentiating Eq. (26) one obtains

$$\langle(\nabla p)(\partial p'/\partial t)|\tau\rangle = 2\pi\int\int \omega k\mathbf{q}|g|^2\cos[k\mathbf{q}\cdot\boldsymbol{\xi}-\omega\tau]d\omega d\mathbf{q},$$

whence

$$\langle(\nabla p)(\partial p/\partial t)\rangle = 2\pi\int\int \omega k\mathbf{q}|g|^2 d\omega d\mathbf{q}.$$

Although uniform, the field is thus not statistically isotropic unless $|g|$ is independent of $\mathbf{q}$. More detailed examples of such calculations are to be found in reference 6.

The function $|g|^2$ may be called the spatio-temporal spectrum of p; in Planck's terminology, it is the spectral component of the specific energy density.[8] In the above discussion, it appears as an arbitrary function, and would be rendered unique only by a more complete formulation of the problem.

VII. DISSIPATIVE SYSTEMS

As an example of the treatment of the inhomogeneous Eq. (1.1), it is simplest to take the case of a frictionally dissipative system, for which the Eq. (2) has no real eigenvalues. Then, to the function

$$a=\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)], \quad (27.1)$$

the Eq. (1.1) associates

$$p=Z(\mathbf{k}\omega)\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)] \quad (27.2)$$

the function Z having no poles or other infinities for real values of $\mathbf{k}$ and ω. The ensemble

$$a(\mathbf{k})=\int g(\omega)\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]d\omega,$$
$$p(\mathbf{k})=\int Zg(\omega)\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]d\omega, \quad (27.3)$$

is constructed, $\mathbf{k}$ now taking the place of q. Statistically uniform solutions can be constructed as before, but it is instructive to use Eq. (24.2) with $w(\mathbf{k}_1,\mathbf{k}_2)$ replacing $w_2(q_1,q_2)$. Then

$$\langle aa'|\tau\rangle = 2\pi\mathcal{R}\int\int\int w(\mathbf{k}_1\mathbf{k}_2)|g(\omega)|^2 \cdot\exp[i(\mathbf{k}_1\cdot\mathbf{x}-\mathbf{k}_2\cdot\mathbf{x}'-\omega\tau)]d\mathbf{k}_1d\mathbf{k}_2d\omega, \quad (28.1)$$

$$\langle pp'|\tau\rangle = 2\pi\mathcal{R}\int\int\int w(\mathbf{k}_1\mathbf{k}_2)|g(\omega)|^2 Z(\mathbf{k}_1,\omega)Z^*(\mathbf{k}_2,\omega) \cdot\exp[i(\mathbf{k}_1\cdot\mathbf{x}-\mathbf{k}_2\cdot\mathbf{x}'-\omega\tau)]d\mathbf{k}_1d\mathbf{k}_2d\omega, \text{ etc.} \quad (28.2)$$

Since these solutions depend on both $\mathbf{x}$ and $\mathbf{x}'$, and not merely on $\mathbf{x}-\mathbf{x}'$, they represent statistically nonuniform solutions. Thus the mean square of p is

$$\langle p^2\rangle = \langle pp'|0\rangle = 2\pi\mathcal{R}\int\int\int w(\mathbf{k}_1,\mathbf{k}_2)|g|^2 Z(\mathbf{k}_1,\omega)Z^*(\mathbf{k}_2,\omega) \cdot\exp[i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}]d\mathbf{k}_1d\mathbf{k}_2d\omega, \quad (28.3)$$

and varies from point to point. Again, more detailed examples of such fields have been given in reference 6, though without using the systematic methods here developed.

VIII. THE SCATTERING OF RADIATION BY FLUCTUATIONS OF THE INDEX OF REFRACTION

If a plane monochromatic wave passes through a region in which the index of refraction varies irregularly with space and time, the radiation will be scattered and converted into nonmonochromatic radiation. This process will be treated in detail as a last example of the above methods. If the scattered radiation is p, and the index of refraction is $1+a$, while the incident wave is $p_0=\cos(\boldsymbol{\kappa}\cdot\mathbf{x}-\nu t)$, with $\nu=\kappa c$, the Eq. (1.1) takes the form[9]

$$c^{-2}\partial^2 p/\partial t^2-\nabla^2 p = 2\kappa^2 a\cos(\boldsymbol{\kappa}\cdot\mathbf{x}-\nu t). \quad (29)$$

It will be supposed that the inhomogeneity a can be represented by an ensemble of functions

$$a_{\mathbf{q}}=a_0(\mathbf{x}-\mathbf{q},t), \quad (30)$$

$a_0(\mathbf{x},t)$ having its maximum or center at the origin, and $\mathbf{q}$ therefore being the point in space at which $a_{\mathbf{q}}$ is centered. The vector $\mathbf{q}$ will be taken as the ensemble variable; the significance of the weight function $w(\mathbf{q})$ will appear below.

If

$$a_0(\mathbf{x},t)=\int\int A(\mathbf{k},\omega)\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]d\mathbf{k}d\omega, \quad (31)$$

[8] M. Planck, *Vorlesungen über die Theorie der Wärmestrahlung* (J. A. Barth, Leipzig, 1912), Chap. I.

[9] Squares of a have been neglected.

Eq. (12) yields

$$\langle aa'|\tau\rangle=(2\pi)^4\int\int\int A(\mathbf{k},\omega)A^*(\mathbf{k}',\omega)W(\mathbf{k}-\mathbf{k}')\cdot\exp[i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}'-\omega\tau)]d\mathbf{k}d\mathbf{k}'d\omega, \quad (32)$$

where

$$W(\boldsymbol{\alpha})=\int w(\mathbf{q})\exp(-i\boldsymbol{\alpha}\cdot\mathbf{q})d\mathbf{q}/(2\pi)^3. \quad (33)$$

If, moreover, $W(\boldsymbol{\alpha})$ has a narrow peak at $\boldsymbol{\alpha}=0$, it will be justifiable to replace $\mathbf{k}$ and $\mathbf{k}'$, in the A factors, both by

$$\mathbf{m}=\tfrac{1}{2}(\mathbf{k}+\mathbf{k}').$$

Using the notation,

$$\boldsymbol{\alpha}=\mathbf{k}-\mathbf{k}',\quad \boldsymbol{\xi}=\mathbf{x}-\mathbf{x}',\quad \mathbf{r}=\tfrac{1}{2}(\mathbf{x}+\mathbf{x}'),$$

Eq. (32) then becomes

$$\langle aa'|\tau\rangle=(2\pi)^4w(\mathbf{r})\int\int|A(\mathbf{m},\omega)|^2\times\exp[i(\mathbf{m}\cdot\boldsymbol{\xi}-\omega\tau)]d\mathbf{m}d\omega, \quad (34)$$

$$=w(\mathbf{r})\phi(\boldsymbol{\xi},\tau). \quad (34.1)$$

If the function a_0 is normalized so that $\phi(0,0)=1$, then $w(\mathbf{r})$ represents the mean-square fluctuation of the refractive index at the point $\mathbf{r}$. If the distance in which this changes appreciably is much greater than the dimensions of the region in which a_0 varies, the above approximation is justified. The function

$$\Phi(\mathbf{m},\omega)=(2\pi)^4|A(\mathbf{m},\omega)|^2 \quad (35)$$

in the spatio-temporal spectrum of the fluctuation, and is independent of position.[10]

To proceed further, it is necessary to determine $p_{\mathbf{q}}$, the outgoing-wave solution of Eq. (29) when a is replaced by $a_{\mathbf{q}}$. The right side of that equation may be written

$$2\boldsymbol{\kappa}^2a_{\mathbf{q}}\cos(\boldsymbol{\kappa}\cdot\mathbf{x}-\nu t)=\Re\int\int B(\mathbf{q},\mathbf{k},\omega)\times\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]d\mathbf{k}d\omega/c^2, \quad (36)$$

where the integration over ω (here and hereafter) is from $0\rightarrow\infty$ only, and

$$B(\mathbf{q},\mathbf{k},\omega)=2\nu^2\{A(\mathbf{k}-\boldsymbol{\kappa},\omega-\nu)\exp[-i(\mathbf{k}-\boldsymbol{\kappa})\cdot\mathbf{q}]+A(\mathbf{k}+\boldsymbol{\kappa},\omega+\nu)\exp[-i(\mathbf{k}+\boldsymbol{\kappa})\cdot\mathbf{q}]\}. \quad (37)$$

The required solution is then

$$p_{\mathbf{q}}=\Re\int\int\int i[B(\mathbf{q},\mathbf{k},\omega)/(kc+\omega)]\cdot\exp\{i[\mathbf{k}\cdot\mathbf{x}-\omega t-u(kc-\omega)]\}d\mathbf{k}d\omega du, \quad (38)$$

the integration over the auxiliary variable u also being always from $0\rightarrow\infty$. That this is a solution of Eq. (29) is most readily seen by direct substitution on the left side; the integration with respect to u can then be performed, and the result is seen to be the right side of Eq. (36). That this is the outgoing solution becomes obvious on making the change of variable $s=t-u$, since, then, the integration over s is from $-\infty\rightarrow t$, and is seen to vanish identically for $t=-\infty$, but not so for $t=+\infty$.

The convolution of $p_{\mathbf{q}}$ is then given by the ninefold integral

$$\{p_{\mathbf{q}},p_{\mathbf{q}}'|\tau\}=2\pi\Re\int\int\int\int\int[B(\mathbf{q},\mathbf{k},\omega)B^*(\mathbf{q},\mathbf{k}',\omega)/(kc+\omega)(k'c+\omega)]e^{i\Theta}d\mathbf{k}d\mathbf{k}'d\omega du du'/c^4,$$

where

$$\Theta=\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}-\omega\tau-u(kc+\omega)+u'(k'c+\omega).$$

This must be multiplied by $w(\mathbf{q})$ and integrated over the variables $\mathbf{q}$ in order to obtain the correlation. Since only the factors B depend on $\mathbf{q}$, one may first evaluate

$$C=2\pi\int B(\mathbf{q},\mathbf{k},\omega)B(\mathbf{q},\mathbf{k}',\omega)w(\mathbf{q})d\mathbf{q}.$$

Using Eq. (33), this is

$$\begin{aligned}C=4\nu^4(2\pi)^4\{&A(\mathbf{k}-\boldsymbol{\kappa},\omega-\nu)A^*(\mathbf{k}'-\boldsymbol{\kappa},\omega-\nu)W(\mathbf{k}-\mathbf{k}')\\&+A(\mathbf{k}+\boldsymbol{\kappa},\omega+\nu)A^*(\mathbf{k}'+\boldsymbol{\kappa},\omega+\nu)W(\mathbf{k}-\mathbf{k}')\\&+A(\mathbf{k}-\boldsymbol{\kappa},\omega-\nu)A^*(\mathbf{k}'+\boldsymbol{\kappa},\omega+\nu)W(\mathbf{k}-\mathbf{k}'-2\boldsymbol{\kappa})\\&+A(\mathbf{k}+\boldsymbol{\kappa},\omega+\nu)A^*(\mathbf{k}'-\boldsymbol{\kappa},\omega-\nu)W(\mathbf{k}-\mathbf{k}'+2\boldsymbol{\kappa})\}.\end{aligned}$$

Because $W(\boldsymbol{\alpha})$ has a sharp maximum at $\boldsymbol{\alpha}=0$, this is, to a sufficient approximation,

$$\begin{aligned}C=4\nu^4(2\pi)^4&\\ \times\{&[|A(\mathbf{m}-\boldsymbol{\kappa},\omega-\nu)|^2+|A(\mathbf{m}+\boldsymbol{\kappa},\omega+\nu)|^2]W(\boldsymbol{\alpha})\\&+A(\mathbf{m},\omega-\nu)A^*(\mathbf{m},\omega+\nu)W(\boldsymbol{\alpha}-2\boldsymbol{\kappa})\\&+A(\mathbf{m},\omega+\nu)A^*(\mathbf{m},\omega-\nu)W(\boldsymbol{\alpha}+2\boldsymbol{\kappa})\}.\end{aligned}$$

The integration over ω will be only from $\omega=0\rightarrow\infty$; if $A(\mathbf{k},\omega)$ has a maximum at $\omega=0$ of width less than ν, only the first term of this expression will contribute appreciably to the integral, so that the approximation:

$$C=4\nu^4\Phi(\mathbf{m}-\boldsymbol{\kappa},\omega-\nu)W(\boldsymbol{\alpha})$$

may be used. Then one obtains

$$\langle pp'|\tau\rangle=4\nu^4\Re\int\int\int\int\int[\Phi(\mathbf{m}-\boldsymbol{\kappa},\omega-\nu)/(mc+\omega)^2]W(\boldsymbol{\alpha})e^{i\Theta}d\mathbf{k}d\mathbf{k}'d\omega du du'. \quad (39)$$

Making the change of variables

$$v=u-u',\quad \sigma=\tfrac{1}{2}c(u+u')$$

and, in Θ, the approximations

$$k=m+\tfrac{1}{2}\boldsymbol{\alpha}\cdot\mathbf{m}_1,\quad k'=m-\tfrac{1}{2}\boldsymbol{\alpha}\cdot\mathbf{m}_1,\quad \mathbf{m}_1=\mathbf{m}/m$$

[10] This results from the assumption that the members of the ensemble differ only as to their center. The introduction of additional ensemble variables would generalize the Eq. (35).

Eq. (39) becomes

$$\langle pp'|\tau\rangle = (4\nu^4/c)\Re\int\int\int\int\int [\Phi(\mathbf{m}-\boldsymbol{\kappa}, \omega-\nu)/(mc+\omega)^2]W(\boldsymbol{\alpha}) \cdot \exp\{i[\boldsymbol{\alpha}\cdot\mathbf{r}+\mathbf{m}\cdot\boldsymbol{\xi}-\omega\tau+v(mc-\omega)+\sigma\boldsymbol{\alpha}\cdot\mathbf{m}_1]\}d\mathbf{m}d\boldsymbol{\alpha}d\omega dv d\sigma, \quad (40)$$

the integration over v being from $-2\sigma/c\to+2\sigma/c$ and that over σ from $0\to\infty$. The integrations over $\boldsymbol{\alpha}$ and v can then be performed at once, leading to

$$\langle pp'|\tau\rangle = (8\nu^4/c)\int\int\int[\Phi(\mathbf{m}-\mathbf{k}, \omega-\nu)/(mc+\omega)^2]w(\mathbf{r}-\sigma\mathbf{m}_1) \cdot\{\sin[2\sigma(m-\omega/c)]/(mc-\omega)\} \cos[\mathbf{m}\cdot\boldsymbol{\xi}-\omega\tau]d\mathbf{m}d\omega d\sigma. \quad (41)$$

If the point $\mathbf{r}$ is well outside the scattering region, the factor $w(\mathbf{r}-\sigma\mathbf{m}_1)$ will be different from zero only when σ is very large; this entails that the factor immediately following w will have a sharp maximum at $m=\omega/c$. The integration over m can therefore be performed with a high degree of precision by replacing m by ω/c everywhere except in this factor; then

$$\langle pp'|\tau\rangle = (2\pi\kappa^4)\int\int\Phi[(\omega/c)\mathbf{m}_1-\boldsymbol{\kappa}, \omega-\nu]S(\mathbf{r}, \mathbf{m}_1) \cdot\cos[(\omega/c)(\mathbf{m}_1\cdot\boldsymbol{\xi}-c\tau)]d\mathbf{m}_1 d\omega, \quad (42)$$

where $d\mathbf{m}_1$, is the element of solid angle associated to $\mathbf{m}_1$, and the "optical depth" is

$$S(\mathbf{r}, \mathbf{m}_1) = \int_{\sigma=0}^{\infty} w(\mathbf{r}-\sigma\mathbf{m}_1)d\sigma. \quad (43)$$

The scattered radiation therefore consists (to this approximation, at least) entirely of free waves traveling with the velocity c—in marked contrast to the fluctuations of the index of refraction, that involve waves of all velocities. The intensity of the scattered radiation is obtained by setting $\boldsymbol{\xi}=0$, $\tau=0$ in Eq. (42) and is a function of $\mathbf{r}$; the radiation field is therefore not homogeneous. The spatio-temporal spectrum also depends on $\mathbf{r}$, in contrast to that of the fluctuations causing the scattering. The scattered radiation is not monochromatic, even when the direction of propagation is disregarded, but does have a peak at $\omega=\nu$. Apart from these generalities, the Eq. (42) contains information concerning the line shape and other coherence properties of the scattered radiation, which it would not be be profitable to discuss here.

IX. SOME UNSOLVED PROBLEMS

It appears that the ensemble method, as described above, is capable of yielding useful results in many problems. However, as has been noted, it evades the analytic representation of the solution being averaged. There are many reasons why it would be desirable to have such representations. Without them, it is difficult to understand the precise relation between the ensemble and the physical reality whose average is being calculated. It is almost certain that there will be many Class AC solutions that yield the same correlations: do these differ only in the arrangement of one and the same set of events, or, are there more fundamental differences? Is it justified to consider the ensemble as an artificially ordered set of Class I solutions, whose random superposition in space-time constitutes the Class AC solution? How does the weight function enter into the analytic representation of the Class AC solutions? One may attempt to answer these questions on the basis of common sense, but reflection always raises doubt.

Another set of problems arises when one considers higher-order correlations, such as $\langle f(\mathbf{x}, t)g(\mathbf{x}', t-\tau)\times h(\mathbf{x}'', t-\sigma)\rangle$. Such correlations occur in the formulation of many physical problems.[4] and the complete set of all orders presumably determines the Class AC solution uniquely, in some sense of the word. If one attempts to generalize the method of ensembles so that these higher correlations may be calculated, many difficulties arise. These are connected with the generalization of the Schwarzian inequalities. There are not only a great number of such inequalities, but there are some that are satisfied only by correlations, and not by convolutions. For this reason, it has not been possible to extend the method of ensembles in this direction.

The reason for these difficulties is not too clear. In one sense, they exist in even the simplest case, for $\langle 1\rangle=1$ while $\{1\}=\infty$, and this prevents one from calculating $\langle p\rangle$ by the method of ensembles. Yet, as has been shown, this does not prevent the calculation of $\langle pp'|\tau\rangle$ by that method, providing that $\langle p\rangle=0$. Therefore, it is possible that some, at least, of the higher-order correlations can be calculated by this same method, providing sufficient ingenuity is exercised. It is also possible that the difficulty is more fundamental, and that the higher correlations can be calculated only after an analytic representation of the field has been found.