This theorem can be exploited to give a uniform expression for the results of all problems involving transition probabilities. Thus, in the integration over the extended region in (44), it is supposed that the current is constant in the exterior region. If we were to replace these constant currents by currents decreasing adiabatically to zero at infinity, the null contribution from the external region would not be affected. But we would have succeeded in substituting for the original problem one in which the current vanishes on the boundaries of the extended region. Accordingly, we can integrate by parts in (44) and regain the form (33) appropriate to null currents on the boundaries. The most general problem requiring the evaluation of transition probabilities between stationary states, involves initial and final currents that are time-independent with respect to different reference systems. When modified with the aid of the adiabatic device, this situation also falls into the class of problems covered by (34).

The adiabatic device is also applicable to eigenvalue problems. Thus, we can use the transformation function (34), appropriate to zero current on the boundary surfaces, to construct the energy eigenvalues for the situation of a time-independent current. We suppose that the current, which is zero on the surface $\sigma_{-\infty}$, grows adiabatically and maintains a constant value between surfaces σ_2 and σ_1, and reduces adiabatically to zero on σ_∞. The designations $\sigma_{\pm\infty}$ refer to the fact that the adiabatic theorem involves the limit of infinite temporal separation between σ_∞ and σ_1, and between σ_2 and $\sigma_{-\infty}$. Then

$$(n\sigma_\infty|n'\sigma_{-\infty})=\delta(n,n')\exp[i\mathcal{W}_0+iP(n)(x_\infty-x_{-\infty})],$$

where [reversing the integration by parts in the first term of (46)],

$$\mathcal{W}_0=-\int_{-\infty}^{\infty}dx_0E(0,x_0),$$

and

$$\exp(i\mathcal{W}_0)=\exp\left(-i\int_{t_1}^{\infty}dx_0E(0,x_0)\right)$$
$$\times\exp(-iE(0)(t_1-t_2))\exp\left(-i\int_{-\infty}^{t_2}dx_0E(0,x_0)\right).$$

On recalling the composition property of transformation functions, we recognize immediately that

$$(n\sigma_1|n'\sigma_2)=\delta(n,n')\times\exp[-iE(0)(t_1-t_2)+iP(n)(x_1-x_2)],$$

which shows that, in the presence of a time-independent current, the energy eigenvalues of the radiation field are displaced by $E(0)$.

The methods discussed in this paper and illustrated for the electromagnetic field are equally applicable to other Bose-Einstein systems, such as the symmetrical pseudoscalar meson field.

PHYSICAL REVIEW VOLUME 91, NUMBER 3 AUGUST 1, 1953

Convergence of the Adiabatic Nuclear Potential

ABRAHAM KLEIN*
Harvard University, Cambridge, Massachusetts
(Received March 30, 1953)

The leading term of the potential of o.der $4n$, n an arbitrary integer, of the pseudoscalar theory with pseudoscalar coupling is computed by a method which takes advantage of the special pair character of this term. It is shown that the coefficients increase with n as $n!(n-1)!$; thus, despite the decreasing ranges, the unrenormalized potentials grow enormously with n, even in the nonrelativistic domain. The breakdown of the expansion in μ/M for a given order is also indicated. A weakness of the discussion is that it does not include radiative corrections.

I. INTRODUCTION

IN a previous paper,[1] it was shown how the relativistic two-body equation could be made the exclusive basis for the discussion of nuclear forces in the nonrelativistic domain. The methods developed were extensively illustrated by the computation of various terms of the two-, three-, and four-body potentials in the symmetric pseudoscalar theory with pseudoscalar (direct) coupling. No serious attempt was made, however, to present theoretical evidence that would render plausible the implied model[2] of, for example, the neutron-proton system.

To justify the model, there are at least three points that require serious investigation: the hard core, higher order potentials, and radiative corrections. One should like to justify the assumption of a hard core, a *sine qua non*, because of the singular character of the potentials in the asymptotic region. On this matter, we shall be silent. Even assuming a reasonable cut-off radius,

* Jr. Fellow, Society of Fellows.
[1] A. Klein, Phys. Rev. **90**, 1101 (1953). Hereafter referred to as A.
[2] M. M. Lévy, Phys. Rev. **88**, 725 (1952).

$\mu r \lesssim 0.5$, one must show that the potentials of higher order than those taken into account become progressively smaller in the nonrelativistic domain. In view of recent discussions of the S matrix,[3] one would tend to be skeptical of this possibility.

In an effort to obtain a definite result, we have focused attention on the potentials of order $4n$ in the symmetric $ps-ps$ theory. In Sec. II we compare the fourth- and eighth-order potentials, each computed to relative order μ/M compared to the leading term of the given order. All but the μ/M term of eighth order were calculated in **A**, and the latter is obtained here by the same three-dimensional methods. Because of its coefficient and functional dependence, this term proves to be larger in magnitude than the "leading" term for significant separations of the two particles, heralding a breakdown of the expansion in powers of μ/M for a given power of $g^2/4\pi$. This tendency is evident even in fourth order, where the combination of two-pair and one-pair potentials effectively cancels, so that there is no possibility of fitting the singlet $n-p$ scattering data[4] with their sum.

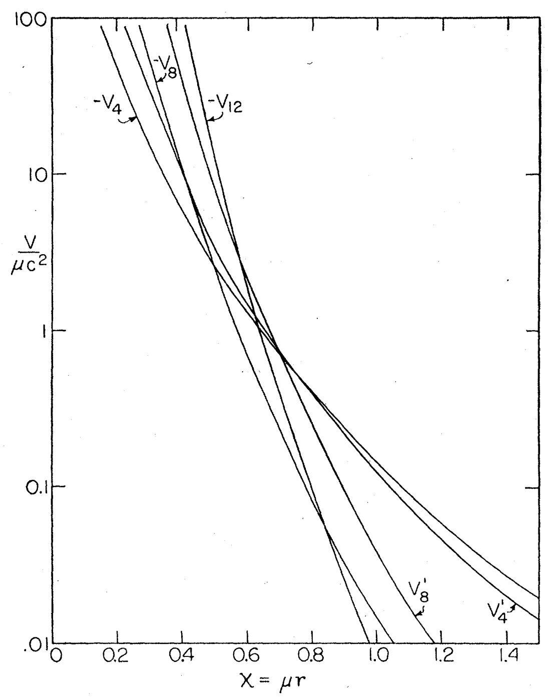

FIG. 1. Plot of the leading terms of the fourth-, eighth-, and twelfth-order potentials for $g^2/4\pi=10$. The unprimed potentials are generated by matrix elements with the maximum number of pairs, the primed potentials by matrix elements with one pair fewer.

In Sec. III a method is presented which, although not a rigorous one, renders plausible nevertheless a definite result for the $2n$-pair potential of order $4n$, n arbitrary. Aside from the satisfaction of achieving an analytic form for this term, the main result, based upon the combinatorial aspects of the problem, is that the coefficients increase with n as $n!(n-1)!$. This dependence soon overwhelms the reduction brought about by the higher powers of the effective coupling constant,[5] $(g^2/4\pi)\times(\mu/2M)$ and by the decreasing range of the functions

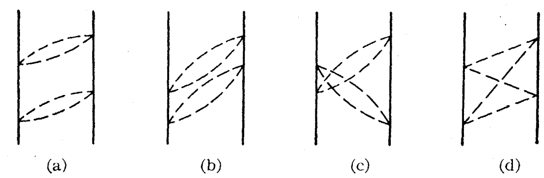

FIG. 2. Diagrams representative of the four types of four-pair matrix elements in eighth order.

involved, even for $\mu r \gtrsim 1$, and is aided in this at distances $\mu r<1$ by the increasingly stronger singularities at the origin. The mere enumeration of possibilities also makes it apparent that the breakdown of the μ/M expansion becomes progressively worse for higher order terms, though the methods employed do not suffice to determine the exact coefficients.

A weakness in the above discussion is that it refers only to "bare" potentials, i.e., the possible damping due to radiative corrections has not been taken into account. It has been pointed out by Wentzel[6] and more recently by Brueckner,[7] however, that nucleon self-energy effects may severely depress virtual nucleon pair formation. Although it is hardly possible to alter the conclusions of the previous paragraph on the basis of their work alone, one can, in an optimistic moment, envisage that at the very least the strong damping of the pair terms may serve to reduce the effective coupling constant to the point where the theory approaches the probable asymptotic status of electrodynamics.[8] For the no-pair terms have a reasonably small effective expansion parameter,[2] $(g^2/4\pi)(\mu/2M)^2<0.1$. It is hardly more likely, however, that higher order potentials of the no-pair variety form a convergent sequence than do the terms discussed in this paper. It would appear, therefore, that the justification by meson-theoretical methods of an adiabatic potential based upon low order terms only has yet to be provided.

The appendix contains a derivation, by the methods of Sec. III, of the n-pair term of the n-body force.

II. EIGHTH-ORDER POTENTIALS

We begin this section with a statement of results and shall discuss some of the relevant aspects of the

[3] C. A. Hurst, Phys. Rev. **85**, 920 (1952); S. Hori, Progr. Theoret. Phys. **8**, 569 (1952); W. Thirring, Helv. Phys. Acta **26**, 33 (1953).

[4] Contrary to what was implied in the concluding discussion of *A*. The remarks there are applicable if one includes the terms of relative order $(\mu/M)^2$ compared to the two-pair terms. As indicated by our later discussion, these may be the decisive terms anyway because of radiative corrections to the pair terms.

[5] Assuming $g^2/4\pi\sim 10$. If $g^2/4\pi\geq 15$, $(g^2/4\pi)(\mu/2M)\geq 1$ and the situation is so much the worse.

[6] G. Wentzel, Phys. Rev. **86**, 802 (1952).

[7] K. Brueckner, Phys. Rev. **90**, 476 (1953).

[8] F. J. Dyson, Phys. Rev. **85**, 631 (1952).

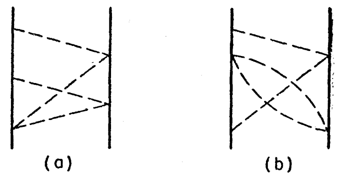

FIG. 3. Diagrams representative of the three-pair matrix elements in eighth order. Diagram (a) is one of six possible configurations, determined by the position of the pair vertex relative to the single vertices and by which particle has the single vertices.

derivation thereafter. It was shown in **A** that to relative order μ/M the fourth-order potential consists of two terms, the two-pair term ($x=\mu r$),

$$V_4(x)=-3\mu(g^2/4\pi)^2(\mu/2M)^2(2/\pi)K_1(2x)/x^2, \quad (1)$$

and the one-pair term[9]

$$V_4'(x)=6\mu(g^2/4\pi)^2(\mu/2M)^3(1+1/x)^2e^{-2x}/x^2. \quad (2)$$

In eighth order the corresponding potentials are the leading four-pair term, also derived in **A**,

$$V_8(x)=-6\mu(g^2/4\pi)^4(\mu/2M)^4(2/\pi)K_1(4x)/x^4, \quad (3)$$

and the leading three-pair term

$$V_8'(x)=24\mu(g^2/4\pi)^4(\mu/2M)^5(1+1/x)^2e^{-4}x/x^4. \quad (4)$$

Equation (4) can be derived by the method employed in **A**, and it will be indicated below that it is the only term of its order which need be considered in the nonrelativistic domain.

Equations (1)–(4) are plotted in Fig. 1, together with $V_{12}(x)$, the leading six-pair term of twelfth-order, which is derived in the next section. The graph is confined to the region $x<1.5$. Several facts emerge clearly. It is seen first that the sum of $V_4(x)$ and $V_4'(x)$ is much smaller in magnitude than $V_4(x)$ and becomes repulsive for $x<0.75$. There is then no hope of fitting the singlet $n-p$ scattering data with such a potential. Further, it is seen that $V_8'(x)$ is almost uniformly larger than $|V_8(x)|$ and that their sum is of greater magnitude than the corresponding fourth-order result. We are confronted with a breakdown of the significance of a factor of μ/M in front of a potential, which is due, as will be seen below, to the greater number of ways of realizing a matrix element with the fewer number of pairs; there is also no evidence of the convergence of the power series in $g^2/4\pi$. In fact, the opposite tendency is evident upon comparison of V_8 and V_{12}, with the latter predominating for $x<0.9$. The nonconvergence indicated here will be established in the next section.

We turn now to a consideration of the features associated with the derivations of Eqs. (3) and (4) which will form the basis of certain generalizations of relevance in our later work. To derive Eq. (3), for example, we recorded all possible four-pair matrix elements involving at most one pair at a time. These could be enumerated without ambiguity by reference to the 4! distinct Feynman diagrams of eighth order for the exchange of four mesons. One then finds only four distinct sets of matrix elements in the nonrelativistic limit, prototypes of which are given in Fig. 2. In order to clarify the essential features, we have preferred to use the pictorial representation associated with the equivalent pair theory.

According to the theory of **A**, matrix elements of the type of Fig. 2(a) were not considered since they were reducible in the three-dimensional sense defined there. It was next found that the sum of all matrix elements of types (b) and (c) were canceled (in the nonrelativistic limit) by iteration of velocity-dependent corrections[10] to the leading two-pair fourth-order potential. Equation (3) then resulted from the summation of all matrix elements of type (d), those for which, in the terminology of pair theory,[11] the meson lines form a closed perimeter.

The cancellation of the matrix elements represented by Figs. 2(b) and (c) is a phenomenon which has been found to occur in every analogous situation encountered in our calculations.[10] Although it hardly appears possible within the present context to prove[12] that it will always occur, that it should happen is eminently reasonable upon physical grounds; for both (b) and (c) can be regarded as retardation corrections to (a), so that one might expect them to disappear in the adiabatic limit. In any case, in our work in the next section we shall consider only potentials of order $4n$ in which all $2n$ lines are connected and thus form a perimeter.

In the derivation of Eq. (4) one encounters a similar situation. Thus, it is matrix elements of the kind represented by Fig. 3(a) that add up to Eq. (4), whereas those like Fig. 3(b) can be canceled if one defines a suitable unperturbed problem and then considers velocity-dependent corrections thereto. It is clear that as one pushes the computation of potentials to higher order in $g^2/4\pi$ and μ/M, one must suitably extend the set of potentials to be included in the unperturbed non-

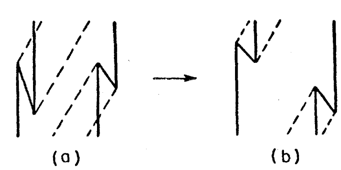

FIG. 4. Diagrams illustrating part of four-pair matrix element with two pairs allowed at a given time [diagram (a)] which is not canceled by leading four-pair terms because of its relation to a [diagram (b)] reducible four-pair term.

[9] The existence of this term has been questioned by some authors. Eq. (2) has been independently verified by Dr. S. Drell using a method based upon the canonical transformation of Dyson, F. J. Dyson, Phys. Rev. **73**, 929 (1948). The universal agreement of the existence of the term in the neutral theory would seem to guarantee that there is a corresponding term in the symmetrical theory.

[10] See Sec. IV C of reference 1.

[11] G. Wentzel, Helv. Phys. Acta **15**, 111 (1942).

[12] It is easy enough to verify, however, that the signs are always appropriate for the concellation. This problem is under further investigation. See Y. Nambu, Prog. Theoret. Phys. **5**, 614 (1950).

relativistic problem. Thus, one considers here, for the first time, velocity-dependent corrections to the one-pair fourth-order potential.

A few final remarks are in order. The assertion that Eq. (4) is the only μ/M "correction" to Eq. (3) must be substantiated by a proof that there are no four-pair static potentials of relative order μ/M. One method of procedure would be to fall back upon the pair analogy. With the pair method of computation one merely never sees such terms, at least in fourth order. By our techniques, the proof consists in demonstrating that the first-order expansion of the pair denominators of the leading pair terms cancels the nonrelativistic limit of matrix elements which can have two pairs at a time, as was shown to be the case in fourth order in **A**. This occurs in eighth order everywhere except where the leading pair matrix element is a reducible one, but the μ/M term with which it should combine is not. Thus, one can think of a matrix element with overlapping pairs, such that when one removes the overlap, there remains an intermediate state with only two nucleons and no mesons as illustrated in Fig. 4. The matrix elements related to Fig. 4 (a) give rise to a large μ/M potential given by

$$V_8^{(b)}(x)=-36\mu(g^2/4\pi)^4(\mu/2M)^5(2/\pi)^2 \times[K_1(2x)]^2/x^4. \quad (5)$$

But even this potential can be omitted in the adiabatic limit if one considers velocity-dependent corrections to the leading two-pair potential of fourth order which arise from the expansion of pair denominators[13] and uses the iteration method of **A**. If one were to extend these methods to four-pair terms of still higher order in μ/M, one should be led to the conclusion that the largest pair contribution to the potential of order $4n$ accounts for the complete *nonrelaitvistic* potential of that order arising from matrix elements with the maximum number of pairs. The correctness of this result will be assumed in the developments that follow.[14]

III. POTENTIAL OF ORDER 4n

A. Preliminaries

We attack in this section the problem of obtaining the potential of order $4n$ which is the sum of matrix elements with $2n$ pairs. This problem has two aspects, that of obtaining an analytic form and that of obtaining the general coefficient. Since we are interested only in the adiabatic limit, it is a more concise and useful procedure to define the potential directly from the appropriate form of the relativistic two-body equation rather than through the intermediary of the three-dimensional formalism. We shall then rely on our previous experience, as discussed in the last section, to help us select the terms of interest.

Our starting point is the equation[15]

$$[\tfrac{1}{2}W+p_0-E_p][\tfrac{1}{2}W-p_0-E_p]\psi_{++}(x) = \int d^4x'd^4X' \exp[iW(X_0-X_0')] \times I(x,x';X-X')\psi_{++}(x'), \quad (6)$$

from which we have eliminated negative-energy components of the wave function by the method explained in **A**, and therefore the interaction kernel $I(x,x';X-X')$ contains reducible elements in both the four-dimensional and the three-dimensional sense. Multiplying through by the sum of the inverses of the operators that stand on the left-hand side of Eq. (6), we obtain the equation

$$(W-2E_p)\psi_{++}(x)=(2\pi)^{-4}\int d^4p'd^4x''d^4x'd^4X'e^{ip'(x-x'')}\{[\tfrac{1}{2}W+p_0'-E_{p'}]^{-1}+[\tfrac{1}{2}W-p_0'-E_{p'}]^{-1}\} \times\exp[iW(X_0-X_0')]I(x'',x';X-X')\psi_{++}(x')=-i(2\pi)^{-3}\int d\mathbf{p}'d^4x''d^4x'd^4X' \times\exp[i\mathbf{p}'\cdot(\mathbf{r}-\mathbf{r}'')-i(E_{p'}-\tfrac{1}{2}W)|x_0-x_0''|]\exp[iW(X_0-X_0')]I(x'',x';X-X')\psi_{++}(x'). \quad (7)$$

According to Appendix A of **A**, the wave function on the right-hand side of Eq. (7) can be represented as

$$\psi_{++}(x)=(2\pi)^{-2}\int d\mathbf{p} \times\exp[i\mathbf{p}\cdot\mathbf{r}-i(E_p-\tfrac{1}{2}W)|x_0|]\phi_{++}(\mathbf{p}). \quad (8)$$

Taking the nonrelativistic limit consists in the observation that if the wave function $\phi_{++}(p)$ is negligible for $p\gtrsim M$, we may set $E_p\cong M$ and further, ignoring the binding energy, place $E_p-\tfrac{1}{2}W\cong 0$. The wave function $\psi_{++}(x')$ in Eq. (7) is then independent of x_0, i.e., we have neglected retardation effects. If analogous ap-

[13] In all previous cases it was only the energy denominator associated with intermediate states without pairs that had to be considered. Here also is the first instance of the relevancy of the difference between the pair denominators of **A** and those of Lévy, reference 2. The ambiguity, based upon the fact that our representation is, in a limited sense, a mixed (free and bound) representation, will be even more pronounced in higher order. The remarks of the text are based upon the use of the Lévy denominators.

[14] This assumption may be unjustified, since if one examines the Dyson-transformed Hamiltonian or the similar Foldy Hamiltonian, Berger, Foldy, and Osborn, Phys. Rev. **87**, 1061 (1952), one finds that the theory contains multiple-pair-vertices with smaller effective coupling constants than the single-pair vertices which generate the leading pair terms. The relation of the former to the method of the next section is not completely clear. It is not unlikely, however, that they account for terms like Eq. (5).

[15] The notation follows reference 1. In Eq. (6), $\mathbf{p}$, p_0 are to be understood as differential operators.

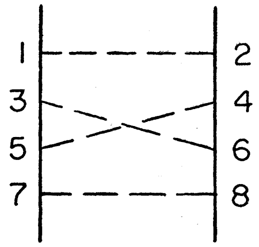

FIG. 5. Eighth-order Feynman diagram which contributes to potential illustrated by Fig. 2(d).

proximations are made uniformly in the right-hand side of Eq. (7), it is easily reduced to the form

$$(W-2E_p)\psi_{++}(\mathbf{r})=-i\int dt d^4x' d^4X' \times\exp[i2M(X_0-X_0')]I(x,x';X-X')\psi_{++}(\mathbf{r}'), \quad (9)$$

in which we have recognized explicitly that the left-hand side is independent of x_0 and therefore renamed x_0'' to t. We are now in a position to recognize the potential since the right-hand side of Eq. (9) has the form

$$\int V(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}',$$

where

$$V(\mathbf{r},\mathbf{r}')=-i\int dt dt' d^4X' \times\exp[i2M(X_0-X_0')]I(x,x';X-X'). \quad (10)$$

When we introduce explicit forms for $I(x,x';X-X')$ and make the requisite nonrelativistic approximations in it, $V(\mathbf{r},\mathbf{r}')$ will reduce to the required $V(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')$.

B. Fourth- and Eighth-Order Potentials

We illustrate the special use we shall make of Eq. (10) by rederiving old results. Consider first the two-pair fourth-order potential. It will prove sufficient to compute for a single Feynman diagram, Fig. 1(b) of **A**, for example, and then to multiply by the sum of the isotopic operators of all contributing diagrams which we shall designate in the general case by T_l, where l is the order of the potential. Thus, for the two-pair part of the fourth-order potential V_{--} we have

$$V_{--}(\mathbf{r},\mathbf{r}')=-i(-ig^2)^2T_4\int dt dt' d^4X' \exp[i2M(X_0-X_0')]\Delta(X-X'-\tfrac{1}{2}(x+x'))\Delta(X-X'+\tfrac{1}{2}(x+x')) \times\langle\gamma_0\gamma_5G_-(X-X'+\tfrac{1}{2}(x-x'))\gamma_5\rangle^{(1)}\langle\gamma_0\gamma_5G_-(X-X'-\tfrac{1}{2}(x-x'))\gamma_5\rangle^{(2)}, \quad (11)$$

where $\langle\,\rangle$ denotes spin matrix element and G_- the value of the Green's function for negative values of its time argument; the time integrations are also limited by this ordering. We recall that

$$G(x)=\frac{i}{(2\pi)^3}\int d\mathbf{p}\exp[i\mathbf{p}\cdot\mathbf{r}-iE_p|t|]\frac{[H(\mathbf{p})+E_p\,\mathrm{sgn}t]}{2E_p}\gamma_0. \quad (12)$$

In the adiabatic limit, we set $E_p\cong M$. For $t<0$, therefore, we obtain

$$\langle\gamma_0\gamma_5G_-(x)\gamma_5\rangle\cong-i\delta(\mathbf{r})e^{iMt}, \quad (13)$$

in virtue of the properties[16] of γ_5. Inserting Eq. (13) in Eq. (11) and imposing the spatial connections implied by the δ functions, we obtain

$$V_{--}(\mathbf{r},\mathbf{r}')\to V_4(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'), \quad (14)$$

$$V_4(\mathbf{r})=-ig^4T_4\int_{-\infty}^{\infty}dt\int dt_3(t_3>t_1)\int dt_4(t_4>t_2)\exp[i2M(t_1+t_2-t_3-t_4)]\Delta(\mathbf{r},t_1-t_4)\Delta(\mathbf{r},t_2-t_3). \quad (15)$$

In Eq. (15) we have preferred to return to individual time coordinates ($t=t_1-t_2$) as we shall do in all subsequent cases. Now the limitations on the allowed values of t_3 and t_4 merely restrict us to the entire two-pair potential, i.e., they specify the number of pairs associated with a single nucleon line. If we are to derive the leading two-pair potential we should establish additional relationships between the time coordinates of the two particles in order to ensure no overlap in time between the pairs. We would then be applying the method of Appendix A of **A** which subdivides Eq. (15) into the sum of all its distinct three-dimensional matrix elements. This is precisely what we must avoid if we are eventually to obtain by reasonable labor a result for arbitrary order. To carry out our program, we must invoke the special property of the pair interactions described in Sec. II, that the sum of *all* pair terms yields in the adiabatic limit only the *leading* pair potential. In short, no error should result from Eq. (15).

Introducing the four-dimensional momentum representations of the meson propagation functions and reversing the orders of integration, we require the time

[16] See Eq. (28) of reference 1.

integrals

$$\int_{-\infty}^{\infty} dt \int_{t_1}^{\infty} dt_3 \int_{t_2}^{\infty} dt_4 \exp[it_1(2M-k_{01})+it_2(2M-k_{02})-it_3(2M-k_{02})-it_4(2M-k_{01})]$$
$$=2\pi(-i)^2(2M-k_{01})^{-1}(2M-k_{02})^{-1}\delta(k_{01}-k_{02}). \quad (16)$$

We may replace $2M-k_0$ by $2M$ since the poles thus neglected do not adequately account for the behavior at small distances anyway because of previous nonrelativistic approximations. This replacement will be made in the future without further comment. As a consequence of Eq. (16) we are left with

$$V_4(r)=i\frac{g^4}{(2M)^2}\frac{1}{(2\pi)^7}T_4\int d\mathbf{k}_1 d\mathbf{k}_2 e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}\int\frac{dk_0}{(k_0^2-\omega_1^2)(k_0^2-\omega_2^2)}$$
$$=-3\mu(g^2/4\pi)^2(\mu/2M)^2(2/\pi)K_1(2x)/x^2. \quad (17)$$

The integral involved is one of a class evaluated in the appendix. We have also substituted the value $T_4=6$ (two Feynman diagrams).

We demonstrate next that the same technique easily reproduces the potential $V_8(x)$ of Eq. (3). We may begin appositely by computing the quantity T_8. This is best done by reference to Fig. 2(d). The number of Feynman diagrams that contribute (equally) to the potential represented by this figure is 2^4, for each pair-vertex corresponds to two γ_5 vertices and the sequence of these may be interchanged independently of all other vertices. For the potential of order $4n$, the result is 2^{2n}. The reader will then easily convince himself that the quantity T_{4n} has the value 3×2^{2n}, i.e., just three times the value for the neutral theory. Thus $T_8=48$.

We shall carry out the calculation for the interaction of Fig. 5. We consider, therefore,

$$V_8(\mathbf{r},\mathbf{r}')=-i(-ig^2)^4T_8\int dtd^4x_3\cdots d^4x_6dt_7dt_8d\mathbf{R}'$$
$$\times\exp[iM(t_1+t_2-t_7-t_8)]\Delta(x_1-x_2)\Delta(x_3-x_6)\Delta(x_5-x_4)\Delta(x_7-x_8)\langle\ \rangle^{(1)}\langle\ \rangle^{(2)}, \quad (18)$$

with

$$\langle\ \rangle^{(1)}=\langle\gamma_0\gamma_5G_-(x_1-x_3)\gamma_5G_+(x_3-x_5)\gamma_5G_-(x_5-x_7)\gamma_5\rangle^{(1)}$$
$$\cong(-i)^2i\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_3-\mathbf{r}_5)\delta(\mathbf{r}_5-\mathbf{r}_7)\exp[iM(t_1-2t_3+2t_5-t_7)], \quad (19)$$

with a similar expression holding for $\langle\ \rangle^{(2)}$. Carrying out the spatial integrations, we find for the equation analogous to (15),

$$V_8(\mathbf{r})=ig^8T_8\int dtdt_3\cdots dt_8(t_3>t_1,\ t_1<t_7,\ t_7>t_5\cdots)$$
$$\times\exp[i2M(t_1+t_2-t_3-t_4+t_5+t_6-t_7-t_8)]\Delta(\mathbf{r},t_1-t_2)\Delta(\mathbf{r},t_3-t_6)\Delta(\mathbf{r},t_5-t_4)\Delta(\mathbf{r},t_7-t_8). \quad (20)$$

The time integrations analogous to Eq. (16) will be done in two steps as follows:

$$\int dtdt_3\cdots dt_8\exp[it_1(2M-k_{01})+\cdots]=(-i/2M)^4\int_{-\infty}^{\infty}dt\exp[-i(t_1-t_2)(k_{01}+k_{02}+k_{03}+k_{04})]$$
$$\times\int_{-\infty}^{0}dt_5\exp[-it_5(k_{03}+k_{04})]\int_{-\infty}^{0}dt_6\exp[it_6(k_{02}+k_{04})]\rightarrow$$
$$(-i/2M)^4 2(\pi)^3\delta(k_{01}+k_{02}+k_{03}+k_{04})\delta(k_{03}+k_{04})\delta(k_{02}+k_{04}). \quad (21)$$

The arrow is used in the last stage to indicate that it is a permissible step only in virtue of the symmetry properties of the entire integral, Eq. (20). Physically, it corresponds to the fact that having established the pair-vertices by prior time integrations, reversal of the time sequence of the two-pair-vertices of each particle leaves the interaction [or Fig. 2(d)] invariant.[17] As a consequence of Eq. (21), we are left with

$$V_8(\mathbf{r})=i\frac{g^8}{(2M)^4}\frac{1}{(2\pi)^{12}}\frac{T_8}{(2)^3\pi}\int d\mathbf{k}_1\cdots d\mathbf{k}_4\exp[i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{r}]\times\int dk_0[(k_0^2-\omega_1^2)(k_0^2-\omega_2^2)(k_0^2-\omega_3^2)(k_0^2-\omega_4^2)]^{-1}. \quad (22)$$

In the appendix it is shown that the integral

$$I_{2l}\equiv\frac{i}{\pi}\frac{1}{(2\pi^2)^{2l}}\int d\mathbf{k}_1\cdots d\mathbf{k}_{2l}\exp[i(\mathbf{k}_1+\cdots+\mathbf{k}_{2l})\cdot\mathbf{r}]\int dk_0[(k_0^2-\omega_1^2)\cdots(k_0^2-\omega_{2l}^2)]^{-1}=-(2/\pi)\mu K_1(2l\mu r)/r^{2l}. \quad (23)$$

Equation (22) yields, therefore,

$$V_8(x)=-6\mu(g^2/4\pi)^4(\mu/2M)^4(2/\pi)K_1(4x)/x^4. \quad (24)$$

C. General Result

On the basis of the previous results, we conjecture that the $2n$-pair potential of order $4n$ has the form

$$V_{4n}(x)=-\mu(g^2/4\pi)^{2n}(\mu/2M)^{2n} N_{4n}T_{4n}2^{-2n-1}(2/\pi)K_1(2nx)/x^{2n}. \quad (25)$$

Here N_{4n} is the number of distinct perimeters, for starting with twelfth order, this number is greater than unity and as we shall see below increases rapidly with n. Figure 6 shows two of the six perimeters in twelfth order. In writing Eq. (25), we have assumed that each perimeter contributes equally to the potential, at least *on the average*. The inclusion of the factor 2^{-2n-1} [see the factor 2^{-3} in Eq. (22)] assumes that the contributions from the time intervals between the established pair vertices yield half a delta function each, again at least on the average.

To find N_{4n}, we ask for the number of ways of constructing distinct meson-line perimeters, starting from n pair vertices for each particle. The result is

$$N_{4n}=\binom{n}{2}[(n-1)(n-2)][(n-2)(n-3)]\cdots=\tfrac{1}{2}n!(n-1)!. \quad (26)$$

We then have

$$N_{4n}T_{4n}2^{-2n-1}=3\times n!(n-1)!, \quad (27)$$

FIG. 6. Two of six twelfth-order perimeters.

and Eq. (25) becomes

$$V_{4n}(x)=-\mu(g^2/4\pi)^{2n}(\mu/2M)^{2n}3\times n!(n-1)!(2/\pi)K_1(2nx)/x^{2n}. \quad (28)$$

It can now be stated that in virtue of Eq. (23), there remains but a single point that must be established in order to completely verify Eq. (28); namely, that whenever we encounter an integral of the form

$$\int_{-\infty}^{0}dte^{-it\lambda}=\pi\delta(\lambda)+iP(1/\lambda), \quad (29)$$

which arises from time intervals between pair-vertices, we can replace it by merely $\pi\delta(\lambda)$ as a consequence of the structure of the entire expression of which it is a part, as in Eq. (21). Unfortunately, we have been unable thus far to find a general proof of this statement. It is, in fact, not true in the simple form analogous to Eq. (21). What appears to be true is that integrals containing an even number[18] of linear denominators from equations like (29) cancel out when one adds the analogous contributions from different perimeters. This has actually been checked in twelfth order, which is the first nontrivial case. It appears to be a consequence of the topological invariance of the group of perimeters as a whole under the inversion in time order of selected pair vertices, but we have been unable to relate this in a general way to the structure of the integrals encountered.

We believe that this lack of rigor hardly invalidates the general inference to be drawn from Eq. (28). Based alone on the combinatorial aspects of the work, there should be little doubt that the coefficients of higher order potentials increase so rapidly as to render unthinkable the convergence of the unrenormalized potentials.

We have also tentatively applied the same technique of calculation to the set of potentials exemplified by Eqs. (2) and (4), where we claim no validity for it. It is found that the method reproduces the correct analytic

[17] In detail, we carry out the sequence of transformations $k_{01}\leftrightarrow k_{03}$ (and $\mathbf{k}_1\leftrightarrow\mathbf{k}_3$), $k_{02}\leftrightarrow k_{04}$ (and $\mathbf{k}_2\leftrightarrow\mathbf{k}_4$) and then note that $k_{01}+k_{02}=-k_{03}-k_{04}$ in virtue of the integral with respect to t. Half the sum of the original plus the equivalent transformed expression yields the factor $\pi\delta(k_{03}+k_{04})$. Similar steps provide the other δ function.

[18] Those involving an odd number vanish identically.

form of the potential, but starting in eighth order, overestimates the coefficients.[19] Again, however, there can be little doubt that the tendency toward breakdown of the expansion in μ/M for a given power of $g^2/4\pi$ becomes increasingly apparent in higher orders.

Note added in proof:—The central result of this paper, Eq. (28), is incorrect. The correct result is obtained by replacing the factor $n!(n-1)!$ by $2^{2(n-1)}/n$. A derivation of the correct general expression for the pair potentials as well as for certain other series of potentials involving a restricted number of gradient couplings, together with the sum of each series, will be submitted for publication shortly. The series of potentials for pair theory can also be deduced directly from the work of reference 11. The "derivation" given in the present paper goes wrong only at the last stage, Eq. (29) and associated discussion. The special results obtained in Secs. II, IIIb, and the Appendix are, however, subject to no objection. Our conclusions must correspondingly be modified from a bald assertion of nonconvergence to the statement that perturbation theory yields a series of potentials with a finite radius of convergence in inverse distance space. However, for values of the coupling constant presently contemplated, $(g^2/4\pi)\geq 10$, there is an important region outside the cut-off radius where the series either doesn't converge or else the convergence is painfully slow. This will be shown in greater detail in the forthcoming publication.

APPENDIX

In Sec. VI of **A**, the general character of the n-pair term of the n-body force was established, and in addition specific expressions were given for the three and four-body forces. We are now in a position to obtain

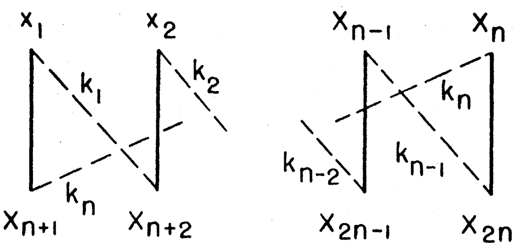

FIG. 7. Representative diagram for the calculation of the n-pair term of the n-body potential.

the result for arbitrary n by means of the technique employed in Sec. III, which applies here without question.

We recall that this force consists of $(n-1)!/2$ terms, each of similar structure, one for every possible perimeter. Furthermore, it is sufficient to calculate for a single Feynman diagram that contributes to some standard perimeter and multiply the result by $T_n=3\times 2^n$, the isotopic factor for the perimeter. From this single term it is a trivial matter of changing functional arguments to obtain all $(n-1)!/2$ terms.

Consider therefore the interaction of Fig. 7. In complete analogy to Eqs. (18) or (11), it can be shown to contribute the potential

$$V_n(\xi_1\xi_2\cdots\xi_{n-1};\xi_1'\cdots\xi_{n-1}')=(-1)^n(-i)^{n-1}(-ig^2)^nT_n\int dt_2dt_3\cdots dt_nd\mathbf{R}'\exp[iM(t_1+t_2+\cdots-t_{n+1}-\cdots t_{2n}]$$
$$\times\Delta(x_1-x_{n+2})\Delta(x_2-x_{n+3})\cdots\Delta(x_n-x_{n+1})\langle\gamma_0\gamma_5G_-(x_1-x_{n+1})\gamma_5\rangle^{(1)}\cdots\langle\gamma_0\gamma_5G_-(x_n-x_{2n})\gamma_5\rangle^{(n)},\quad (A.1)$$

where $\xi_1\cdots\xi_{n-1}$ is the sequence of independent coordinate differences,[20] which defines the particular perimeter considered, $\xi_1=\mathbf{r}_1-\mathbf{r}_2$, $\xi_2=\mathbf{r}_2-\mathbf{r}_3$, $\cdots\xi_n=\mathbf{r}_n-\mathbf{r}_1$, and the time integrations are restricted by the condition that only the n-pair part is to be included ($t_1<t_{n+1}$, etc.). By means of the relationship,

$$\langle\gamma_0\gamma_5G_-(x_1-x_{n+1})\gamma_5\rangle^{(1)}\cdots\langle\gamma_0\gamma_5G_-(x_n-x_{2n})\gamma_5\rangle^{(n)}$$
$$\cong(-i)^n\delta(\mathbf{r}_1-\mathbf{r}_{n+1})\cdots\delta(\mathbf{r}_n-\mathbf{r}_{2n})\cdot e^{iM(t_1-t_{n+1})}\cdots e^{iM(t_n-t_{2n})},\quad (A.2)$$

Eq. (A.1) is reduced to a point potential,

$$V_n(\xi_1\cdots\xi_{n-1},\xi_1'\cdots\xi_{n-1}']\to V_n(\xi_1\cdots\xi_{n-1})\delta(\xi_1-\xi_1')\cdots\delta(\xi_{n-1}-\xi_{n-1}'),\quad (A.3)$$

$$V_n(\xi_1\cdots\xi_{n-1})=(-i)^{n-1}g^{2n}T_n\int dt_2\cdots dt_{2n}\exp[i2M(t_1+t_2+\cdots$$
$$-t_{n+1}-\cdots t_{2n})]\Delta(\xi_1,t_1-t_{n+2})\cdots\Delta(\xi_n,t_n-t_{n+1}).\quad (A.4)$$

Introducing four-dimensional momentum representations of the Δ functions, we encounter the time integrations

$$\int_{-\infty}^{\infty}dt_2\cdots dt_n\int_{t_1}^{\infty}dt_{n+1}\cdots\int_{t_n}^{\infty}dt_{2n}\exp[it_1(2M-k_{01})+\cdots+it_n(2M-k_{0n})$$
$$-it_{n+1}(2M-k_{0n})-it_{n+2}(2M-k_{01})-\cdots-it_{2n}(2M-k_{0\ n-1})]$$
$$=(-i/2M)^n(2\pi)^{n-1}\delta(k_{01}-k_{02})\delta(k_{02}-k_{03})\cdots\delta(k_{0\ n-1}-k_{0n}).\quad (A.5)$$

[19] One cannot, in full justice, rule out this possibility for Eq. (28).
[20] Note that ξ_n is not independent of the other ξ_i, but satisfies $\Sigma_i\xi_i=0$.

As a consequence of Eq. (A.5), Eq. (A.4) has been reduced to the form

$$V_n(\xi_1\cdots\xi_{n-1})=i(-1)^n\left(\frac{g^2}{4\pi}\right)^n\left(\frac{1}{2M}\right)^n T_n\left(\frac{1}{2\pi^2}\right)^n\frac{1}{2\pi}\int d\mathbf{k}_1\cdots d\mathbf{k}_n \times\exp[i(\mathbf{k}_1\cdot\xi_1+\cdots+\mathbf{k}_n\cdot\xi_n)]\int dk_0[(\omega_1{}^2-k_0{}^2)\cdots(\omega_n{}^2-k_0{}^2)]^{-1}. \quad (A.6)$$

It remains only to demonstrate that, for $n\geq 2$,

$$I_n\equiv\frac{i}{\pi}\frac{1}{(2\pi^2)^n}\int d\mathbf{k}_1\cdots d\mathbf{k}_n\exp[i(\mathbf{k}_1\cdot\xi_1+\cdots+\mathbf{k}_n\cdot\xi_n)]\int dk_0[(\omega_1{}^2-k_0{}^2)\cdots(\omega_n{}^2-k_0{}^2)]^{-1} =-\mu(2/\pi)K_1[\mu(\xi_1+\xi_2+\cdots+\xi_n)]/\xi_1\xi_2\cdots\xi_n, \quad (A.7)$$

in order to establish the result[21]

$$V_n(\xi_1\cdots\xi_{n-1})=(-1)^{n-1}3\times 2^{n-1}(g^2/4\pi)^n(\mu/2M)^n(2/\pi)K_1[\mu(\xi_1+\cdots+\xi_n)]/\mu^{n-1}\xi_1\cdots\xi_n, \quad (A.8)$$

in agreement with a previous derivation.[11]

We may remark that the same method suffices to derive the μ/M corrections to Eq. (A.8) which are analagous to the one-pair potential of fourth order, but we shall not go into detail. We turn rather to the proof of (A.7).

We employ spherical polar coordinates for the three-momentum integrations and immediately carry out the angular integrations. We are left with

$$I_n=\frac{i}{\pi}\left(\frac{2}{\pi}\right)^n\frac{1}{\xi_1\xi_2\cdots\xi_n}\int_{-\infty}^{\infty}dk_0\int_0^{\infty}dk_1k_1\sin k_1\xi_1\cdots dk_nk_n\sin k_n\xi_n[(\omega_1{}^2-k_0{}^2)\cdots(\omega_n{}^2-k_0{}^2)]^{-1}. \quad (A.9)$$

Next, consider the factors

$$[\omega_1{}^2-k_0{}^2]^{-1}[\omega_2{}^2-k_0{}^2]^{-1}=[k_1{}^2-k_2{}^2]^{-1}\{[\omega_2{}^2-k_0{}^2]^{-1}-[\omega_1{}^2-k_0{}^2]^{-1}\}. \quad (A.10)$$

By means of the formula

$$\int_0^{\infty}\frac{\sin(lk\xi)kdk}{k^2-k_1{}^2}=\tfrac{1}{2}\pi\cos lk_1\xi,\quad \xi>0, \quad (A.11)$$

we can carry out the k_1 integral in the first term and the k_2 integral in the second term. We label as k the remaining one of the momenta k_1, k_2 and combine the two terms again, with the result

$$I_n=\frac{i}{\pi}\left(\frac{2}{\pi}\right)^{n-1}\frac{1}{\xi_1\xi_2\cdots\xi_n}\int_{-\infty}^{\infty}dk_0\int_0^{\infty}kdk\sin k(\xi_1+\xi_2)k_3dk_3\sin k_3\xi_3\cdots[(\omega^2-k_0{}^2)(\omega_3{}^2-k_0{}^2)\cdots(\omega_n{}^2-k_0{}^2)]^{-1}. \quad (A.12)$$

We now perform the same sequence of operations with the factors $[(\omega^2-k_0{}^2)(\omega_3{}^2-k_0{}^2)]^{-1}$ and so on, a total of $(n-2)$ such sequences for Eq. (A.11), until we have reduced its structure to

$$I_n=\frac{i}{\pi}\frac{2}{\pi}\frac{1}{\xi_1\xi_2\cdots\xi_n}\int_0^{\infty}dkk\sin k(\xi_1+\cdots+\xi_n)\int_{-\infty}^{\infty}dk_0(\omega^2-k_0{}^2)^{-1}=-\mu(2/\pi)K_1[\mu(\xi_1+\cdots+\xi_n)]/\xi_1\cdots\xi_n. \quad (A.13)$$

The last form which is the result desired follows from the equations

$$\int_{-\infty}^{\infty}dk_0[\omega^2-i\eta-k_0{}^2]=\pi i/\omega, \quad (A.14)$$

and

$$\int_{-\infty}^{\infty}\frac{dkk\sin k(\xi_1+\cdots+\xi_n)}{\omega}=\mu K_1[\mu(\xi_1+\cdots+\xi_n)].$$

If we take n even and set $\xi_i=r$ for all i, we obtain Eq. (23) of the text.

[21] The demonstration holds also for $n=2$, if we agree that there is only half a perimeter in this case and consequently halve the result. For instead of $2^n=4$ Feynman diagrams, we have only two, the "ladder" and "crossed-quantum" diagrams.