PHYSICAL REVIEW VOLUME 91, NUMBER 3 AUGUST 1, 1953

The Theory of Quantized Fields. II

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts
(Received February 19, 1953)

The arguments leading to the formulation of the action principle for a general field are presented. In association with the complete reduction of all numerical matrices into symmetrical and antisymmetrical parts, the general field is decomposed into two sets, which are identified with Bose-Einstein and Fermi-Dirac fields. The spin restriction on the two kinds of fields is inferred from the time reflection invariance requirement. The consistency of the theory is verified in terms of a criterion involving the various generators of infinitesimal transformations. Following a discussion of charged fields, the electromagnetic field is introduced to satisfy the postulate of general gauge invariance. As an aspect of the latter, it is recognized that the electromagnetic field and charged fields are not kinematically independent. After a discussion of the field strength commutation relations, the independent dynamical variables of the electromagnetic field are exhibited in terms of a special gauge.

THE general program of this series[1] is the construction of a theory of quantized fields in terms of a single fundamental dynamical principle. We shall first present a revised account of the developments contained in the initial paper.

THE DYNAMICAL PRINCIPLE

The transformation functions connecting various representations have the two fundamental properties

$$(\alpha'|\gamma')=\int(\alpha'|\beta')d\beta'(\beta'|\gamma'),$$
$$(\alpha'|\beta')^*=(\beta'|\alpha'),$$

where $\int d\beta'$ symbolizes both integration and summation over the eigenvalue spectrum. If $\delta(\alpha'|\beta')$ is any infinitesimal alteration of the transformation function, we may write

$$\delta(\alpha'|\beta')=i(\alpha'|\delta W_{\alpha\beta}|\beta'), \tag{1}$$

which serves as the definition of the infinitesimal operator $\delta W_{\alpha\beta}$. The requirement that any infinitesimal alteration maintain the multiplicative composition law of transformation functions implies an additive composition law for the infinitesimal operators,

$$\delta W_{\alpha\gamma}=\delta W_{\alpha\beta}+\delta W_{\beta\gamma}. \tag{2}$$

If the α and β representations are identical, we infer that

$$\delta W_{\alpha\alpha}=0,$$

which expresses the fixed orthonormality requirements on the eigenvectors of a given representation. On identifying the α and γ representations, we learn that

$$\delta W_{\beta\alpha}=-\delta W_{\alpha\beta}.$$

The second property of transformation functions implies that

$$-i(\alpha'|\delta W_{\alpha\beta}|\beta')^*=-i(\beta'|\delta W_{\alpha\beta}{}^\dagger|\alpha')$$
$$=i(\beta'|\delta W_{\beta\alpha}|\alpha'),$$

or

$$\delta W_{\alpha\beta}{}^\dagger=\delta W_{\alpha\beta};$$

the infinitesimal operators $\delta W_{\alpha\beta}$ are Hermitian.

The $\delta W_{\alpha\beta}$ possess another additivity property referring to the composition of two dynamically independent systems. Thus, if I and II designate such systems,

$$(\alpha_{\mathrm{I}}'\alpha_{\mathrm{II}}'|\beta_{\mathrm{I}}'\beta_{\mathrm{II}}')=(\alpha_{\mathrm{I}}'|\beta_{\mathrm{I}}')(\alpha_{\mathrm{II}}'|\beta_{\mathrm{II}}'),$$

and if $\delta W_{\alpha\beta}{}^{\mathrm{I}}$ and $\delta W_{\alpha\beta}{}^{\mathrm{II}}$ are the operators characterizing infinitesimal changes of the separate transformation functions, that of the composite system is

$$\delta W_{\alpha\beta}=\delta W_{\alpha\beta}{}^{\mathrm{I}}+\delta W_{\alpha\beta}{}^{\mathrm{II}}.$$

Infinitesimal alterations of eigenvectors that preserve the orthonormality properties have the form

$$\delta\Psi(\alpha')=-iG_\alpha\Psi(\alpha'),$$
$$\delta\Psi(\alpha')^\dagger=i\Psi(\alpha')^\dagger G_\alpha,$$

where the generator G_α is an infinitesimal Hermitian operator which possesses an additivity property for the composition of dynamically independent systems. If the two eigenvectors of a transformation function are varied independently, the resulting change of the transformation function has the general structure (1), with

$$\delta W_{\alpha\beta}=G_\alpha-G_\beta.$$

The vector

$$\Psi(\alpha')+\delta\Psi(\alpha')=(1-iG_\alpha)\Psi(\alpha'),$$

can be characterized as an eigenvector of the operator set

$$\bar{\alpha}=(1-iG_\alpha)\alpha(1+iG_\alpha)=\alpha-\delta\alpha,$$

with the eigenvalues α'. Here

$$\delta\alpha=-i[\alpha,G_\alpha].$$

This infinitesimal unitary transformation of the eigenvector $\Psi(\alpha')$ induces a transformation of any operator F such that

$$(\alpha'|F|\alpha'')=(\bar{\alpha}'|\bar{F}|\bar{\alpha}'').$$

[1] J. Schwinger, Phys. Rev. 82, 914 (1951), Part I.

We write this in the form

$$(\bar{\alpha}'|F|\bar{\alpha}'')-(\alpha'|F|\alpha'')=(\bar{\alpha}'|(F-\bar{F})|\bar{\alpha}''),$$

or, in virtue of the infinitesimal nature of the transformation,

$$\delta(\alpha'|F|\alpha'')=(\alpha'|\delta F|\alpha''),$$

where the left side refers to the change in the eigenvectors for a fixed F, while the right side provides an equivalent variation of the operator F, given by

$$\delta F=F-\bar{F}=-i[F, G_\alpha].$$

If the change consists in the alteration of some parameter τ, upon which the dynamical variables depend, and which may occur explicitly in F, we have

$$\bar{F}=F-(\delta F)_\tau$$
$$=F+\delta_\tau F-\partial_\tau F,$$

where $\delta_\tau F$ is the total alteration in F, from which is subtracted $\partial_\tau F$, the change in F associated with the explicit appearance of τ, since the latter cannot be produced by an operator transformation. We thereby obtain the "equation of motion" with respect to the parameter τ,

$$\delta_\tau F=\partial_\tau F+i[F, G_\tau]. \tag{3}$$

For dynamical systems obeying the postulate of local action, complete descriptions are provided by sets of physical quantities, ζ, associated with space-like surfaces, σ. An infinitesimal alteration of the general transformation function $(\zeta_1'\sigma_1|\zeta_2''\sigma_2)$ is characterized by

$$\delta(\zeta_1'\sigma_1|\zeta_2''\sigma_2)=i(\zeta_1'\sigma_1|\delta W_{12}|\zeta_2''\sigma_2). \tag{4}$$

Here the indices 1 and 2 refer both to the choice of complete set of commuting operators ζ, and to the space-like surface σ. We can, in particular, consider transformations between the same set of operators on different surfaces, or between different sets of commuting operators on the same surface, as in

$$\delta(\zeta'\sigma|\bar{\zeta}'\sigma)=i(\zeta'\sigma|\delta W|\bar{\zeta}'\sigma). \tag{5}$$

One type of change of the general transformation function consists in the introduction, independently on σ_1 and on σ_2, of infinitesimal unitary transformations of the operators, including displacements of these surfaces. The transformations will be generated by operators G_1 and G_2, constructed from dynamical variables on σ_1 and σ_2, respectively, and

$$\delta W_{12}=G_1-G_2. \tag{6}$$

When the transformation function connects two different sets of operators on the same surface, which are subjected to infinitesimal transformations generated by G and $\bar{G}$, respectively, we have, referring to (5),

$$\delta W=G-\bar{G}. \tag{7}$$

Since physical phenomena at distinct points on a space-like surface are dynamically independent, a generator G must have the additive form

$$G=\int_\sigma d\sigma G_{(0)}(x)=\int_\sigma d\sigma_\mu G_\mu(x),$$

where $d\sigma$ is the numerical measure of an element of space-like area and $G_{(0)}(x)$ is to be regarded as the time-like component of a vector in a local coordinate system based on σ in order to give the surface integral an invariant form. If one can interpret $G_\mu(x)$ on σ_1, and on σ_2, as the values of a vector defined at all points, the difference of surface integrals in (6) can be transformed into the volume integral

$$\delta W_{12}=\int_{\sigma_2}^{\sigma_1}(dx)\partial_\mu G_\mu(x),$$
$$(\partial_\mu=\partial/\partial x_\mu).$$

A second type of transformation function alteration is obtained on considering that the transformation connecting ζ_1, σ_1, and ζ_2, σ_2 can be constructed through the intermediary of an infinite succession of transformations relating operators on infinitesimally neighboring surfaces. According to the general additivity property (2),

$$\delta W_{12}=\sum_{\sigma_2}^{\sigma_1}\delta W_{\sigma+d\sigma,\,\sigma},$$

where $\delta W_{\sigma+d\sigma,\,\sigma}$ characterizes a modification of the transformation function connecting infinitesimally differing complete sets of operators on the infinitesimally separated surfaces σ and $\sigma+d\sigma$. If the choice of intermediate operators depends continuously upon the surface, we shall have

$$\delta W_{\sigma,\,\sigma}=0,$$

and, referring again to the dynamical independence of phenomena at points separated by a space-like interval, with the consequent additivity property, we see that $\delta W_{\sigma+d\sigma,\,\sigma}$ will have the general form

$$\delta W_{\sigma+d\sigma,\,\sigma}=\int_\sigma^{\sigma+d\sigma}(dx)\delta\mathfrak{L}(x).$$

Therefore

$$\delta W_{12}=\int_{\sigma_2}^{\sigma_1}(dx)\delta\mathfrak{L}(x). \tag{8}$$

The combination of these two types of modifications is described by

$$\delta W_{12}=G_1-G_2+\int_{\sigma_2}^{\sigma_1}(dx)\delta\mathfrak{L}(x),$$

which involves dynamical variables on the surfaces σ_1, σ_2, and in the interior of the volume bounded by these surfaces. On the other hand, we can write this as the volume integral

$$\delta W_{12}=\int_{\sigma_2}^{\sigma_1}(dx)[\partial_\mu G_\mu(x)+\delta\mathfrak{L}(x)],$$

which indicates, conversely, that any part of $\delta\mathfrak{L}(x)$, possessing the form of a divergence, contributes only to the generation of unitary transformations on σ_1 and σ_2.

The fundamental dynamical principle is contained in the postulate that there exists a class of transformation function alterations for which the characterizing operators δW_{12} are obtained by appropriate variation of a single operator W_{12},

$$\delta W_{12}=\delta(W_{12}).$$

Of course, this principle must be implemented by the explicit specification of that class.

The operator W_{12}, the action integral operator, evidently possess the form

$$W_{12}=\int_{\sigma_2}^{\sigma_1}(dx)\mathfrak{L}(x).$$

The Hermitian requirement on δW_{12} is satisfied if W_{12} is Hermitian, which implies the same property for $\mathfrak{L}(x)$, the Lagrange function operator. In order that relations between states on σ_1 and σ_2 be invariantly characterized, the Lagrange function must be a scalar with respect to the transformations of the orthochronous[2] Lorentz group, which preserve the temporal order of σ_1 and σ_2. A dynamical system is specified by exhibiting the Lagrange function in terms of a set of fundamental dynamical variables in the infinitesimal neighborhood of the point x. Contained in this Lagrange function will be certain numerical parameters, which may be functions of x. Any change of these parameters modifies the structure of the Lagrange function and is thus an alteration of the dynamical system. Accordingly, infinitesimal changes of the dynamical system are described by

$$\delta W_{12}=\int_{\sigma_2}^{\sigma_1}(dx)\delta\mathfrak{L}(x),$$

where $\delta\mathfrak{L}=\delta(\mathfrak{L})$, and the numerical parameters are the object of variation. This form is in agreement with (8). For a fixed dynamical system, W_{12} can be altered by displacing the surfaces σ_1, σ_2 and by varying the dynamical variables contained in the Lagrange function. The transformation function $(\zeta_1'\sigma_1|\zeta_2''\sigma_2)$ describes the relation between two states of the given system so that a change in the transformation function can only arise from alterations of the states on σ_1 and σ_2. Hence, for a fixed dynamical system we must have

$$\delta W_{12}=G_1-G_2,$$

where $\delta W_{12}=\delta(W_{12})$ and the objects of variation here are σ_1, σ_2, and the dynamical variables of which $\mathfrak{L}$ is a function.

The latter statement is the operator principle of stationary action. It asserts that W_{12} must be stationary with respect to variations of the dynamical variables in the interior of the region defined by σ_1 and σ_2, since G_1 and G_2 only contain dynamical variables associated with the boundaries of the region. This principle implies equations of motion for the dynamical variables, that is to say, field equations, and provides expressions for the generators G_1 and G_2. The class of variations to which our postulate refers can now be defined through the requirement that this information concerning field equations and infinitesimal unitary transformations be self-consistent.

There exists much freedom within this class, as may be inferred from the remark that two Lagrange functions, differing by the divergence of a vector, describe the same dynamical system. Thus

$$\bar{\mathfrak{L}}(x)=\mathfrak{L}(x)-\partial_\mu f_\mu(x),$$

yields

$$\bar{W}_{12}=W_{12}-(W_1-W_2), \tag{9}$$

where, on each surface,

$$W=\int_\sigma d\sigma_\mu f_\mu=\int_\sigma d\sigma f_{(0)}.$$

Accordingly, the stationary action principle for $\bar{W}_{12}$ is satisfied if it is obeyed by W_{12}, since

$$\delta\bar{W}_{12}=\bar{G}_1-\bar{G}_2.$$

Here

$$\delta W_1=G_1-\bar{G}_1,\quad \delta W_2=G_2-\bar{G}_2,$$

define $\bar{G}_1$ and $\bar{G}_2$, which are new generators of infinitesimal unitary transformations on σ_1 and σ_2, respectively. The latter equations possess the form (7), and thus characterize transformation functions connecting two different representations on a common surface. Indeed, with a suitably elaborate notation, we recognize in (9) the additivity property of action operators,

$$W(\bar{\zeta}_1\sigma_1,\bar{\zeta}_2\sigma_2)=W(\bar{\zeta}_1\sigma_1,\zeta_1\sigma_1)+W(\zeta_1\sigma_1,\zeta_2\sigma_2)+W(\zeta_2\sigma_2,\bar{\zeta}_2\sigma_2),$$

where, for example,

$$W_1=-W(\bar{\zeta}_1\sigma_1,\zeta_1\sigma_1)=W(\zeta_1\sigma_1,\bar{\zeta}_1\sigma_1),$$

and

$$W_2=W(\zeta_2\sigma_2,\bar{\zeta}_2\sigma_2).$$

To be consistent with the postulate of local action, the field equations must be differential equations of finite order. One can always convert such equations into systems of first order equations by suitable adjunction of variables. We shall designate the fundamental dynamical variables that obey first-order field equations by $\chi_r(x)$, which form the components of the general field operator $\chi(x)$. With no loss in generality, we take

[2] This name was suggested by H. J. Bhabha, Revs. Modern Phys. **21**, 451 (1949).

$\chi(x)$ to be a Hermitian operator,

$$\chi_r(x)^\dagger = \chi_r(x).$$

If the Lagrange function is to yield field equations of the desired structure, it must be linear in the first derivatives of the field operators with respect to the space-time coordinates. Furthermore, if these field equations are to emerge as explicit equations of motion for field components, that part of the Lagrange function containing first coordinate derivatives must be bilinear in the field components. With these preliminary remarks, we write the following general expression for the Lagrange function,

$$\mathfrak{L} = \tfrac{1}{2}(\chi\mathfrak{A}_\mu\partial_\mu\chi - \partial_\mu\chi\mathfrak{A}_\mu\chi) - \mathfrak{H}(\chi), \tag{10}$$

in which a matrix notation is employed,

$$\chi\mathfrak{A}_\mu\partial_\mu\chi = \chi_r(\mathfrak{A}_\mu)_{rs}\partial_\mu\chi_s.$$

The derivative terms have been symmetrized with respect to the operation of integration by parts, a process which adds a divergence to the Lagrange function, and is thus without effect on the structure of the dynamical system. In order that $\mathfrak{L}$ be a Hermitian operator, the general function $\mathfrak{H}$ must possess this character,

$$\mathfrak{H}(\chi)^\dagger = \mathfrak{H}(\chi),$$

and the numerical matrices $\mathfrak{A}_\mu$; $\mu=0, 1, 2, 3$ ($x_4 = ix_0$, $\mathfrak{A}_4 = i\mathfrak{A}_0$) must be skew-Hermitian,

$$\mathfrak{A}_\mu^\dagger = \mathfrak{A}_\mu^{\mathrm{tr}*} = -\mathfrak{A}_\mu; \quad \mu=0, 1, 2, 3.$$

Although we are interested in complete dynamical systems, it is advantageous mathematically to employ devices based upon the properties of external sources. Accordingly, we add to (10) a term designed to describe the generation of the field $\chi(x)$ by an external source $\xi(x)$, which is to be regarded as a field quantity of the same general nature as $\chi(x)$,

$$\mathfrak{L}_{\mathrm{source}} = \tfrac{1}{2}(\xi\mathfrak{B}\chi + \chi\mathfrak{B}\xi). \tag{11}$$

This is a Hermitian operator if $\mathfrak{B}$ is a Hermitian matrix,

$$\mathfrak{B}^\dagger = \mathfrak{B}.$$

For the source concept to be meaningful, all components of χ must occur coupled with the source components in (11), which requires that $\mathfrak{B}$ be a nonsingular numerical matrix.

An orthochronous Lorentz transformation

$$'x_\mu = r_{\mu\nu}x_\nu + t_\mu,$$
$$r^{\mathrm{tr}}r = 1, \quad r_{44} > 0,$$

induces a linear transformation on the field components,

$$'\chi = L\chi = \chi L^{\mathrm{tr}},$$

where L must be a real matrix,

$$L^* = L,$$

to maintain the Hermiticity of $'\chi$. The scalar requirement on $\mathfrak{L}$ is satisfied if $\mathfrak{H}$ is a scalar,

$$\mathfrak{H}(L\chi) = \mathfrak{H}(\chi),$$

and if

$$L^{\mathrm{tr}}\mathfrak{A}_\mu L = r_{\mu\nu}\mathfrak{A}_\nu. \tag{12}$$

We shall suppose that the source possesses the same transformation properties as the field. The condition for the source term of the Lagrange function to be a scalar is then given by

$$L^{\mathrm{tr}}\mathfrak{B}L = \mathfrak{B}. \tag{13}$$

Note that $\mathfrak{A}_\mu^{\mathrm{tr}}$ and $\mathfrak{B}^{\mathrm{tr}}$ also obey Eqs. (12) and (13), respectively, and that these equations can be combined into

$$L^{-1}(\mathfrak{B}^{-1}\mathfrak{A}_\mu)L = r_{\mu\nu}(\mathfrak{B}^{-1}\mathfrak{A}_\nu),$$

in view of the nonsingular character of $\mathfrak{B}$.

For an infinitesimal Lorentz transformation,

$$'x_\mu = x_\mu - \epsilon_{\mu\nu}x_\nu + \epsilon_\mu, \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu},$$

the matrix L can be written

$$L = 1 - i\tfrac{1}{2}\epsilon_{\mu\nu}S_{\mu\nu}, \tag{14}$$

where

$$S_{\mu\nu}^* = -S_{\mu\nu}; \quad \mu, \nu = 0, \cdots 3. \tag{15}$$

The infinitesimal version of (13) is

$$-S_{\mu\nu}^{\mathrm{tr}} = \mathfrak{B}S_{\mu\nu}\mathfrak{B}^{-1} = S_{\mu\nu}^\dagger$$

or

$$(\mathfrak{B}S_{\mu\nu})^\dagger = (\mathfrak{B}S_{\mu\nu}),$$

in which the complex conjugate statements refer to the components indicated in (15). Similarly,

$$\mathfrak{A}_\mu S_{\nu\lambda} - S_{\nu\lambda}^\dagger\mathfrak{A}_\mu = i(\delta_{\mu\lambda}\mathfrak{A}_\nu - \delta_{\mu\nu}\mathfrak{A}_\lambda) \tag{16}$$

and

$$[\mathfrak{B}^{-1}\mathfrak{A}_\mu, S_{\nu\lambda}] = i(\delta_{\mu\lambda}\mathfrak{B}^{-1}\mathfrak{A}_\nu - \delta_{\mu\nu}\mathfrak{B}^{-1}\mathfrak{A}_\lambda).$$

If one views $'\chi = (1 - i\tfrac{1}{2}\epsilon_{\mu\nu}S_{\mu\nu})\chi$ as a field in the original coordinate system and thus subject to the same dependence upon that coordinate system as χ, it is inferred that

$$L^{-1}S_{\mu\nu}L = r_{\mu\lambda}r_{\nu\kappa}S_{\lambda\kappa}.$$

For infinitesimal transformations, this reads

$$i[S_{\mu\nu}, S_{\lambda\kappa}] = \delta_{\mu\kappa}S_{\nu\lambda} - \delta_{\nu\kappa}S_{\mu\lambda} + \delta_{\nu\lambda}S_{\mu\kappa} - \delta_{\mu\lambda}S_{\nu\kappa}.$$

In performing the variation of the action integral, we shall treat the two types of quantities, coordinates and field variables, on somewhat the same footing, although the former are numbers and the latter operators. We introduce an arbitrary variation of the coordinates, δx_μ, throughout the interior of the region, but subject to the condition that the boundaries remain plane surfaces,

$$\partial_\mu\delta x_\nu + \partial_\nu\delta x_\mu = 0, \tag{17}$$

on σ_1 and σ_2. The field components $\chi_r(x)$ are dependent both upon the coordinate system and the "intrinsic field." Under a rotation of the coordinate system, the field components are altered in the manner described by (14). Accordingly, we write the general variation of the field as the sum of an intrinsic field variation, and of the variation induced by the local rotation of the coordinate system,

$$\delta(\chi)=\delta\chi-i\tfrac{1}{2}(\partial_\mu\delta x_\nu)S_{\mu\nu}\chi,$$

where the antisymmetry of $S_{\mu\nu}$ ensures that only the rotation part of the coordinate displacement is effective. For the source field, a prescribed function of the coordinates, we have

$$\delta(\xi)=\delta x_\mu\partial_\mu\xi. \qquad (18)$$

We also remark that

$$\delta(dx)=(dx)\partial_\mu\delta x_\mu,$$

and

$$\delta(\partial_\mu)=-(\partial_\mu\delta x_\nu)\partial_\nu,$$

whence

$$\delta(\partial_\mu\chi)=\partial_\mu\delta(\chi)-(\partial_\mu\delta x_\nu)\partial_\nu\chi. \qquad (19)$$

The Lorentz invariance of $\mathfrak{L}$ produces a significant simplification, in computing the contribution to $\delta(\mathfrak{L})$ from the coordinate induced variation of χ. Thus, if $\partial_\mu\delta x_\nu$ were antisymmetrical and constant, its coefficient in the variation of the Lagrange function would vanish identically, save for the source term since the rotation induced change of ξ is not present in (18). Accordingly, for the general coordinate variation of (10), there remains only those terms in which $\partial_\mu\delta x_\nu$ is differentiated, or occurs in the dilation combination, $\partial_\mu\delta x_\nu+\partial_\nu\delta x_\mu$. Both types are contained entirely in (19), which leads to

$$\begin{aligned}\delta(\mathfrak{L})=\delta\mathfrak{L}-\tfrac{1}{2}(\partial_\mu\delta x_\nu+\partial_\nu\delta x_\mu)\tfrac{1}{2}(\chi\mathfrak{A}_\mu\partial_\nu\chi-\partial_\nu\chi\mathfrak{A}_\mu\chi)\\ -i\tfrac{1}{2}(\partial_\mu\partial_\nu\delta x_\lambda)\tfrac{1}{2}\chi(\mathfrak{A}_\mu S_{\nu\lambda}+S_{\nu\lambda}{}^\dagger\mathfrak{A}_\mu)\chi\\ -i\tfrac{1}{4}(\partial_\mu\delta x_\nu)(\xi\mathfrak{B}S_{\mu\nu}\chi-\chi S_{\mu\nu}{}^\dagger\mathfrak{B}\xi).\end{aligned}$$

In virtue of the symmetry of the second derivative,

$$\begin{aligned}&(\partial_\mu\partial_\nu\delta x_\lambda)\chi(\mathfrak{A}_\mu S_{\nu\lambda}+S_{\nu\lambda}{}^\dagger\mathfrak{A}_\mu)\chi\\ &=(\partial_\mu(\partial_\nu\delta x_\lambda+\partial_\lambda\delta x_\nu))\chi(\mathfrak{A}_\nu S_{\mu\lambda}+S_{\mu\lambda}{}^\dagger\mathfrak{A}_\nu)\chi\\ &\rightarrow-(\partial_\nu\delta x_\lambda+\partial_\lambda\delta x_\nu)\partial_\mu[\chi(\mathfrak{A}_\nu S_{\mu\lambda}+S_{\mu\lambda}{}^\dagger\mathfrak{A}_\nu)\chi],\end{aligned}$$

where the last step expresses the result of an integration by parts, for which the integrated term vanishes, since the dilation tensor is zero on the boundaries (Eq. (17)). Collecting the coefficients of $\partial_\mu\delta x_\nu$ into the tensor $T_{\mu\nu}$, we have

$$\begin{aligned}\delta(W_{12})&=\int_{\sigma_2}^{\sigma_1}(dx)[\delta\mathfrak{L}+(\partial_\mu\delta x_\nu)T_{\mu\nu}]\\ &=\int_{\sigma_2}^{\sigma_1}(dx)[\delta\mathfrak{L}-\delta x_\nu\partial_\mu T_{\mu\nu}+\partial_\mu(T_{\mu\nu}\delta x_\nu)],\end{aligned}$$

where

$$\begin{aligned}T_{\mu\nu}=\mathfrak{L}\delta_{\mu\nu}-\tfrac{1}{2}(\chi\mathfrak{A}_{(\mu}\partial_{\nu)}\chi-\partial_{(\nu}\chi\mathfrak{A}_{\mu)}\chi)\\ -i\tfrac{1}{4}(\xi\mathfrak{B}S_{\mu\nu}\chi-\chi S_{\mu\nu}{}^\dagger\mathfrak{B}\xi)\\ +i\tfrac{1}{2}\partial_\lambda[\chi(\mathfrak{A}_{(\mu}S_{\lambda\nu)}+S_{\lambda(\nu}{}^\dagger\mathfrak{A}_{\mu)})\chi],\end{aligned} \qquad (20)$$

and we have employed a notation for the symmetrical part of a tensor,

$$\mathfrak{A}_{(\mu}\partial_{\nu)}=\tfrac{1}{2}(\mathfrak{A}_\mu\partial_\nu+\mathfrak{A}_\nu\partial_\mu).$$

The expression for $\delta\mathfrak{L}$ is

$$\begin{aligned}\delta\mathfrak{L}=\delta\chi\mathfrak{A}_\mu\partial_\mu\chi-\partial_\mu\chi\mathfrak{A}_\mu\delta\chi-\delta\mathfrak{H}+\tfrac{1}{2}(\delta\chi\mathfrak{B}\xi+\xi\mathfrak{B}\delta\chi)\\ +\delta x_\mu\tfrac{1}{2}(\chi\mathfrak{B}\partial_\mu\xi+\partial_\mu\xi\mathfrak{B}\chi)+\partial_\mu[\tfrac{1}{2}(\chi\mathfrak{A}_\mu\delta\chi-\delta\chi\mathfrak{A}_\mu\chi)].\end{aligned}$$

Hence, on applying the principle of stationary action to coordinate and field variations, separately, we obtain

$$\partial_\mu T_{\mu\nu}=\tfrac{1}{2}(\chi\mathfrak{B}\partial_\mu\xi+\partial_\mu\xi\mathfrak{B}\chi),$$

and

$$\delta\mathfrak{H}=\delta\chi\mathfrak{A}_\mu\partial_\mu\chi-\partial_\mu\chi\mathfrak{A}_\mu\delta\chi+\tfrac{1}{2}(\delta\chi\mathfrak{B}\xi+\xi\mathfrak{B}\delta\chi), \qquad (21)$$

while the surface terms yield, on σ_1 and σ_2, the infinitesimal generator

$$G=\int_\sigma d\sigma_\mu[\tfrac{1}{2}(\chi\mathfrak{A}_\mu\delta\chi-\delta\chi\mathfrak{A}_\mu\chi)+T_{\mu\nu}\delta x_\nu].$$

The operator $\mathfrak{H}$ is an arbitrary, invariant function of the field χ. If its variation is to possess the form (21), with $\delta\chi$ appearing on the left and on the right, the latter must possess elementary operator properties, characterizing the class of variations to which the action principle refers. Thus, we should be able to displace $\delta\chi$ entirely to the left, or to the right, in the structure of $\delta\mathfrak{H}$,

$$\delta\mathfrak{H}=\delta\chi(\partial_l\mathfrak{H}/\partial\chi)=(\partial_r\mathfrak{H}/\partial\chi)\delta\chi,$$

which defines the left and right derivatives of $\mathfrak{H}$ with respect to χ. In view of the complete symmetry between left and right in the process of multiplication, we infer that the expressions with $\delta\chi$ on the left and on the right are, in fact, identical. The field equations, therefore, possess the two equivalent forms

$$\begin{aligned}2\mathfrak{A}_\mu\partial_\mu\chi&=(\partial_l\mathfrak{H}/\partial\chi)-\mathfrak{B}\xi,\\ -\partial_\mu\chi2\mathfrak{A}_\mu&=(\partial_r\mathfrak{H}/\partial\chi)-\xi\mathfrak{B},\end{aligned}$$

and G can be equivalently written

$$\begin{aligned}G&=\int_\sigma d\sigma_\mu[\chi\mathfrak{A}_\mu\delta\chi+T_{\mu\nu}\delta x_\nu]\\ &=\int_\sigma d\sigma_\mu[-\delta\chi\mathfrak{A}_\mu\chi+T_{\mu\nu}\delta x_\nu].\end{aligned} \qquad (22)$$

In keeping with the restriction of the stationary action principle to fixed dynamical systems, the external source has not been altered. If we now introduce an infinitesimal variation of ξ, and extend the argument of the previous paragraph to $\delta\xi$, we obtain the two equivalent expressions for the change induced in W_{12},

$$\delta_\xi W_{12}=\int_{\sigma_2}^{\sigma_1}(dx)\delta\xi\mathfrak{B}\chi=\int_{\sigma_2}^{\sigma_1}(dx)\chi\mathfrak{B}\delta\xi.$$

The corresponding modification in the relation between states on σ_1 and on σ_2 can be ascribed to the individual states only if one introduces a convention, of the nature of a boundary condition. Thus, we may suppose that the state on σ_2 is unaffected by varying the external source in the region between σ_1 and σ_2. In this "retarded" description, $\delta_\xi W_{12}$ generates the infinitesimal transformation of the state on σ_1. An alternative, "advanced" description corresponds to $-\delta_\xi W_{12}$ generating the change in the state on σ_2, with a fixed state on σ_1. These are just the simplest of possible boundary conditions.

The suitability of the designations, retarded and advanced, can be seen by considering the matrix of an operator constructed from dynamical variables on some surface σ, intermediate between σ_1 and σ_2,

$$(\zeta_1'\sigma_1|F(\sigma)|\zeta_2''\sigma_2) = \int (\zeta_1'\sigma_1|\zeta'\sigma)d\zeta'(\zeta'\sigma|F(\sigma)|\zeta''\sigma)d\zeta''(\zeta''\sigma|\zeta_2''\sigma_2).$$

An infinitesimal change of the source ξ produces the following change in the matrix element,

$$\begin{aligned}\delta_\xi(\zeta_1'\sigma_1|F(\sigma)|\zeta_2''\sigma_2) &= (\zeta_1'\sigma_1|(\partial_\xi F(\sigma)+i\delta_\xi W_{1\sigma}F(\sigma)+iF(\sigma)\delta_\xi W_{\sigma 2})|\zeta_2''\sigma_2)\\ &= (\zeta_1'\sigma_1|(\partial_\xi F(\sigma)+i(F(\sigma)\delta_\xi W_{12})_+)|\zeta_2''\sigma_2),\end{aligned}$$

in which we have allowed for the possibility that $F(\sigma)$ may be explicitly dependent upon the source, and introduced a notation for temporally ordered products. The matrix element depends upon the external source through the operator $F(\sigma)$, and the eigenvectors on σ_1 and σ_2. One thereby gets various expressions for $\delta_\xi F(\sigma)$, depending upon the boundary conditions that are adopted. Thus, if the state on σ_2 is prescribed, we find

$$\begin{aligned}\delta_\xi F(\sigma)]_{\text{ret}} &= \partial_\xi F(\sigma)+i(F(\sigma)\delta_\xi W_{12})_+ - i\delta W_{12}F(\sigma)\\ &= \partial_\xi F(\sigma)+i[F(\sigma),\delta_\xi W_{\sigma 2}],\end{aligned} \tag{23}$$

which only involves changes in the source prior to, or on σ. The opposite convention yields the analogous result

$$\begin{aligned}\delta_\xi F(\sigma)]_{\text{adv}} &= \partial_\xi F(\sigma)+i(F(\sigma)\delta_\xi W_{12})_+ - iF(\sigma)\delta W_{12}\\ &= \partial_\xi F(\sigma)-i[F(\sigma),\delta_\xi W_{1\sigma}].\end{aligned}$$

Note that

$$\delta_\xi F(\sigma)]_{\text{ret}} - \delta_\xi F(\sigma)]_{\text{adv}} = i[F(\sigma),\delta_\xi W_{12}].$$

The operator G of Eq. (22) consists of two parts,

$$G = G_\chi + G_x,$$

where

$$G_\chi = \int_\sigma d\sigma_\mu \chi \mathfrak{A}_\mu \delta\chi = -\int_\sigma d\sigma_\mu \delta\chi \mathfrak{A}_\mu \chi,$$

and

$$G_x = \int d\sigma_\mu T_{\mu\nu}\delta x_\nu = \epsilon_\nu P_\nu + \tfrac{1}{2}\epsilon_{\mu\nu}J_{\mu\nu}.$$

The latter form of G_x is a consequence of the restriction to plane space-like surfaces, limiting displacements to infinitesimal translations and rotations,

$$\delta x_\nu = \epsilon_\nu + \epsilon_{\mu\nu}x_\mu,$$

with the associated operators, the energy-momentum vector

$$P_\nu = \int d\sigma_\mu T_{\mu\nu},$$

and angular momentum tensor

$$J_{\mu\nu} = \int d\sigma_\lambda M_{\lambda\mu\nu},$$

$$M_{\lambda\mu\nu} = x_\mu T_{\lambda\nu} - x_\nu T_{\lambda\mu}.$$

The operator G_x evidently generates the infinitesimal transformation of an eigenvector, produced by the displacement of the surface to which it refers. With the notation

$$\delta_x\Psi(\zeta'\sigma) = (\epsilon_\nu\delta_\nu + \tfrac{1}{2}\epsilon_{\mu\nu}\delta_{\mu\nu})\Psi(\zeta'\sigma),$$

we have

$$i\delta_\nu\Psi(\zeta'\sigma) = P_\nu\Psi(\zeta'\sigma), \quad -i\delta_\nu\Psi(\zeta'\sigma)^\dagger = \Psi(\zeta'\sigma)^\dagger P_\nu,$$

and

$$i\delta_{\mu\nu}\Psi(\zeta'\sigma) = J_{\mu\nu}\Psi(\zeta'\sigma), \quad -i\delta_{\mu\nu}\Psi(\zeta'\sigma)^\dagger = \Psi(\zeta'\sigma)^\dagger J_{\mu\nu}.$$

If $F(\sigma)$ is an arbitrary function of dynamical variables on σ, and possibly of nondynamical parameters dependent on σ, we use the notation

$$\begin{aligned}\delta_x F(\sigma) &= (\epsilon_\nu\delta_\nu + \tfrac{1}{2}\epsilon_{\mu\nu}\delta_{\mu\nu})F(\sigma),\\ \partial_x F(\sigma) &= (\epsilon_\nu\partial_\nu + \tfrac{1}{2}\epsilon_{\mu\nu}\partial_{\mu\nu})F(\sigma),\end{aligned}$$

to distinguish between the total change on displacement, and that occasioned by the explicit appearance of nondynamical parameters. On referring to Eq. (3), we see that

$$\begin{aligned}\delta_\nu F(\sigma) &= \partial_\nu F(\sigma) + i[F(\sigma), P_\nu],\\ \delta_{\mu\nu}F(\sigma) &= \partial_{\mu\nu}F(\sigma) + i[F(\sigma), J_{\mu\nu}].\end{aligned}$$

The proper interpretation of the generating operator G_χ can be obtained by noting its equivalence with an appropriately chosen infinitesimal variation of the external source. Consider the following infinitesimal surface distribution on the negative side of σ,

$$\mathfrak{B}\delta\xi = \mathfrak{A}_{(0)}\delta\chi\delta(x_{(0)}), \tag{24}$$

which is not incompatible with the operator properties of these variations. We have assumed, for simplicity, that the equation of the surface σ is $x_{(0)}=0$. With this choice,

$$\delta_\xi W_{12} = \int_\sigma d\sigma \chi \mathfrak{A}_{(0)}\delta\chi = G_\chi.$$

The change that is produced in χ can be deduced from the variation of the field equatons,

$$2\mathfrak{A}_\mu\partial_\mu\delta_\xi\chi-\delta_\xi(\partial_l\mathfrak{H}/\partial\chi)=-\mathfrak{B}\delta\xi$$
$$=-\mathfrak{A}_{(0)}\delta\chi\delta(x_{(0)}).$$

Evidently there is a discontinuity in $\delta_\xi\chi$, on crossing the surface distribution $\delta\xi$, which is given by

$$2\mathfrak{A}_{(0)}\delta_\xi\chi]=-\mathfrak{A}_{(0)}\delta\chi.$$

In the retarded description, say, $\delta_\xi\chi$ is zero prior to the source bearing surface, so that the discontinuity in $\delta_\xi\chi$ is the change induced in χ on (the positive side of) σ. Thus, the surface variation of the external source simulates the transformation generated by G_χ, in which $\mathfrak{A}_{(0)}\chi$ on σ is replaced by

$$\begin{aligned}\mathfrak{A}_{(0)}\bar{\chi}&=\mathfrak{A}_{(0)}\chi+\mathfrak{A}_{(0)}\delta_\xi\chi\\&=\mathfrak{A}_{(0)}\chi-\tfrac{1}{2}\mathfrak{A}_{(0)}\delta\chi.\end{aligned}\tag{25}$$

The matrix $\mathfrak{A}_{(0)}$ has been retained in this statement since it is a singular matrix, in general. The number of components of χ that appear independently in (25) equals the rank of the matrix $\mathfrak{A}_{(0)}$, and this is the number of independent component field equations that are equations of motion, in that they contain time-like derivatives. The expression of (25) in terms of the generator G_χ is

$$[\mathfrak{A}_{(0)}\chi, G_\chi]=i\tfrac{1}{2}\mathfrak{A}_{(0)}\delta\chi.\tag{26}$$

The factor of $\frac{1}{2}$ that appears in this result stems from the treatment of all components of $\mathfrak{A}_{(0)}\chi$ on the same footing; we have not divided them into two sets of which one is fixed and the other varied.* If F is an arbitrary function of $\mathfrak{A}_{(0)}\chi$ on σ, we write

$$[F, G_\chi]=i(\delta F)_\chi=i\tfrac{1}{2}\delta F,$$

in which the components of $\mathfrak{A}_{(0)}\chi$ are the objects of variation. When the field equations that are equations of constraint prove sufficient to express all components of χ in terms of $\mathfrak{A}_{(0)}\chi$, we can extend (26) into

$$[\chi, G_\chi]=i\tfrac{1}{2}\delta\chi.$$

Of course, one must distinguish between these variations, in which only the $\mathfrak{A}_{(0)}\chi$ are independent, and the independent variations of all components of χ which produce the equations of constraint from the action principle.

In order to facilitate the explicit construction of the field commutation relations, we shall introduce a reducibility hypothesis, which is associated with the Lorentz invariant process of separating the matrices $\mathfrak{A}_\mu$, $\mathfrak{B}$ into symmetrical and antisymmetrical parts. We require that the field and the source decompose into two sets, of the first kind $\chi^{(1)}=\phi$, $\xi^{(1)}=\zeta$, and of the second kind, $\chi^{(2)}=\psi$, $\xi^{(2)}=\eta$, as a concomitant of the decomposition

$$\mathfrak{A}_\mu=\mathfrak{A}_\mu^{(1)}+\mathfrak{A}_\mu^{(2)},\qquad \mathfrak{B}=\mathfrak{B}^{(1)}+\mathfrak{B}^{(2)},$$
$$\mathfrak{A}_\mu^{(1)\,\mathrm{tr}}=-\mathfrak{A}_\mu^{(1)},\qquad \mathfrak{B}^{(1)\,\mathrm{tr}}=\mathfrak{B}^{(1)},$$
$$\mathfrak{A}_\mu^{(2)\,\mathrm{tr}}=\mathfrak{A}_\mu^{(2)},\qquad \mathfrak{B}^{(2)\,\mathrm{tr}}=-\mathfrak{B}^{(2)}.$$

The matrices of the first kind are real ($\mu=0,\cdots 3$), and those of the second kind are imaginary. We shall not write the distinguishing index when no confusion is possible.

According to this reducibility hypothesis, the field equations in the two equivalent forms

$$2\mathfrak{A}_\mu\partial_\mu\chi=(\partial_l\mathfrak{H}/\partial\chi)-\mathfrak{B}\xi,$$
$$-2\mathfrak{A}_\mu^{\mathrm{tr}}\partial_\mu\chi=(\partial_r\mathfrak{H}/\partial\chi)-\mathfrak{B}^{\mathrm{tr}}\xi,$$

separate into the two sets

$$2\mathfrak{A}_\mu\partial_\mu\phi=(\partial\mathfrak{H}/\partial\phi)-\mathfrak{B}\zeta,\quad(\partial_l\mathfrak{H}/\partial\phi)=(\partial_r\mathfrak{H}/\partial\phi),$$

and

$$2\mathfrak{A}_\mu\partial_\mu\psi=(\partial_l\mathfrak{H}/\partial\psi)-\mathfrak{B}\eta,\quad(\partial_l\mathfrak{H}/\partial\psi)=-(\partial_r\mathfrak{H}/\partial\psi).$$

Furthermore, the generator

$$G_\chi=\int d\sigma\chi\mathfrak{A}_{(0)}\delta\chi=\int d\sigma(-\mathfrak{A}_{(0)}{}^{\mathrm{tr}}\delta\chi)\chi,$$

decomposes into $G_\phi+G_\psi$, where

$$G_\phi=\int d\sigma\phi\mathfrak{A}_{(0)}\delta\phi=\int d\sigma(\mathfrak{A}_{(0)}\delta\phi)\phi,$$

and

$$G_\psi=\int d\sigma\psi\mathfrak{A}_{(0)}\delta\psi=\int d\sigma(-\mathfrak{A}_{(0)}\delta\psi)\psi.\tag{27}$$

These results reflect the form assumed by the Lagrange function,

$$\mathfrak{L}=\tfrac{1}{2}\{\phi\mathfrak{A}_\mu,\partial_\mu\phi\}+\tfrac{1}{2}[\psi\mathfrak{A}_\mu,\partial_\mu\psi]-\mathfrak{H}(\phi,\psi)$$
$$+\tfrac{1}{2}\{\zeta\mathfrak{B},\phi\}+\tfrac{1}{2}[\eta\mathfrak{B},\psi].$$

The equivalence between left and right derivatives of the arbitrary function $\mathfrak{H}$, with respect to field components of the first kind, and of the two expressions for G_ϕ, shows that $\delta\phi$ commutes with all fields at the same point. It is compatible with the field equations to extend this statement to fields at arbitrary points,

$$[\phi(x),\delta\phi(x')]=[\psi(x),\delta\phi(x')]=0,$$

provided the source components are included,

$$[\zeta(x),\delta\phi(x')]=[\eta(x),\delta\phi(x')]=0.$$

It follows from (27) that the relation between ψ and $\delta\psi$ is one of anticommutivity. The opposite signs of the left and right derivatives of $\mathfrak{H}$ with respect to ψ is then accounted for by

$$[\phi(x),\delta\psi(x')]=\{\psi(x),\delta\psi(x')\}=0,$$

provided only that $\mathfrak{H}$ is an even function of the vari-

* *Note added in proof:*—Further discussion of this point will be found in a paper submitted to the *Philosophical Magazine.*

ables of the second kind. The inclusion of the source components

$$[\zeta(x), \delta\psi(x')] = \{\eta(x), \delta\psi(x')\} = 0,$$

insures compatibility with the field equations. We have now obtained the explicit characterization of the class of variations to which our fundamental postulate refers.

Let us also notice that

$$\delta_\xi W_{12} = \int_{\sigma_2}^{\sigma_1} (dx)\chi\mathfrak{B}\delta\xi = \int_{\sigma_2}^{\sigma_1} (dx)(\mathfrak{B}^{\mathrm{tr}}\delta\xi)\chi,$$

decomposes into $\delta_\zeta W_{12} + \delta_\eta W_{12}$, where

$$\delta_\zeta W_{12} = \int_{\sigma_2}^{\sigma_1} (dx)\phi\mathfrak{B}\delta\zeta = \int_{\sigma_2}^{\sigma_1} (dx)(\mathfrak{B}\delta\zeta)\phi,$$

and

$$\delta_\eta W_{12} = \int_{\sigma_2}^{\sigma_1} (dx)\psi\mathfrak{B}\delta\eta = \int_{\sigma_2}^{\sigma_1} (dx)(-\mathfrak{B}\delta\eta)\psi.$$

We can conclude that source variations have the same operator properties as field variations, as already exploited in Eq. (24).

The operator properties of $\mathfrak{A}_{(0)}\chi$ on a given σ can now be deduced from (26), with the results

$$\begin{aligned} &[\mathfrak{A}_{(0)}\phi(x), \phi(x')\mathfrak{A}_{(0)}] = i\tfrac{1}{2}\mathfrak{A}_{(0)}\delta_\sigma(x-x'), \\ &[\mathfrak{A}_{(0)}\phi(x), \psi(x')\mathfrak{A}_{(0)}] = 0, \\ &\{\mathfrak{A}_{(0)}\psi(x), \psi(x')\mathfrak{A}_{(0)}\} = i\tfrac{1}{2}\mathfrak{A}_{(0)}\delta_\sigma(x-x'), \end{aligned} \tag{28}$$

in which $\delta_\sigma(x-x')$ is the three-dimensional delta function appropriate to the surface σ. The numerical forms of these commutators and anticommutators insures their consistency with the operator properties of $\delta\mathfrak{A}_{(0)}\phi$ and $\delta\mathfrak{A}_{(0)}\psi$. The dynamical variables of the first and second kind thus describe Bose-Einstein and Fermi-Dirac fields, respectively, which are unified in the general field χ.

Since the rank of the antisymmetrical matrix $\mathfrak{A}_{(0)}{}^{(1)}$ is necessarily even, there are an even number of independent field components of the first kind, say $2n^{(1)}$. One can always arrange the matrix $\mathfrak{A}_{(0)}{}^{(1)}$ so that all elements are zero beyond the first $2n^{(1)}$ rows and columns. We shall denote this nonsingular submatrix of dimensionality $2n^{(1)}$ by $\boldsymbol{\mathfrak{A}}_{(0)}{}^{(1)}$, and the associated independent components of ϕ by $\boldsymbol{\phi}$. The first commutation relation of (28) can then be written

$$[\boldsymbol{\phi}(x), \boldsymbol{\phi}(x')] = i\tfrac{1}{2}\boldsymbol{\mathfrak{A}}_{(0)}{}^{-1}\delta_\sigma(x-x').$$

The matrix $\mathfrak{B}^{(2)}$, associated with Fermi-Dirac fields, is antisymmetrical and nonsingular. Hence the total number of field components of the second kind is even. If we allow for the possibility that $\mathfrak{A}_{(0)}{}^{(2)}$ may be singular, and arrange the rows and columns so that the nonsingular submatrix $\boldsymbol{\mathfrak{A}}_{(0)}{}^{(2)}$ is associated with the independent components $\boldsymbol{\psi}$, we obtain

$$\{\boldsymbol{\psi}(x), \boldsymbol{\psi}(x')\} = i\tfrac{1}{2}\boldsymbol{\mathfrak{A}}_{(0)}{}^{-1}\delta_\sigma(x-x'),$$

which requires that the real, symmetrical matrix $i\boldsymbol{\mathfrak{A}}_{(0)}{}^{(2)-1}$ be positive definite.

We shall argue that the number of independent field components of the second kind, the dimensionality of $\boldsymbol{\mathfrak{A}}_{(0)}{}^{(2)}$, must be even, $2n^{(2)}$. Let us imagine that, by a suitable real transformation, $\boldsymbol{\mathfrak{A}}_{(0)}{}^{(2)}$ is brought into diagonal form. If the number of components in $\boldsymbol{\psi}$ is odd, the product of all these components at a given point commutes with $\boldsymbol{\psi}$ at that point. Thus, as far as the algebra of operators at a given point is concerned, this product is a multiple of the unit operator (the necessary commutivity with $\boldsymbol{\psi}$ at other points on σ can always be achieved), which contradicts the assumption that all components of $\boldsymbol{\psi}$ are independent.

The relation between invariance under time reflection, and the connection between spin and statistics, may be noted here. The time reflection transformation

$$'x_4 = -x_4, \quad 'x_k = x_k,$$

induces a transformation of the field

$$'\chi = L_4\chi,$$

such that

$$L_4{}^{\mathrm{tr}}\mathfrak{A}_4 L_4 = -\mathfrak{A}_4, \quad L_4{}^{\mathrm{tr}}\mathfrak{A}_k L_4 = \mathfrak{A}_k, \tag{29}$$

and

$$L_4{}^{\mathrm{tr}}\mathfrak{B}L_4 = \mathfrak{B}, \quad \mathcal{H}(L_4\chi) = \mathcal{H}(\chi).$$

However, this preservation of the form of the Lagrange function is only apparent, for fields of the second kind. Since $-i\mathfrak{A}_{(0)}{}^{(2)}$ is a non-negative matrix, one can only satisfy the first equation of (29) with an imaginary $L_4{}^{(2)}$ which produces skew-Hermitian field components $'\chi^{(2)}$. But the invariance of the Lagrange function is not the correct criterion for invariance under time reflection. The reversal of the time sense inverts the order of σ_1 and σ_2, and thus introduces a minus sign in the action integral, which can only be compensated by changing the sign of i in (4). We shall describe this as a transformation from the algebra of the operators χ to the complex conjugate algebra of operators χ^*. Since the linear transformation designed to maintain the form of $\mathcal{L}(\phi, \partial_\mu\phi; \psi, \partial_\mu\psi)$ has effectively replaced $\mathcal{L}$ with $\mathcal{L}(\phi, \partial_\mu\phi; i\psi, i\partial_\mu\psi)$, the criterion for invariance reads

$$\mathcal{L}(\phi, \partial_\mu\phi; i\psi, i\partial_\mu\psi)^* = \mathcal{L}(\phi^*, \partial_\mu\phi^*; \psi^*, \partial_\mu\psi^*).$$

The derivative term in $\mathcal{L}$ is indeed invariant since the matrices $\mathfrak{A}_\mu{}^{(1)}$ and $\mathfrak{A}_\mu{}^{(2)}$ are real and imaginary, respectively. We describe this by saying that the theory is kinematically invariant under time reflection. In order that it be dynamically invariant, $\mathcal{H}$ must be such that

$$\mathcal{H}(\phi, i\psi)^* = \mathcal{H}(\phi^*, \psi^*).$$

Since $\mathcal{H}$ is an even function of the components of ψ, the latter are to be paired with the aid of imaginary matrices, characteristic of the variables of the second kind. The source term is invariant if source and field transform in the same way.

The correlation between spin and statistics enters on observing that an imaginary L_4 is characteristic of half-integral spin fields. We can prove this by remarking that all the transformation properties of L_4 are satisfied by

$$L_4=\exp(-\tfrac{1}{2}\pi iS_{14})L_1\exp(\tfrac{1}{2}\pi iS_{14})=\exp(-\pi iS_{14})L_1,$$

where L_1 is the matrix describing the reflection of the first space axis. The latter form is a consequence of

$$L_1^{-1}S_{14}L_1=-S_{14}.$$

The essential point with regard to the reality of L_4 is that $S_{14}=iS_{10}$ is a real matrix, whence

$$L_4^*=\exp(\pi iS_{14})L_1=\exp(2\pi iS_{14})L_4.$$

Now S_{14} must possess the same eigenvalues as S_{12}, say, which implies that L_4 is real for an integral spin field, and imaginary for a half-integral spin field. The requirement of time reflection invariance thus restricts fields of the first (B.E.) and second (F.D.) kind to integral and half-integral spins, respectively. This correlation is also satisfactory in that it identifies the double-valued, half-integral spin fields with fields of the second kind, of which $\mathfrak{L}$ is an even function.

We have introduced several kinds of generators of infinitesimal transformations. A criterion for consistency is obtained from the alternative evaluations of the commutator of two such generators,

$$[G_a, G_b]=i(\delta G_a)_b=-i(\delta G_b)_a,$$

namely

$$(\delta G_a)_b+(\delta G_b)_a=0.$$

As a first example, we consider the two generators

$$G_x=\epsilon_\nu P_\nu(\sigma_1)+\tfrac{1}{2}\epsilon_{\mu\nu}J_{\mu\nu}(\sigma_1),$$

and

$$G_\xi=\int_{\sigma_2}^{\sigma_1}(dx)\chi\mathfrak{B}\delta\xi,$$

in the retarded description. In preparation for the test, we remark that

$$P_\nu(\sigma_1)-P_\nu(\sigma_2)=\int_{\sigma_2}^{\sigma_1}(dx)\partial_\mu T_{\mu\nu}$$
$$=\int_{\sigma_2}^{\sigma_1}(dx)\tfrac{1}{2}(\chi\mathfrak{B}\partial_\nu\xi+\partial_\nu\xi\mathfrak{B}\chi),$$

and that

$$J_{\mu\nu}(\sigma_1)-J_{\mu\nu}(\sigma_2)=\int_{\sigma_2}^{\sigma_1}(dx)\partial_\lambda M_{\lambda\mu\nu}$$
$$=\int_{\sigma_2}^{\sigma_1}(dx)[x_\mu\partial_\lambda T_{\lambda\nu}-x_\nu\partial_\lambda T_{\lambda\mu}+T_{\mu\nu}-T_{\nu\mu}].$$

Since

$$T_{\mu\nu}-T_{\nu\mu}=-i\tfrac{1}{2}(\xi\mathfrak{B}S_{\mu\nu}\chi-\chi S_{\mu\nu}{}^\dagger\mathfrak{B}\xi),$$

we have

$$J_{\mu\nu}(\sigma_1)-J_{\mu\nu}(\sigma_2)=\int_{\sigma_2}^{\sigma_1}(dx)\tfrac{1}{2}[\chi\mathfrak{B}(x_\mu\partial_\nu-x_\nu\partial_\mu+iS_{\mu\nu})\xi+(x_\mu\partial_\nu-x_\nu\partial_\mu+iS_{\mu\nu})\xi\mathfrak{B}\chi].$$

In the absence of an external source, $T_{\mu\nu}$ is symmetrical and divergenceless, and P_ν, $J_{\mu\nu}$ are conserved. For simplicity, we shall confine our verification to the situation of no source, in which the infinitesimal $\delta\xi$ is distributed in the region between σ_1 and σ_2. Hence

$$\delta_\xi P_\nu(\sigma_1)=-\int_{\sigma_2}^{\sigma_1}(dx)\partial_\nu\chi\mathfrak{B}\delta\xi,$$

and

$$\delta_\xi J_{\mu\nu}(\sigma_1)=-\int_{\sigma_2}^{\sigma_1}(dx)(x_\mu\partial_\nu-x_\nu\partial_\mu+iS_{\mu\nu})\chi\mathfrak{B}\delta\xi.$$

The consistency requirement

$$(\delta G_\xi)_x=\int_{\sigma_2}^{\sigma_1}(dx)(\delta\chi)_x\mathfrak{B}\delta\xi=\delta_\xi G_x,$$

then demands that

$$-(\delta\chi)_x=\epsilon_\nu\partial_\nu\chi+\tfrac{1}{2}\epsilon_{\mu\nu}(x_\mu\partial_\nu-x_\nu\partial_\mu+iS_{\mu\nu})\chi, \tag{30}$$

which is indeed true in virtue of the equivalence between $(\delta\chi(x))_x$, induced by the displacement δx_μ, and $'\chi(x)-\chi(x)$, induced by the coordinate transformation $'x_\mu=x_\mu+\delta x_\mu$.

Alternative forms of P_ν and $J_{\mu\nu}$ are convenient for testing the consistency of G_x and G_χ. The following relations derived from (16),

$$\chi\mathfrak{A}_\nu\partial_\mu\chi-\chi\mathfrak{A}_\mu\partial_\nu\chi=i\chi(\mathfrak{A}_\lambda S_{\mu\nu}-S_{\mu\nu}{}^\dagger\mathfrak{A}_\lambda)\partial_\lambda\chi,$$
$$\partial_\mu\chi\mathfrak{A}_\nu\chi-\partial_\nu\chi\mathfrak{A}_\mu\chi=i\partial_\lambda\chi(\mathfrak{A}_\lambda S_{\mu\nu}-S_{\mu\nu}{}^\dagger\mathfrak{A}_\lambda)\chi,$$

enable us to write $T_{\mu\nu}$ as

$$T_{\mu\nu}=\mathfrak{L}\delta_{\mu\nu}-\tfrac{1}{2}(\chi\mathfrak{A}_\mu\partial_\nu\chi-\partial_\nu\chi\mathfrak{A}_\mu\chi)+\partial_\lambda s_{\lambda\mu\nu}+\rho_{\mu\nu},$$

where

$$s_{\lambda\mu\nu}=-s_{\mu\lambda\nu}=i\tfrac{1}{4}\chi(2\mathfrak{A}_{(\mu}S_{\lambda\nu)}+2S_{\lambda(\nu}{}^\dagger\mathfrak{A}_{\mu)}-\mathfrak{A}_\lambda S_{\mu\nu}-S_{\mu\nu}{}^\dagger\mathfrak{A}_\lambda)\chi,$$

and

$$\rho_{\mu\nu}=-i\tfrac{1}{4}[S_{\mu\nu}\chi(\partial_l\mathcal{H}/\partial\chi)+(\partial_r\mathcal{H}/\partial\chi)S_{\mu\nu}\chi].$$

In virtue of the antisymmetry of $s_{\lambda\mu\nu}$ in the first two indices, $\partial_\lambda s_{\lambda\mu\nu}$ is automatically divergenceless and does not contribute to the energy-momentum vector P_ν,

$$P_\nu=\int d\sigma_\mu[\mathfrak{L}\delta_{\mu\nu}-\tfrac{1}{2}(\chi\mathfrak{A}_\mu\partial_\nu\chi-\partial_\nu\chi\mathfrak{A}_\mu\chi)+\rho_{\mu\nu}],$$

but does enter in

$$J_{\mu\nu}=\int d\sigma_\lambda[-\tfrac{1}{2}\chi\mathfrak{A}_\lambda(x_\mu\partial_\nu-x_\nu\partial_\mu+iS_{\mu\nu})\chi+\tfrac{1}{2}(x_\mu\partial_\nu-x_\nu\partial_\mu+iS_{\mu\nu})\chi\mathfrak{A}_\lambda\chi+x_\mu\rho_{\lambda\nu}-x_\nu\rho_{\lambda\mu}]+\int(d\sigma_\nu x_\mu-d\sigma_\mu x_\nu)\mathfrak{L}.$$

The components of P_ν in a local coordinate system are

$$P_{(0)}=\int d\sigma[\mathcal{H}-\chi\mathfrak{A}_{(k)}\partial_{(k)}\chi-\tfrac{1}{2}(\xi\mathfrak{B}\chi+\chi\mathfrak{B}\xi)],$$
$$P_{(k)}=\int d\sigma[-\chi\mathfrak{A}_{(0)}\partial_{(k)}\chi+\rho_{(0)(k)}], \tag{31}$$

while those of $J_{\mu\nu}$ are

$$J_{(0)(k)}=x_{(0)}P_{(k)}-\int d\sigma x_{(k)}[\mathcal{H}-\tfrac{1}{2}(\chi\mathfrak{A}_{(l)}\partial_{(l)}\chi-\partial_{(l)}\chi\mathfrak{A}_{(l)}\chi)-\tfrac{1}{2}(\xi\mathfrak{B}\chi+\chi\mathfrak{B}\xi)]-\tfrac{1}{2}i\int d\sigma\chi(\mathfrak{A}_{(0)}S_{(0)(k)}+S_{(0)(k)}{}^{\dagger}\mathfrak{A}_{(0)})\chi, \tag{32}$$

$$J_{(k)(l)}=\int d\sigma[-\chi\mathfrak{A}_{(0)}(x_{(k)}\partial_{(l)}-x_{(l)}\partial_{(k)}+iS_{(k)(l)})\chi+x_{(k)}\rho_{(0)(l)}-x_{(l)}\rho_{(0)(k)}].$$

The quantity $\rho_{\mu\nu}$ is closely related to the infinitesimal expression of the scalar character of $\mathcal{H}$,

$$\mathcal{H}(\chi-i\tfrac{1}{2}\epsilon_{\mu\nu}S_{\mu\nu}\chi)-\mathcal{H}(\chi)=0.$$

We can, indeed, conclude that

$$\rho_{\mu\nu}=0,$$

if $\mathcal{H}$ is no more than quadratic in the components of various independent fields. We shall also prove this without the latter restriction, but, for simplicity, with the limitation that there are no equations of constraint. The commutation relations equivalent to (30),

$$[\chi, P_\nu]=-i\partial_\nu\chi,$$
$$[\chi, J_{\mu\nu}]=-i(x_\mu\partial_\nu-x_\nu\partial_\mu+iS_{\mu\nu})\chi,$$

imply that

$$[\chi, N_{\mu\nu}]=S_{\mu\nu}\chi,$$

where

$$N_{\mu\nu}=J_{\mu\nu}-x_\mu P_\nu+x_\nu P_\mu.$$

This enables one to express the scalar requirement on $\mathcal{H}$ in the form

$$[\mathcal{H}, N_{\mu\nu}]=0.$$

The components

$$N_{(0)(k)}=\int d\sigma'(x_{(k)}-x_{(k)}')[\mathcal{H}(x')-\tfrac{1}{2}(\chi\mathfrak{A}_{(l)}\partial_{(l)}'\chi-\partial_{(l)}'\chi\mathfrak{A}_{(l)}\chi)-\tfrac{1}{2}(\xi\mathfrak{B}\chi+\chi\mathfrak{B}\xi)]-\tfrac{1}{2}i\int d\sigma\chi(\mathfrak{A}_{(0)}S_{(0)(k)}+S_{(0)(k)}{}^{\dagger}\mathfrak{A}_{(0)})\chi,$$

do not involve the unknown $\rho_{(0)(k)}$. According to our simplifying assumption of no constraint equations, the commutators (anticommutators) of all field components at x and x' contain the three-dimensional delta function $\delta_\sigma(x-x')$ and therefore vanish when multiplied by $x_{(k)}-x_{(k)}'$. Furthermore,

$$[\mathcal{H}(x), \chi(x')]\mathfrak{A}_{(0)}=\tfrac{1}{2}i(\partial_r\mathcal{H}/\partial\chi)\delta_\sigma(x-x'),$$

and

$$\mathfrak{A}_{(0)}[\chi(x'), \mathcal{H}(x)]=\tfrac{1}{2}i(\partial_l\mathcal{H}/\partial\chi)\delta_\sigma(x-x'),$$

from which we obtain

$$[\mathcal{H}, N_{(0)(k)}]=2i\rho_{(0)(k)}=0.$$

With this information, the proof is easily extended to all components of $\rho_{\mu\nu}$.

The consistency of the generators G_x and G_χ requires that

$$\tfrac{1}{2}\delta_\chi(\epsilon_\nu P_\nu+\tfrac{1}{2}\epsilon_{\mu\nu}J_{\mu\nu})=-\int d\sigma(\delta\chi)_x\mathfrak{A}_{(0)}\delta\chi,$$

or

$$\delta_\chi P_\nu=\int d\sigma\partial_\nu\chi 2\mathfrak{A}_{(0)}\delta\chi,$$

$$\delta_\chi J_{\mu\nu}=\int d\sigma(x_\mu\partial_\nu-x_\nu\partial_\mu+iS_{\mu\nu})\chi 2\mathfrak{A}_{(0)}\delta\chi,$$

which can now be verified from the expressions (31) and (32), with $\rho_{(0)(k)}=0$.

CHARGED FIELDS

Our considerations thus far specifically exclude the electromagnetic field (and the gravitational field). We introduce the concept of charge by requiring that the Lagrange function be invariant under constant phase (special gauge) transformations, the infinitesimal version of which is

$$'\chi=(1-i\delta\lambda\mathcal{E})\chi.$$

Here $\delta\lambda$ is a constant, and $\mathcal{E}$ is an imaginary matrix which can be viewed as a rotation matrix referring to a space other than the four-dimensional world. The invariance requirement implies that

$$\mathcal{E}^{\dagger}=\mathfrak{B}\mathcal{E}\mathfrak{B}^{-1},$$

or

$$(\mathfrak{B}\mathcal{E})^{\dagger}=\mathfrak{B}\mathcal{E},$$

and

$$[\mathcal{E}, \mathfrak{B}^{-1}\mathfrak{A}_\mu]=[\mathcal{E}, S_{\mu\nu}]=0,$$

and that

$$\mathcal{H}(\chi-i\delta\lambda\mathcal{E}\chi)-\mathcal{H}(\chi)=0.$$

We now write the general variation as

$$\delta(\chi)=\delta\chi-i\tfrac{1}{2}(\partial_\mu\delta x_\nu)S_{\mu\nu}\chi-i\delta\lambda\mathcal{E}\chi,$$

where $\delta\lambda$, characterizing a local phase transformation, is an arbitrary function of x, consistent with constant values on σ_1 and on σ_2. The additional contribution to $\delta(\mathfrak{L})$ thereby produced is

$$j_\mu\partial_\mu\delta\lambda-i\tfrac{1}{2}(\xi\mathfrak{B}\mathcal{E}\chi-\chi\mathfrak{B}\mathcal{E}\xi)\delta\lambda,$$

where

$$j_\mu=-i\chi\mathfrak{A}_\mu\mathcal{E}\chi$$

is the charge-current vector. The stationary action principle requires that

$$\partial_\mu j_\mu = -i\tfrac{1}{2}(\xi\mathfrak{B}\mathcal{E}\chi - \chi\mathfrak{B}\mathcal{E}\xi), \tag{33}$$

and yields as the phase transformation generator

$$G_\lambda = \int_\sigma d\sigma_\mu j_\mu \delta\lambda = Q\delta\lambda,$$

where Q is the charge operator.

The integral statement derived from (33),

$$Q(\sigma_1) - Q(\sigma_2) = \int_{\sigma_2}^{\sigma_1} (dx) i\tfrac{1}{2}(\chi\mathfrak{B}\mathcal{E}\xi - \xi\mathfrak{B}\mathcal{E}\chi),$$

becomes the conservation of charge in the absence of an external source. If an infinitesimal source is introduced in the region bounded by σ_1 and σ_2, we then have, in the retarded description,

$$\delta_\xi Q(\sigma_1) = -i\int_{\sigma_2}^{\sigma_1} (dx)\delta\xi\mathfrak{B}\mathcal{E}\chi$$

$$= i[Q(\sigma_1), G_\xi],$$

whence

$$[\chi, Q] = \mathcal{E}\chi.$$

This commutation relation also follows directly from the significance of G_λ, indicating the consistency of the latter with G_ξ.

We shall suppose that the matrix $\mathfrak{B}$ is an element of the algebra generated by $\mathfrak{B}^{-1}\mathfrak{A}_\mu$ and $S_{\mu\nu}$. It follows that $\mathfrak{B}$ commutes with $\mathcal{E}$, and therefore that the latter is explicitly Hermitian,

$$\mathcal{E}^\dagger = \mathcal{E}.$$

Such an antisymmetrical, imaginary matrix possesses real eigenvalues which are symmetrically distributed about zero; nonvanishing eigenvalues occur in oppositely signed pairs. Since $\mathcal{E}$ commutes with all members of the above-mentioned algebra, the charge-bearing character of a given field depends upon the reducibility of this algebra. Thus, if the algebra for a certain kind of field is irreducible, the only matrix commuting with all members of the algebra is the symmetrical unit matrix. Hence $\mathcal{E}=0$, and the field is electrically neutral. If, however, the matrix algebra is reducible to two similar algebras, as in

$$\mathfrak{A}_\mu = \begin{pmatrix} \mathfrak{A}_\mu & 0 \\ 0 & \mathfrak{A}_\mu \end{pmatrix}, \tag{34}$$

the matrix $\mathcal{E}$ exists and has the form (with the same partitioning)

$$\mathcal{E} = e\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{35}$$

This describes a charged field, composed of particles with charges $\pm e$, the eigenvalues of $\mathcal{E}$. If three similar algebras are involved, the field contains particles with charges $0, \pm e$.

To present $\mathcal{E}$ as a diagonal matrix, we must forego the choice of Hermitian field components. Thus, for the example of a charged F.D. field, where the field components decompose into $\psi_{(1)}, \psi_{(2)}$, corresponding to the structures (34) and (35), the mutually Hermitian conjugate operators

$$\psi_{(+)} = \psi_{(1)} - i\psi_{(2)}, \quad \psi_{(-)} = \psi_{(1)} + i\psi_{(2)},$$

are associated with eigenvalues $+e$ and $-e$, respectively. On introducing these field components, the derivative term in the Lagrange function, the electric current vector, and the commutation relations, respectively, read

$$\tfrac{1}{4}[\psi_{(-)}\mathfrak{A}_\mu, \partial_\mu\psi_{(+)}] + \tfrac{1}{4}[\psi_{(+)}\mathfrak{A}_\mu, \partial_\mu\psi_{(-)}], \tag{36}$$

$$-ie\tfrac{1}{2}(\psi_{(-)}\mathfrak{A}_\mu\psi_{(+)} - \psi_{(+)}\mathfrak{A}_\mu\psi_{(-)}), \tag{37}$$

and

$$\begin{aligned} \{\mathfrak{A}_{(0)}\psi_{(+)}(x), \psi_{(+)}(x')\mathfrak{A}_{(0)}\} &= \{\mathfrak{A}_{(0)}\psi_{(-)}(x), \psi_{(-)}(x')\mathfrak{A}_{(0)}\} = 0, \\ \{\mathfrak{A}_{(0)}\psi_{(+)}(x), \psi_{(-)}(x')\mathfrak{A}_{(0)}\} &= \{\mathfrak{A}_{(0)}\psi_{(-)}(x), \psi_{(+)}(x')\mathfrak{A}_{(0)}\} = i\mathfrak{A}_{(0)}\delta_\sigma(x-x'). \end{aligned} \tag{38}$$

There is evident symmetry with respect to the substitution $\psi_{(+)} \leftrightarrow \psi_{(-)}$, $e \leftrightarrow -e$.

Since $\psi_{(+)}$ and $\psi_{(-)}$ are Hermitian conjugate operators, we can arbitrarily select one as the primary non-Hermitian field. We shall write

$$\mathfrak{B}^{-1}\mathfrak{A}_\mu = i\gamma_\mu,$$

and

$$\psi_{(+)} = \psi, \quad \psi_{(-)}\mathfrak{B} = \psi^\dagger\mathfrak{B} = \bar{\psi}.$$

This yields the following forms for (36), (37), and (38):

$$\begin{aligned} &\tfrac{1}{4}[\bar{\psi}\gamma_\mu, i\partial_\mu\psi] - \tfrac{1}{4}[i\partial_\mu\bar{\psi}\gamma_\mu, \psi], \\ &e\tfrac{1}{2}[\bar{\psi}\gamma_\mu, \psi], \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\{\gamma_{(0)}\psi(x), \gamma_{(0)}\psi(x')\} = \{\bar{\psi}(x)\gamma_{(0)}, \bar{\psi}(x')\gamma_{(0)}\} = 0, \\ &\{\gamma_{(0)}\psi(x), \bar{\psi}(x')\gamma_{(0)}\} = \gamma_{(0)}\delta_\sigma(x-x'). \end{aligned} \tag{40}$$

To express the now slightly obscured symmetry between positive and negative charge, we call $\psi_{(-)}$ the charge conjugate field

$$\psi^c = (-\mathfrak{B}^{-1})\bar{\psi}, \tag{41}$$

and state this symmetry as invariance under the substitution $\psi \leftrightarrow \psi^c$, $e \leftrightarrow -e$.

The matrices γ_μ; $\mu = 0, \cdots 3$, obey

$$\gamma_\mu{}^\dagger = \mathfrak{B}\gamma_\mu\mathfrak{B}^{-1},$$

and

$$\gamma_\mu{}^{\mathrm{tr}} = -\mathfrak{B}\gamma_\mu\mathfrak{B}^{-1}, \tag{42}$$

since they are purely imaginary matrices. One should also recall that $\mathfrak{B}$ is an antisymmetrical, imaginary matrix. If we were to depart from these special structures by subjecting all matrices to an arbitrary unitary transformation, we should find that the only formal changes occur in (41) and (42), where the matrix $\mathfrak{B}$ appears modified by an orthogonal, rather than a unitary transformation. Hence, in a general representation these equations read

$$\psi^c = C\bar{\psi},$$
$$\gamma_\mu{}^{\mathrm{tr}} = -C^{-1}\gamma_\mu C,$$

where C still exhibits the symmetry of $\mathfrak{B}$, appropriate to the example of a half-integral spin field,

$$C^{\mathrm{tr}} = -C.$$

The commutation relations (40) are in the canonical form which corresponds to the division of the independent field components into two sets, such that one has vanishing anticommutators (commutators, for an integral spin field) among members of the same set. The generator of changes in ψ and $\bar{\psi}$, Eq. (27) in the notation of the charged half-integral spin field example, is

$$G(\psi, \bar{\psi}) = \tfrac{1}{2}i\int d\sigma(\bar{\psi}\gamma_{(0)}\delta\psi - \delta\bar{\psi}\gamma_{(0)}\psi),$$

which can be deduced directly from the Lagrange function derivative term (39). Associated with the freedom of altering the Lagrange function by the addition of a divergence, are various expressions for generating operators of changes in the field components. Thus, we have the following two simple possibilities for the derivative term and the associated generating operator,

$$\tfrac{1}{2}[\bar{\psi}\gamma_\mu, i\partial_\mu\psi],$$
$$G(\psi) = i\int d\sigma\bar{\psi}\gamma_{(0)}\delta\psi,$$

and

$$-\tfrac{1}{2}[i\partial_\mu\bar{\psi}\gamma_\mu, \psi],$$
$$G(\bar{\psi}) = -i\int d\sigma\delta\bar{\psi}\gamma_{(0)}\psi.$$

Evidently $G(\psi)$, for example, in the generator of alterations in the components $\gamma_{(0)}\psi$, with no change in $\bar{\psi}\gamma_{(0)}$. The associated commutation relations,

$$[\gamma_{(0)}\psi, G(\psi)] = i\gamma_{(0)}\delta\psi,$$
$$[\bar{\psi}\gamma_{(0)}, G(\psi)] = 0,$$

are satisfied in virtue of (40), and, conversely, in conjunction with the analogous statements for $G(\bar{\psi})$, imply these operator properties of the field components. The connection with the generator in the symmetrical treatment of all field components is given by

$$G(\psi, \bar{\psi}) = \tfrac{1}{2}G(\psi) + \tfrac{1}{2}G(\bar{\psi}),$$

which indicates the origin of the factor (1/2) in the general Eq. (26).

THE ELECTROMAGNETIC FIELD

The postulate of general gauge invariance motivates the introduction of the electromagnetic field. If all fields and sources are subjected to the general gauge transformation,

$$'\chi = \exp(-i\lambda(x)\mathcal{E})\chi = \chi\exp(i\lambda(x)\mathcal{E}),$$

the Lagrange function we have been considering alters in the following manner,

$$'\mathfrak{L} = \mathfrak{L} + j_\mu\partial_\mu\lambda.$$

The addition of the electromagnetic field Lagrange function,

$$\mathfrak{L}_{\mathrm{emf}} = \tfrac{1}{2}\{j_\mu, A_\mu\} - \tfrac{1}{4}\{F_{\mu\nu}, \partial_\mu A_\nu - \partial_\nu A_\mu\} + \tfrac{1}{4}F_{\mu\nu}{}^2 + J_\mu A_\mu, \quad (43)$$

provides a compensating quantity through the associated gauge transformation

$$'A_\mu = A_\mu - \partial_\mu\lambda.$$

The term involving the external current J_μ is effectively gauge invariant if

$$\partial_\mu J_\mu = 0,$$

since the modification is in the form of a divergence. In the same sense, there is no objection to employing a form of the Lagrange function in which the second term of (43) is replaced by

$$\tfrac{1}{2}\{\partial_\mu F_{\mu\nu}, A_\nu\}. \quad (44)$$

We write the general variation of A_μ in the form

$$\delta(A_\mu) = \delta A_\mu - (\partial_\mu\delta x_\nu)A_\nu$$
$$= \delta A_\mu - \tfrac{1}{2}(\partial_\mu\delta x_\nu - \partial_\nu\delta x_\mu)A_\nu - \tfrac{1}{2}(\partial_\mu\delta x_\nu + \partial_\nu\delta x_\mu)A_\nu,$$

which ascribes to A_μ the same transformation properties as the gradient of a scalar, thus preserving the possibility of gauge transformations under arbitrary coordinate deformations. In a similar way,

$$\delta(F_{\mu\nu}) = \delta F_{\mu\nu} - (\partial_\mu\delta x_\lambda)F_{\lambda\nu} - (\partial_\nu\delta x_\lambda)F_{\mu\lambda}.$$

With regard to the derivation of the electromagnetic field equations from the action principle, it should be noted that general gauge invariance requires that the sources of charged fields depend implicitly upon the vector potential A_μ. We express this dependence by

$$\delta_A\xi(x') = \int(dx)(\delta\xi(x')/\delta A_\mu(x))\delta A_\mu(x).$$

Since the infinitesimal gauge transformation, $\delta A_\mu = -\partial_\mu\delta\lambda$, must induce the change $\delta\xi = -i\delta\lambda\mathcal{E}\xi$, we learn that

$$\partial_\mu(\delta\xi(x')/\delta A_\mu(x)) = -i\mathcal{E}\xi(x)\delta(x-x'). \quad (45)$$

One obtains the following field equations on varying

$F_{\mu\nu}$ and A_ν in the complete Lagrange function,

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \tag{46}$$

$$\partial_\nu F_{\mu\nu}=j_\mu+k_\mu+J_\mu, \tag{47}$$

where

$$k_\mu(x)=\tfrac{1}{2}\int(dx')[(\delta\xi(x')/\delta A_\mu(x))\mathfrak{B}\chi(x')+\chi(x')\mathfrak{B}(\delta\xi(x')/\delta A_\mu(x))],$$

is the contribution to the total current vector associated with charged field sources. We derive from (45) that

$$\partial_\mu k_\mu=i\tfrac{1}{2}(\xi\mathfrak{B}\mathcal{E}\chi-\chi\mathfrak{B}\mathcal{E}\xi).$$

But the total current vector is divergenceless in consequence of the electromagnetic field equations. Therefore

$$\partial_\mu j_\mu=-i\tfrac{1}{2}(\xi\mathfrak{B}\mathcal{E}\chi-\chi\mathfrak{B}\mathcal{E}\xi),$$

which is in agreement with (33).

After removing the terms in $\delta(\mathfrak{L}_{\text{emf}})$ that contribute to the field equations, we are left with

$$\begin{aligned}\delta(\mathfrak{L}_{\text{emf}})=\tfrac{1}{2}\{\delta j_\mu, A_\mu\}-\partial_\mu(F_{\mu\nu}\delta A_\nu)+A_\mu\partial_\nu J_\mu\delta x_\nu\\ -\tfrac{1}{2}(\partial_\mu\delta x_\nu+\partial_\nu\delta x_\mu)(\tfrac{1}{2}\{j_\mu, A_\nu\}-\tfrac{1}{2}\{F_{\mu\lambda}, F_{\nu\lambda}\})\\ -(\partial_\mu\delta x_\nu)J_\mu A_\nu,\end{aligned} \tag{48}$$

in which

$$\tfrac{1}{2}\{\delta j_\mu, A_\mu\}=-i\delta\chi\mathfrak{A}_\mu\mathcal{E}\{\chi, A_\mu\}=-i\{A_\mu,\chi\}\mathfrak{A}_\mu\mathcal{E}\delta\chi.$$

This term alters the field equations of charged fields,

$$2\mathfrak{A}_\mu(\partial_\mu\chi-i\mathcal{E}\tfrac{1}{2}\{A_\mu,\chi\})=(\partial_l\mathfrak{JC}/\partial\chi)-\mathfrak{B}\xi,$$

$$-(\partial_\mu\chi+i\tfrac{1}{2}\{\chi, A_\mu\}\mathcal{E})2\mathfrak{A}_\mu=(\partial_r\mathfrak{JC}/\partial\chi)-\xi\mathfrak{B}.$$

We have anticipated that not all components of A_μ commute with χ. The tensor $T_{\mu\nu}$ is now obtained as

$$T_{\mu\nu}=\cdots+\tfrac{1}{2}\{F_{\mu\lambda}, F_{\nu\lambda}\}-\tfrac{1}{2}\{j_{(\mu}, A_{\nu)}\}-J_\mu A_\nu, \tag{49}$$

where $\cdots$ stands for (20), but with $\mathfrak{L}$ the complete Lagrange function. The action principle supplies the differential equation

$$\partial_\mu T_{\mu\nu}=\tfrac{1}{2}(\chi\mathfrak{B}\partial_\nu\xi+\partial_\nu\xi\mathfrak{B}\chi)+A_\mu\partial_\nu J_\mu. \tag{50}$$

The divergence term in (48) yields the infinitesimal generator

$$G_A=-\int d\sigma_\mu F_{\mu\nu}\delta A_\nu=-\int d\sigma F_{(0)(k)}\delta A_{(k)}, \tag{51}$$

while the Lagrange function with the derivative term (44) would give

$$G_F=\int d\sigma_\mu\delta F_{\mu\nu}A_\nu=\int d\sigma\delta F_{(0)(k)}A_{(k)}. \tag{52}$$

The change in the action integral produced by a variation of the external current J_μ is given by

$$\delta_J W_{12}=\int_{\sigma_2}^{\sigma_1}(dx)\delta J_\mu A_\mu.$$

If δJ_μ has the explicitly divergenceless form

$$\delta J_\mu=\partial_\nu\delta M_{\mu\nu},\quad M_{\mu\nu}=-M_{\nu\mu}, \tag{53}$$

where $\delta M_{\mu\nu}$ vanishes on σ_1 and σ_2, we find that

$$\delta_J W_{12}=\int_{\sigma_2}^{\sigma_1}(dx)\tfrac{1}{2}\delta M_{\mu\nu}F_{\mu\nu},$$

which makes it unnecessary to introduce an external source that is directly coupled to the field strength tensor $F_{\mu\nu}$.

The special nature of the electromagnetic field[3] is apparent in the form of the operator (52) generating changes in the local electric field components. Since one of the field equations is the equation of constraint

$$\partial_{(k)}F_{(0)(k)}=j_{(0)}+k_{(0)}+J_{(0)}, \tag{54}$$

the three variations $\delta F_{(0)(k)}$ cannot be arbitrarily assigned; the electromagnetic field and charged fields are not kinematically independent. This is evidently an aspect of the gauge invariance that links the two types of fields. Alternatively, we see from (51) that $A_{(0)}$ is not a dynamical variable subject to independent variations. But there is no field equation that expresses $A_{(0)}$ in terms of independent dynamical variables, in virtue of the arbitrariness associated with the existence of gauge transformations. Furthermore, a variation of $A_{(k)}$ in the form of a gradient, that is, a gauge transformation, yields a generating operator which, in consequence of (54), no longer contains electromagnetic field dynamical variables. Thus, in either form, (51) or (52), there are only two kinematically independent variations of the electromagnetic field quantities.

We now apply these generators to deduce commutation properties for the gauge invariant field strength components. According to the effect of a variation $\delta A_{(k)}$, upon the local components of $F_{\mu\nu}$ we have

$$[F_{(0)(k)}, G_A]=0,$$
$$[F_{(k)(l)}, G_A]=i(\partial_{(k)}\delta A_{(l)}-\partial_{(l)}\delta A_{(k)}),$$

whence

$$[F_{(0)(k)}(x), F_{(0)(l)}(x')]=0, \tag{55}$$

and

$$[F_{(k)(l)}(x), F_{(0)(m)}(x')]=i(\delta_{(k)(m)}\partial_{(l)}-\delta_{(l)(m)}\partial_{(k)})\delta_\sigma(x-x'). \tag{56}$$

In using G_F, we must restrict the electric field variation according to

$$\partial_{(k)}\delta F_{(0)(k)}=0,$$

which is identically satisfied on writing

$$\delta F_{(0)(k)}=\partial_{(l)}\delta Z_{(k)(l)},\quad Z_{(k)(l)}=-Z_{(l)(k)}.$$

[3] Papers dealing with the situation peculiar to the electromagnetic field are legion. Of the older literature, the closest in spirit to our procedure is that of W. Pauli, *Handbuch der Physik* (Edwards Brothers, Ann Arbor, 1943), Vol. 24.

This yields the form

$$G_F=\int d\sigma \tfrac{1}{2}F_{(k)(l)}\delta Z_{(k)(l)}.$$

The expression of changes induced by $\delta F_{(0)(k)}$,

$$[F_{(k)(l)}, G_F]=0,$$
$$[F_{(0)(k)}, G_F]=i\delta F_{(0)(k)},$$

then provides the commutation properties

$$\begin{aligned}&[F_{(k)(l)}(x), F_{(m)(n)}(x')]=0,\\ &[F_{(0)(k)}(x), F_{(l)(m)}(x')]\\ &\quad=i(\delta_{(k)(l)}\partial_{(m)}-\delta_{(k)(m)}\partial_{(l)})\delta_\sigma(x-x').\end{aligned} \tag{57}$$

where the latter is equivalent to (56).

An alternative derivation employs an infinitesimal change in the external source, distributed on (the negative side of) σ, $x_{(0)}=0$,

$$\delta M_{\mu\nu}=\delta m_{\mu\nu}\delta(x_{(0)}),$$

for which the associated generator is

$$G_m=\int d\sigma[\tfrac{1}{2}\delta m_{(k)(l)}F_{(k)(l)}-\delta m_{(0)(k)}F_{(0)(k)}].$$

The alteration produced in the field components follows from the field Eq. (47), and the form of (46) given by

$$\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}+\partial_\lambda F_{\mu\nu}=0. \tag{58}$$

Thus,

$$\partial_{(0)}(\delta F_{(0)(l)}-\delta M_{(0)(l)})=-\partial_{(k)}\delta F_{(k)(l)}+\partial_{(k)}\delta M_{(k)(l)},$$
$$\partial_{(0)}\delta F_{(k)(l)}=\partial_{(l)}\delta F_{(0)(k)}-\partial_{(k)}\delta F_{(0)(l)},$$

which yields the following discontinuities in $\delta F_{\mu\nu}$ on crossing the surface,

$$\delta F_{(0)(l)}]=\partial_{(k)}\delta m_{(k)(l)},$$
$$\delta F_{(k)(l)}]=\partial_{(l)}\delta m_{(0)(k)}-\partial_{(k)}\delta m_{(0)(l)}.$$

In the retarded description, these discontinuities are the actual changes in the field components on σ. On referring to the general formula (23), we obtain

$$\partial_{(k)}\delta m_{(k)(l)}=i[F_{(0)(l)}, G_m],$$
$$\partial_{(l)}\delta m_{(0)(k)}-\partial_{(k)}\delta m_{(0)(l)}=i[F_{(k)(l)}, G_m].$$

In view of the arbitrary values of $\delta m_{\mu\nu}$ on σ, these equations imply field strength commutation relations, which are identical with (55) and (57).

We give a related procedure which also illustrates the possibility of evaluating commutators of field quantities at points in time-like relation. The two field Eqs. (47) and (58) can be combined into (we incorporate k_ν with j_ν)

$$-\partial_\lambda{}^2F_{\mu\nu}=\partial_\mu(j_\nu+J_\nu)-\partial_\nu(j_\mu+J_\mu).$$

A change in the external current, of the form (53), yields

$$-\partial_\lambda{}^2\delta_M F_{\mu\nu}-\partial_\mu\delta_M j_\nu+\partial_\nu\delta_M j_\mu=\partial_\mu\partial_\lambda\delta M_{\nu\lambda}-\partial_\nu\partial_\lambda\delta M_{\mu\lambda}, \tag{59}$$

where, in the retarded description

$$\delta_M F_{\mu\nu}(x)=i\left[F_{\mu\nu}(x), \int_{\sigma_2}^{\sigma}(dx')\tfrac{1}{2}\delta M_{\lambda\kappa}(x')F_{\lambda\kappa}(x')\right]$$
$$=\int_{\sigma_2}^{\sigma_1}(dx')\tfrac{1}{2}\delta M_{\lambda\kappa}(x')\eta_+(x-x')i\times[F_{\mu\nu}(x), F_{\lambda\kappa}(x')],$$

and η_+ is the discontinuous function

$$\begin{aligned}\eta_+(x-x')&=1, \quad x_0>x_0'\\ &=0, \quad x_0<x_0'.\end{aligned}$$

We have a similar expression for $\delta_M j_\nu(x)$. On comparing the coefficients of $\delta M_{\lambda\kappa}(x')$ in (59) (our two treatments employing external sources are thus distinguished by surface and volume distributions of $\delta M_{\mu\nu}$, respectively), we find

$$\begin{aligned}&-\partial_\lambda{}^2\eta_+(x-x')i[F_{\mu\nu}(x), F_{\lambda\kappa}(x')]\\ &\quad-\partial_\mu\eta_+(x-x')i[j_\nu(x), F_{\lambda\kappa}(x')]\\ &\quad+\partial_\nu\eta_+(x-x')i[j_\mu(x), F_{\lambda\kappa}(x')]\\ &\quad=(\delta_{\nu\lambda}\partial_\mu\partial_\kappa-\delta_{\nu\kappa}\partial_\mu\partial_\lambda-\delta_{\mu\lambda}\partial_\nu\partial_\kappa\\ &\qquad+\delta_{\mu\kappa}\partial_\nu\partial_\lambda)\delta(x-x').\end{aligned} \tag{60}$$

The value of $i[F_{\mu\nu}(x), F_{\lambda\kappa}(x')]$, for equal times, is then obtained from the coefficient of the differentiated delta function of the time coordinate, with the anticipated result.

In the approximation that neglects the dynamical relation between currents and fields at points in time-like relation, the differential Eq. (60) has the solution

$$\eta_+(x-x')i[F_{\mu\nu}(x), F_{\lambda\kappa}(x')]=(\delta_{\nu\lambda}\partial_\mu\partial_\kappa-\delta_{\nu\kappa}\partial_\mu\partial_\lambda-\delta_{\mu\lambda}\partial_\nu\partial_\kappa+\delta_{\mu\kappa}\partial_\nu\partial_\lambda)D_{\mathrm{ret}}(x-x'),$$

where $D_{\mathrm{ret}}(x-x')$ is the familiar retarded solution of

$$-\partial_\lambda{}^2D_{\mathrm{ret}}=\delta(x-x'). \tag{61}$$

Had we employed the advanced description, η_+ would be replaced by $-\eta_-$, where

$$\begin{aligned}\eta_-(x-x')&=0, \quad x_0>x_0'\\ &=1, \quad x_0<x_0',\end{aligned}$$

and the advanced solution of (61) would appear. Subtracting these two results, we find

$$i[F_{\mu\nu}(x), F_{\lambda\kappa}(x')]=(\delta_{\nu\lambda}\partial_\mu\partial_\kappa-\delta_{\nu\kappa}\partial_\mu\partial_\lambda-\delta_{\mu\lambda}\partial_\nu\partial_\kappa+\delta_{\mu\kappa}\partial_\nu\partial_\lambda)D(x-x'),$$

in which $D(x-x')$ is the homogeneous solution of (61) provided by

$$D=D_{\mathrm{ret}}-D_{\mathrm{adv}}.$$

The kinematical relation between the electromagnetic field and charged fields, on a given σ, is most clearly indicated in a special choice of gauge, the so-called radiation gauge,

$$\partial_{(k)}A_{(k)}=0. \tag{62}$$

With this choice, the constraint equation for the electric field reads

$$\partial_{(k)}F_{(0)(k)}=-\partial_{(k)}{}^2A_{(0)}=j_{(0)}+J_{(0)},$$

so that the scalar potential is completely determined by the charge density,

$$A_{(0)}(x)=\int_\sigma d\sigma'\mathfrak{D}_\sigma(x-x')(j_{(0)}(x')+J_{(0)}(x')),$$

where

$$\mathfrak{D}_\sigma(x-x')=(1/4\pi)[(x_{(k)}-x_{(k)}')^2]^{-\frac{1}{2}}.$$

Evidently, $A_{(0)}$ does not commute with the components of charged fields. In this gauge, then, the dependence of the electric field upon the charged fields is made explicit through the decomposition of the electric field into transverse and longitudinal parts,

$$\begin{aligned}F_{(0)(k)}&=-\partial_{(0)}A_{(k)}-\partial_{(k)}A_{(0)}\\&=F_{(0)(k)}{}^{(T)}+F_{(0)(k)}{}^{(L)}.\end{aligned}$$

The inference that the transverse fields are the independent dynamical variables of the electromagnetic field in this gauge is confirmed on examining the generators G_A and G_F. Indeed,

$$G_A=-\int d\sigma F_{(0)(k)}\delta A_{(k)}=-\int d\sigma F_{(0)(k)}{}^{(T)}\delta A_{(k)},$$

and

$$G_F=\int d\sigma\delta F_{(0)(k)}A_{(k)}=\int d\sigma\delta F_{(0)(k)}{}^{(T)}A_{(k)},$$

in view of the transverse nature of $A_{(k)}$, Eq. (62). We can now derive the commutation properties of these dynamical variables from

$$\begin{aligned}&[A_{(k)},G_A]=i\delta A_{(k)}, &&[F_{(0)(k)}{}^{(T)},G_A]=0,\\&[F_{(0)(k)}{}^{(T)},G_F]=i\delta F_{(0)(k)}{}^{(T)}, &&[A_{(k)},G_F]=0,\end{aligned}$$

on taking into account the restrictions

$$\partial_{(k)}\delta A_{(k)}=\partial_{(k)}\delta F_{(0)(k)}{}^{(T)}=0,$$

produced by the transverse nature of these quantities. The Lagrange multiplier device permits us to deduce that

$$i[A_{(k)}(x),F_{(0)(l)}{}^{(T)}(x')]=\delta_{(k)(l)}\delta_\sigma(x-x')+\partial_{(l)}'\lambda_{(k)}.$$

The divergenceless character of the transverse electric field supplied the information

$$\partial_{(l)}'^2\lambda_{(k)}=\partial_{(k)}\delta_\sigma(x-x'),$$

whence

$$\lambda_{(k)}=-\partial_{(k)}\mathfrak{D}_\sigma(x-x').$$

The resulting commutator

$$\begin{aligned}i[A_{(k)}(x),F_{(0)(l)}{}^{(T)}(x')]&\\=\delta_{(k)(l)}\delta_\sigma(x-x')-\partial_{(k)}\partial_{(l)}'\mathfrak{D}_\sigma(x-x')&\\=(\delta_{(k)(l)}\delta_\sigma(x-x'))^{(T)},&\end{aligned}$$

is also consistent with the transverse nature of $A_{(k)}$. The remaining commutation relations are

$$[A_{(k)}(x),A_{(l)}(x')]=[F_{(0)(k)}{}^{(T)}(x),F_{(0)(l)}{}^{(T)}(x')]=0.$$

We shall use the device of the external current to derive the commutation relations between the electromagnetic field tensor and the displacement generators P_ν, $J_{\mu\nu}$. According to (49) and (50),

$$P_\nu(\sigma_1)-P_\nu(\sigma_2)=\int_{\sigma_2}^{\sigma_1}(dx)[\cdots+A_\lambda\partial_\nu J_\lambda],$$

$$\begin{aligned}J_{\mu\nu}(\sigma_1)-J_{\mu\nu}(\sigma_2)=\int_{\sigma_2}^{\sigma_1}(dx)[\cdots+A_\lambda(x_\mu\partial_\nu-x_\nu\partial_\mu)J_\lambda&\\+A_\mu J_\nu-A_\nu J_\mu],&\end{aligned}$$

in which we have indicated only the terms containing the external current. We consider an infinitesimal change in the latter possessing the form (53). In the retarded description, the resulting changes of P_ν and $J_{\mu\nu}$ on σ_1 are

$$\delta_MP_\nu(\sigma_1)=-\int_{\sigma_2}^{\sigma_1}(dx)\tfrac{1}{2}\delta M_{\lambda\kappa}\partial_\nu F_{\lambda\kappa},$$

$$\begin{aligned}\delta_MJ_{\mu\nu}(\sigma_1)=-\int_{\sigma_2}^{\sigma_1}(dx)[\tfrac{1}{2}\delta M_{\lambda\kappa}(x_\mu\partial_\nu-x_\nu\partial_\mu)F_{\lambda\kappa}&\\+\delta M_{\lambda\nu}F_{\mu\nu}-\delta M_{\lambda\mu}F_{\nu\lambda}].&\end{aligned}$$

When expressed in terms of the generator

$$G_M=\int_{\sigma_2}^{\sigma_1}(dx)\tfrac{1}{2}\delta M_{\lambda\kappa}F_{\lambda\kappa},$$

the following commutators are encountered,

$$\begin{aligned}&i[F_{\lambda\kappa},P_\nu]=\partial_\nu F_{\lambda\kappa},\\&i[F_{\lambda\kappa},J_{\mu\nu}]=(x_\mu\partial_\nu-x_\nu\partial_\mu)F_{\lambda\kappa}+\delta_{\nu\kappa}F_{\mu\lambda}\\&\qquad-\delta_{\mu\kappa}F_{\nu\lambda}+\delta_{\mu\lambda}F_{\nu\kappa}-\delta_{\nu\lambda}F_{\mu\kappa}.\end{aligned}$$

Finally, we remark that the extension of (31) to include the electromagnetic field, in the radiation gauge, is

$$\begin{aligned}P_{(0)}=\int d\sigma[\tfrac{1}{2}(F_{(0)(k)}{}^{(T)})^2+\tfrac{1}{4}(F_{(k)(l)})^2&\\+\mathcal{H}-\chi\mathfrak{A}_{(k)}(\partial_{(k)}-ie A_{(k)})\chi&\\-J_{(k)}A_{(k)}+\tfrac{1}{2}(j_{(0)}+J_{(0)})A_{(0)}-\tfrac{1}{2}(\xi\mathfrak{B}\chi+\chi\mathfrak{B}\xi)],&\end{aligned}$$

and

$$P_{(k)}=\int d\sigma[\tfrac{1}{2}\{F_{(0)(l)}{}^{(T)}, F_{(k)(l)}\}-\chi\mathfrak{A}_{(0)}\partial_{(k)}\chi].$$

In arriving at the expression for $P_{(0)}$, the noncommutivity of $A_{(0)}$ with χ must be taken into consideration, but produces no actual contribution. A variation of each of the independent fields yields

$$\delta P_\mu=\int d\sigma[\delta F_{(0)(k)}{}^{(T)}\partial_\mu A_{(k)}-\delta A_{(k)}\partial_\mu F_{(0)(k)}l^{(T)} -\delta\chi 2\mathfrak{A}_{(0)}\partial_\mu\chi],$$

which confirms the consistency of the translation generator with the various field variation generators.

PHYSICAL REVIEW VOLUME 91, NUMBER 3 AUGUST 1, 1953

The Theory of Quantized Fields. III

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts

(Received March 31, 1953)

In this paper we discuss the electromagnetic field, as perturbed by a prescribed current. All quantities of physical interest in various situations, eigenvalues, eigenfunctions, and transition probabilities, are derived from a general transformation function which is expressed in a non-Hermitian representation. The problems treated are: the determination of the energy-momentum eigenvalues and eigenfunctions for the isolated electromagnetic field, and the energy eigenvalues and eigenfunctions for the field perturbed by a time-independent current; the evaluation of transition probabilities and photon number expectation values for a time-dependent current that departs from zero only within a finite time interval, and for a time-dependent current that assumes non-vanishing time-independent values initially and finally. The results are applied in a discussion of the infrared catastrophe and of the adiabatic theorem. It is shown how the latter can be exploited to give a uniform formulation for all problems requiring the evaluation of transition probabilities or eigenvalue displacements.

INTRODUCTION

We shall approach the general problem of coupled fields through the simpler situation presented by a single field which is externally perturbed. In this paper we illustrate the treatment of a Bose-Einstein system by discussing the Maxwell field with a prescribed electric current. A succeeding paper will be devoted to the Dirac field.

The solution to all dynamical questions is obtained by constructing the transformation function linking two descriptions of the system that are associated with different space-like surfaces. Thus, for a closed system, the general transformation function can be expressed as

$$(\zeta_1'\sigma_1|\zeta_2''\sigma_2)=\sum_{\gamma',\gamma''}(\zeta_1'|\gamma')(\gamma'\sigma_1|\gamma''\sigma_2)(\gamma''|\zeta_2''),$$

where the γ are a complete set of compatible constants of the motion, in terms of which the energy-momentum vector P_μ can be exhibited. In the γ representation, the effect of an infinitesimal translation of σ_1 is given by

$$\delta_\epsilon(\gamma'\sigma_1|\gamma''\sigma_2)=i(\gamma'\sigma_1|P_\mu\delta\epsilon_\mu|\gamma''\sigma_2)=iP_\mu'\delta\epsilon_\mu(\gamma'\sigma_1|\gamma''\sigma_2),$$

where $P_\mu'=P_\mu(\gamma')$. Accordingly, if σ_1 is parallel to σ_2, and is generated from the latter by the translation X_μ, we have

$$(\gamma'\sigma_1|\gamma''\sigma_2)=\delta(\gamma',\gamma'')\exp(iP_\mu'X_\mu),$$

and

$$(\zeta_1'\sigma_1|\zeta_2''\sigma_2)=\sum_{\gamma'}(\zeta_1'|\gamma')\exp(iP_\mu'X_\mu)(\gamma'|\zeta_2''). \quad (1)$$

This shows how a knowledge of the transformation function that relates two conveniently chosen representations on parallel surfaces yields all the eigenvalues and eigenfunctions of P_μ.

Another illustration of the utility of transformation functions relates to the situation in which the same system is externally perturbed, in the interior of the space-time region bounded by σ_1 and σ_2. The transformation function $(\gamma'\sigma_1|\gamma''\sigma_2)$, inferred from the knowledge of $(\zeta_1'\sigma_1|\zeta_2''\sigma_2)$, then yields the probability of a transition from the initial state γ'' to the final state γ',

$$p(\gamma',\gamma'')=|(\gamma'\sigma_1|\gamma''\sigma_2)|^2. \quad (2)$$

Representations of particular convenience are suggested by the characterization of the vacuum state for a complete system. The vacuum is the state of minimum energy. If this natural origin of energy is adjusted to zero, the vacuum can be described as that state presenting identical properties to all observers, $P_\mu\Psi_0=J_{\mu\nu}\Psi_0=0$, and is therefore independent of the surface σ. Now, if the general field component χ is analyzed into contributions of various frequencies, χ_{p0}, we have $[\chi_{p0}, P_0]=p_0\chi_{p0}$, or $P_0\chi_{p0}=\chi_{p0}(P_0-p_0)$. When this relation, involving a positive frequency, $p_0>0$, is