The Analysis of Angular Correlation and Angular Distribution Data

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Consideration of a number of problems which arise in the analysis of angular correlation and angular distribution data is given. These include corrections for finite angular resolution. For a source of constant strength the questions of determination of most probable counting rates, their associated errors and the determination of the most probable coefficients in the Legendre representation of the data, as well as the errors in the determination of these coefficients, are examined. A brief discussion of the corrections for a decaying source is also presented.

I. INTRODUCTION

THE importance of measurements of angular distributions in nuclear reactions and of angular correlation in nuclear cascades has become more and more evident in recent months. As experimental precision improves, the need for a systematic discussion of methods of evaluating the data becomes more pressing. Certain aspects of some of the problems which arise have been discussed.[1] However, a number of questions arise having to do with angular resolution corrections, determination of counting rates and errors in the coefficients, in terms of which the data are finally represented. The analysis of these problems is presented herewith, and the application of the results to angular correlation measurements has been made and described in an accompanying paper.[2]

In succeeding sections we take up the following problems. The effect of the finite solid angle of the detectors on the correlation function is treated in Sec. II. The results are, of course, also applicable in angular distribution measurements where only a single-counting rate need be determined. In Sec. III the question of determination of counting rates at each angle and assignment of weights to each determination is treated. In Sec. IV a discussion is given of the least squares fit of the data in terms of a series of Legendre polynomials, or if desirable, any other functions of the (scattering) angle ϑ or angle between the two radiations. This includes an account of the statistical errors in the coefficients as well as a method whereby the existence of other types of errors may be detected. Finally, in Sec. V the question of a decaying source is briefly considered.

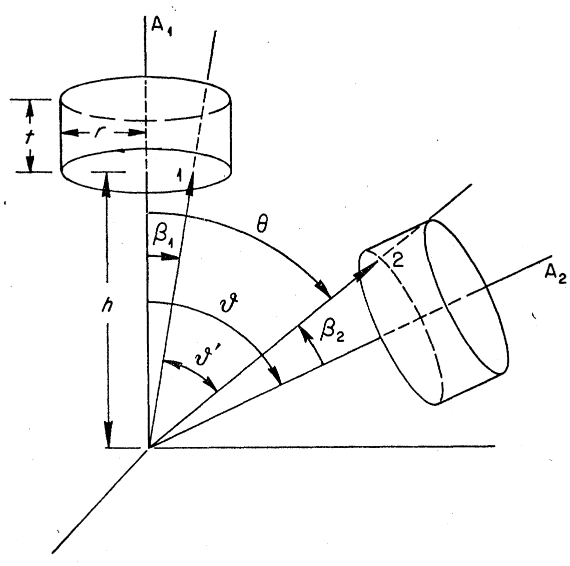

FIG. 1. Geometry for finite resolution in angular correlation. The notation also applies for singles counts as in angular distribution measurements. The azimuth angles φ_1 and φ_2 are measured with respect to the cylinder axes A_1 and A_2, respectively.

II. ANGULAR RESOLUTION CORRECTION

The results of an angular correlation measurement are most conveniently expressed in terms of a Legendre polynomial expansion. That is, the coincidence counting rate per unit solid angle $\sin\vartheta d\vartheta d\varphi$ (where the angles represent the relative orientation of the propagation vectors of the two radiations) is proportional to

$$W(\vartheta)=\sum_{\nu=0}^{\nu_m} \alpha_\nu P_\nu(\cos\vartheta). \quad (1)$$

When the detectors for the two radiations subtend finite solid angles Ω_1 and Ω_2 at the source, it is advisable to modify the theoretical correlation and compare this smeared correlation with the measured one. The geometry envisaged is shown in Fig. 1. The detectors (scintillation counters) are assumed to be crystals cut in the form of right circular cylinders with the base oriented towards the source. The source, at the origin, is on the intersection of the axes of the cylinders. In this case, as the following shows, the form of the correlation function is unchanged[3] and each coefficient α_ν becomes multiplied by an attenuation factor for which one can obtain an exact and very simple expression.

Taking into account the absorption of the radiation

[1] H. Fraunfelder in *Annual Reviews of Nuclear Science* (Annual Reviews, Inc., Stanford, 1953), Vol. 2.

[2] E. Klema and F. McGowan, this issue [Phys. Rev. **91**, 616 (1953)].

[3] This fact, which is *a priori* evident, was also brought out by S. Frankel, Phys. Rev. **83**, 673 (1951) and by E. L. Church and J. J. Kraushaar, Phys. Rev. **88**, 419 (1952). While the geometry considered here is a special one, the results presented below may constitute an incentive for the use of such wherever it is convenient to do so.

in each crystal, we introduce the following notation. The distance from the source to the front face of each crystal is h, the thickness is t, and r is the radius of the crystal. If $x(\beta)$ is the distance traversed by the radiation incident on the crystal at an angle β with the axis, the absorption is proportional to $(1-e^{-\tau x})$. Then the measured correlation function would be

$$\bar{W}(\vartheta)=\frac{\int d\Omega_1 d\Omega_2 W(\vartheta')(1-e^{-\tau x_1})(1-e^{-\tau x_2})}{\int d\Omega_1 d\Omega_2 (1-e^{-\tau x_1})(1-e^{-\tau x_2})}, \qquad (2)$$

where x_1 and x_2 refer to the two crystals, $d\Omega_1$ and $d\Omega_2$ are the solid angle elements for each radiation, and ϑ' is the angle between their propagation vectors, while ϑ is the angle between the cylinder axes.

The required integrals are of the form

$$I_l=\int d\Omega_1 d\Omega_2 P_l(\cos\vartheta')(1-e^{-\tau x_1})(1-e^{-\tau x_2}), \qquad (3)$$

and

$$x(\beta)=t\sec\beta \quad \text{for} \quad 0\leqslant\beta\leqslant\tan^{-1}[r/(h+t)]=\beta',$$
$$x(\beta)=r\cos\beta-h\sec\beta \quad \text{for} \quad \beta'\leqslant\beta\leqslant\tan^{-1}(r/h)=\gamma,$$

where γ is the half-angle subtended on the front face. Using the addition theorem

$$P_l(\cos\vartheta')=\frac{4\pi}{2l+1}\sum_{-l}^{l} Y_l^{m*}(1)Y_l^m(2),$$

the integration over φ_1, the azimuth of direction of radiation 1 can be carried out since x_1 is independent of φ_1. Then only the $m=0$ term contributes and

$$I_l=2\pi\int_0^\gamma \sin\beta_1 d\beta_1 P_l(\cos\beta_1)(1-e^{-\tau x_1})$$
$$\times\int_{-\gamma}^{+\gamma}\int \sin\theta d\theta d\varphi_2 P_l(\cos\theta)(1-e^{-\tau x_2}).$$

Here θ is the angle between radiation 2 and the axis of detector 1. Applying the addition theorem once more to the spherical triangle formed by the axes of the two detectors and the direction of radiation 2 so that

$$P_l(\cos\theta)=P_l(\cos\beta_2)P_l(\cos\vartheta)+\cdots,$$

the dots indicating azimuth dependent terms, we have on integration over φ_2, for which the azimuth dependent terms do not contribute,

$$I_l=4\pi^2 P_l(\cos\vartheta)J_l(1)J_l(2),$$

where

$$J_l=\int_0^\gamma P_l(\cos\beta)(1-e^{-\tau x(\beta)})\sin\beta d\beta. \qquad (5)$$

The result (4) confirms the statement made above with regard to the angular dependence of the correlation function. Of course, ϑ represents the angle between detector axes. The attenuation factor is

$$Q_l=(J_l/J_0)^2 \qquad (6)$$

for similar detectors and for γ rays with similar absorption coefficients. For a single detector, as would be used in an angular distribution measured, with the data represented by Eq. (1), the attenuation factor would be simply

$$Q_l=J_l/J_0. \qquad (6a)$$

For full absorption ($x\tau\rightarrow\infty$) we have

$$\frac{J_l}{J_0}=\frac{P_{l-1}(x_0)-x_0P_l(x_0)}{(l+1)(1-x_0)}, \qquad (6b)$$

where $x_0=\cos\gamma$.

In Table I numerical results are given for the case $h=7$ and 10 cm, $t=2.54$ cm and $r=1.9$ cm, for $l=2$ and 4 ($Q_0=1$) and several values of the absorption coefficient τ ranging from 0.123 cm^{-1} to 40 cm^{-1}. These values are pertinent to the experiments of Klema and McGowan.[2] While the individual J_l vary appreciably with absorption coefficient, the ratios J_l/J_0 are much less sensitive.

In the sequel it will be assumed that these angular resolution corrections have been made so that, for the considerations given below, no further reference to this effect need be made.

TABLE I. Attenuation coefficients J_l/J_0 for angular distribution measurements. For angular correlation measurements these coefficients should be squared or a product of two coefficients should be taken.

	$l=2$		$l=4$	
τ (cm^{-1})	$h=7$	$h=10$	$h=7$	$h=10$
0.123	0.95931	0.97823	0.86873	0.92865
0.130	0.95927	0.97821	0.86862	0.92863
0.150	0.95917	0.97818	0.86833	0.92850
0.200	0.95887	0.97808	0.86758	0.92818
0.300	0.95851	0.97790	0.86620	0.92759
1.00	0.95565	0.97672	0.85737	0.92383
2.00	0.95311	0.97567	0.84938	0.92049
3.00	0.95172	0.97510	0.84508	0.91863
5.00	0.95039	0.97457	0.84096	0.91695
10.0	0.94925	0.97410	0.83746	0.91551
40.0	0.94830	0.97373	0.83457	0.91432

III. COUNTING RATE DETERMINATION

For both angular distribution and angular correlation measurements one wishes to fit a counting rate μ_i, determined at each angle of observation ϑ_i, by a function of the form exhibited in Eq. (1). There are three main questions: (1) the evaluation of the coefficients α_ν, (2) the evaluation of the errors in these coefficients, and (3) the determination of the maximum order of the Legendre polynomial ν_m, assuming that no *a priori* information on this point is available. The latter question is best discussed after questions (1) and (2) are clearly explored. The importance of the maximum $\nu(=\nu_m)$ in angular distribution measurements is pointed up by the dependence of ν_m on the angular momentum of the incident beam of particles.[4] Its importance for angular correlations rests on the fact that $\nu_m \leqslant 2j_i, 2L_1, 2L_2$, where j_i, L_1, L_2 are, respectively, the intermediate state angular momentum and the angular momenta of radiations 1, 2.

For the determination of the counting rate μ_i, at a fixed angle, we consider two possible procedures: (a) The detector(s) is exposed to the radiation(s) for time intervals $t_1, t_2 \cdots t_s$ and the number of counts (singles or coincidences) obtained during each time interval is recorded. (b) The detector records a given number of counts n_j and the time t_j required for this is recorded. For the angular distribution case we clearly deal with a source of constant strength. For certain correlation experiments this assumption may be made with extremely negligible error.[2] Consequently, the fact that the detectors may not have 100 percent efficiency is a trivial effect: the counting rate given below is multiplied by $1/\eta$, where η is the pertinent efficiency. We consider the case of a decaying source in Sec. V. The possible presence of an appreciable background count would alter the results given below. This question is considered elsewhere[2] and will not enter in the following.

(a) Fixed Time of Counting

If the true counting rate for the source of given strength is μ, the probability for n_j counts in a time interval t_j is given by the Poisson law:

$$P_{t_j}(n_j)=\frac{(\mu t_j)^{n_j}}{n_j!}e^{-\mu t_j}, \tag{7}$$

$$\langle n_j\rangle=\mu t_j, \tag{7a}$$

$$\sigma^2(n_j)=\langle(n_j-\langle n_j\rangle)^2\rangle=\mu t_j. \tag{7b}$$

The data, represented by the numbers n_j, t_j are fitted by least squares. That is, μ' is determined so that

$$\textstyle\sum_j w_j(n_j-\mu' t_j)^2=\text{minimum}. \tag{8}$$

The weights w_j are inversely proportional to $\sigma^2(n_j)$ and here, as well as in the following, the normalization of the weights is irrelevant. We take $w_j=1/\sigma^2(n_j)=1/\mu t_j$. Then (8) gives

$$\mu'=\textstyle\sum_j w_j t_j n_j/\sum_j w_j t_j^2=\sum n_j/\sum t_j. \tag{9}$$

The weight attached to this determination of the counting rate can be taken to be

$$w(\mu_i')=1/\sigma^2(\mu_i')=N_i/\mu_i'^2, \tag{10}$$

using (7b). Here the index i refers to a particular angle ϑ_i.

[4] C. N. Yang, Phys. Rev. **74**, 764 (1948).

(b) Fixed Number of Counts

At the specified angle one measures the time t_j required to accumulate n_j counts. Let $P_n(t)dt$ be the probability that the nth count is recorded in the time interval t to $t+dt$. Then

$$P_n(t)dt=dt\int_0^t dt' P_{n-1}(t')K(t-t');\quad n\geqslant 1, \tag{11}$$

where the kernel $K(t-t')dt$ is the probability that a single count occurs in the time interval $t-t'$ to $t-t'+dt$. For a constant strength source this is

$$K(t-t')=\mu e^{-\mu(t-t')}. \tag{12}$$

The solution of (11) with the condition $P_1(t)=K(t)$ is

$$P_n(t)=\mu^n t^{n-1}e^{-\mu t}/(n-1)!. \tag{13}$$

Clearly $P_n(t)$ is normalized to unity and

$$\langle t\rangle=n/\mu, \tag{13a}$$

$$\sigma^2(t)=\langle(t-\langle t\rangle)^2\rangle=n/\mu^2. \tag{13b}$$

The least squares fit of the data now takes the form

$$\textstyle\sum_j w_j(t_j-n_j/\mu')^2=\text{minimum},$$

where $w_j=1/\sigma^2(t_j)$. This gives

$$\mu'=\textstyle\sum w_j n_j^2/\sum w_j n_j t_j=\sum n_j/\sum t_j. \tag{14}$$

In both cases (a) and (b) the most probable counting rate (for the given finite number of counts) is given by the ratio of total counts to total time.

In order to determine the weight to be associated with this counting rate determination the probability that a sequence of t_i measurements shall give $T=\sum_j t_j=(\sum_j n_j)/\mu'$ in an interval dT must be obtained. This problem is slightly different from that considered in (a) above since now the statistically varying quantities t_j are in the denominator of (14). The required probability is

$$P_{N_i}(T)=\mu^{N_i}\frac{T^{N_i-1}e^{-\mu T}}{(N_i-1)!} \tag{15}$$

from (13), and $N_i=\sum_j n_j$. Then the probability that a measurement of the counting rate gives a value μ' to $\mu'+d\mu'$ is

$$P(\mu')d\mu'=P_{N_i}(T)\,|dT/d\mu'|=(N_i/\mu')^2 P_{N_i}(T), \tag{16}$$

and $T=N_i/\mu'$. For convenience a subscript i is omitted from μ' and μ here and above. From this (normalized) probability function we obtain

$$\langle\mu_i'\rangle=\mu_i N_i/(N_i-1)\approx\mu_i, \tag{16a}$$

$$\sigma^2(\mu_i')=\langle(\mu_i'-\langle\mu_i'\rangle)^2\rangle =\mu_i^2 N_i^2/(N_i-1)^2(N_i-2)\approx\mu_i^2/N_i. \tag{16b}$$

For the weight $w(\mu_i')$ we may take

$$w(\mu_i')=N_i/\mu_i'^2. \tag{16c}$$

In (16b) and (16c) we consider that $N_i \gg 1$ at each angle. Of course, for $N_i \to \infty$ we obtain $\langle \mu' \rangle = \mu$ and $\sigma^2(\mu') = 0$. In fact, $P(\mu')$ is then equal to $\delta(\mu' - \mu)$, as is to be expected. It will be noted that for large N_i there is essentially no difference between procedures (a) and (b).

IV. LEAST SQUARES FIT OF THE DATA

Once the counting rates are determined at each angle, and thereby, the associated weights determined, one may proceed to problems (1) and (2) mentioned at the beginning of the previous section. A discussion of problem (3) can then be given (see end of this section).

The determination of the coefficients α_ν is now to be carried by replacing $W(\vartheta)$ in Eq. (1) by $\mu(\vartheta)$, or more specifically,

$$\mu_i = \mu(\vartheta_i) = \sum_\lambda \alpha_\lambda A_{i\lambda}, \tag{17}$$

where $A_{i\lambda} = \varphi_\lambda(\cos\vartheta_i)$ and φ_λ may be taken to be P_λ or any other function. In (17) the left and right sides are not equal but what is meant is that the right side represents $\mu(\vartheta_i)$ as closely as possible. The prime on the symbol μ' has been dropped. The measurement of ϑ_i, it will be assumed, contains no error.

Let m be the number of angles at which data is taken and l be the number of coefficients α_λ which one assumes. Then $l = \nu_m + 1$ in the angular distribution case and in the angular correlation case, since ν is even, $l = \frac{1}{2}\nu_m + 1$. In any case $m \geqslant l$.

The coefficients α_λ, which are the most probable values for the given data, are determined from

$$\sum_i w_i(\mu_i - \sum_\lambda \alpha_\lambda A_{i\lambda})^2 = \text{minimum},$$

yielding the normal equations

$$\sum_i w_i(\mu_i - \sum_\lambda \alpha_\lambda A_{i\lambda}) A_{i\gamma} = 0. \tag{18}$$

Here w_i is given by (10) or (16c). Equation (18) and subsequent results are more compactly expressed by introducing matrix notation. We define a square $(l \times l)$ matrix C whose elements are

$$C_{\lambda\gamma} = \sum_i w_i A_{i\lambda} A_{i\gamma} = C_{\gamma\lambda}.$$

That is,

$$C = \tilde{A} w A,$$

where w is the diagonal $(m \times m)$ matrix with elements w_i and A is the $m \times l$ matrix (nonsquare, in general). The tilde means transposed. Further let $\boldsymbol{\xi}$ be the l component vector

$$\boldsymbol{\xi} = \tilde{A} w \boldsymbol{\mu},$$

where $\boldsymbol{\mu}$ is the m component vector with components μ_i. Then

$$C\boldsymbol{\alpha} = \boldsymbol{\xi},$$

or, in detail,

$$\alpha_\lambda = \sum_\gamma C_{\lambda\gamma}^{-1} \xi_\gamma. \tag{19}$$

Of course, C^{-1} is constructed by forming a matrix k whose λ, ν element is the cofactor of $C_{\lambda\nu}$ in the determinant $|C|$ and dividing by $|C|$. Note, that k like C is symmetric.

Equation (19) gives the coefficients α_λ. It is seen that the normalization of the weights does not enter; that is, if $w_i = b/\sigma^2(\mu_i)$, the constant b cancels out. From (19) we observe that $\boldsymbol{\alpha}$ is a linear homogeneous function of the counting rates μ_i. Hence, due to the existence of a variation in μ_i, expressed by the mean square deviation $\sigma^2(\mu_i)$, there will be a corresponding mean square deviation in α_λ. Writing

$$\alpha_\lambda = \sum_i B_{\lambda i} w_i \mu_i,$$

where

$$B_{\lambda i} = \sum_\nu C_{\nu\lambda}^{-1} A_{i\nu},$$

we have

$$\begin{aligned}\sigma^2(\alpha_\lambda) &= \sum_i B_{\lambda i}^2 w_i^2 \sigma^2(\mu_i)\\ &= b \sum_{i\nu\gamma} C_{\nu\lambda}^{-1} C_{\gamma\lambda}^{-1} A_{i\nu} A_{i\gamma} w_i\\ &= b C_{\lambda\lambda}^{-1}.\end{aligned}$$

Consequently, if we define a matrix G by

$$G_{\lambda\gamma} = \sum_i A_{i\lambda} A_{i\gamma} / \sigma^2(\mu_i) = G_{\gamma\lambda},$$

then

$$\sigma^2(\alpha_\lambda) = G_{\lambda\lambda}^{-1}. \tag{20}$$

That is, the mean square deviation of the coefficient α_λ is given by the λ, λ diagonal matrix element of the inverse of G and this does not depend on the normalization b.[5] However, $\sigma^2(\alpha_\lambda)$ is linear homogeneous in the $\sigma^2(\mu_i)$ as it should be.

Now the square error given by (20) is a measure of the deviations to be expected on purely statistical grounds. Of course, in an actual experiment other sources of error may exist. A clue to the existence of extraneous errors is provided by comparing (20) with the mean square error defined in a different manner. This mean square error, to which we now turn our attention, is based on a comparison between the true coefficients, which we denote by α_λ^0, and the least squares value α_λ given by (19).

Consider the residuals E_i between the measured values of μ_i and the true values represented by $\sum_\lambda \alpha_\lambda^0 A_{i\lambda}$. For convenience we reduce the equations to the form in which the weights are effectively unity by introducing the notation

$$z_i = w_i^{\frac{1}{2}} \mu_i, \tag{21a}$$

$$a_{i\lambda} = w_i^{\frac{1}{2}} A_{i\lambda}, \quad b_{\lambda i} = w_i^{\frac{1}{2}} B_{\lambda i}. \tag{21b}$$

Then, the residual E_i are given by

$$E_i = \sum_\lambda \alpha_\lambda^0 a_{i\lambda} - z_i. \tag{22a}$$

The residual between the measured values and the least squares curve are

$$e_i = \sum_\lambda \alpha_\lambda a_{i\lambda} - z_i. \tag{22b}$$

[5] The result expressed by Eq. (20) is not new. However, it has been reproduced here for two reasons. First, it seems not to be widely known and secondly the customary derivations of the result are somewhat beclouded by a cumbersome notation. The same remark applies to the result given in (30) below.

We make use of the following:

$$\sum_i a_{i\gamma} e_i = \sum_\lambda \alpha_\lambda C_{\gamma\lambda} - \xi_\gamma = 0, \tag{23a}$$

$$\sum_i a_{i\lambda} b_{\gamma i} = \sum_i a_{i\lambda} \sum_\nu C_{\nu\gamma}^{-1} a_{i\nu} = \delta_{\lambda\gamma}, \tag{23b}$$

$$\sum_i e_i b_{\gamma i} = \sum_i e_i \sum_\nu C_{\nu\gamma}^{-1} a_{i\nu} = 0, \tag{23c}$$

where the last result follows from (23a).

Then, from (22a) and (22b),

$$\sum_i (E_i e_i - e_i^2) = \sum_{i\lambda} (\alpha_\lambda^0 - \alpha_\lambda) a_{i\lambda} e_i = 0,$$

by (23a). Also, using the above,

$$\sum_i (E_i^2 - e_i E_i) = \sum_i (E_i^2 - e_i^2) = \sum_\lambda (\alpha_\lambda^0 - \alpha_\lambda) \sum_i E_i a_{i\lambda}, \tag{24}$$

and

$$\sum_i (E_i - e_i) b_{\gamma i} = \sum_i E_i b_{\gamma i} = \sum_{i\lambda} (\alpha_\lambda^0 - \alpha_\lambda) a_{i\lambda} b_{\gamma i} = \alpha_\gamma^0 - \alpha_\gamma, \tag{25}$$

by (23c) and (23b).

If a large number of measurements of the coefficients α_γ are carried out and the values of $(\alpha_\gamma^0 - \alpha_\gamma)$ is squared and averaged over this sequence of experiments, we would obtain the following for the mean square difference of true and least squares coefficients

$$\langle (\alpha_\gamma - \alpha_\gamma^0)^2 \rangle_{\text{Av}} = \sum_{ij} \langle E_i E_j \rangle_{\text{Av}} b_{\gamma i} b_{\gamma j}.$$

Here, and below, $\langle \; \rangle_{\text{Av}}$ is used to denote the average over a large number of experiments. Now, since E_i and E_j are uncorrelated (if $i \neq j$)

$$\langle E_i E_j \rangle_{\text{Av}} = \langle E_i^2 \rangle_{\text{Av}} \delta_{ij} = \epsilon^2 \delta_{ij}.$$

Hence

$$\langle (\alpha_\gamma - \alpha_\gamma^0)^2 \rangle_{\text{Av}} = \epsilon^2 \sum_i b_{\gamma i}^2 = \epsilon^2 C_{\gamma\gamma}^{-1}. \tag{26}$$

To calculate the mean square residual ϵ^2, we have from (24) and (25) that

$$\sum_i (E_i^2 - v_i^2) = \sum_{\lambda ij} E_i E_j b_{\lambda i} a_{j\lambda}.$$

Then, taking averages in the above sense,

$$\sum_i \epsilon^2 - \sum_i \langle v_i^2 \rangle_{\text{Av}} = \epsilon^2 \sum_{\lambda i} b_{\lambda i} a_{i\lambda} = l\epsilon^2,$$

from (23b). Hence

$$\epsilon^2 = \sum_i \langle v_i^2 \rangle_{\text{Av}} / (m - l). \tag{27}$$

Since one does not carry out the large number of experiments envisaged in the averaging process, the mean square $\langle v_i^2 \rangle_{\text{Av}}$ is replaced by the sum of squares of the actual residuals:

$$\sum_i \langle v_i^2 \rangle_{\text{Av}} \approx \sum_i w_i (\mu_i - \sum_\lambda \alpha_\lambda A_{i\lambda})^2 = \sum_i v_i^2, \tag{28}$$

where α_λ are the most probable (least squares) coefficients from (19). Inserting (19) in (28) we find

$$\sum_i v_i^2 = \sum_i w_i \mu_i^2 - \boldsymbol{\alpha} \cdot \boldsymbol{\xi}. \tag{29}$$

The mean square error in α_λ is then

$$\langle (\alpha_\lambda - \alpha_\lambda^0)^2 \rangle_{\text{Av}} \cong [\sum_i v_i^2 / (m - l)] C_{\lambda\lambda}^{-1}, \tag{30}$$

where the result (29) is to be used. Since this result is independent of a normalization constant b, one may use $w_i = 1/\sigma^2(\mu_i)$ throughout. It is clear that for the purpose of quoting mean square errors like that given in (30) one should have an excess of points of observation, i.e., $m > l$.

From the manner in which the result (30) is constructed it would seem that essentially all sources of error are included and not just the statistical errors $\sigma^2(\mu_i)$. In fact $\langle (\alpha_\lambda - \alpha_\lambda^0)^2 \rangle_{\text{Av}}$ is independent of the size of the $\sigma^2(\mu_i)$ and depends only on ratios of these mean square deviations. Thus, a comparison of (30) and (20) would serve to indicate the importance of nonstatistical errors. A value of ϵ appreciably greater than unity implies that errors other than those due to a finite number of counts, taken in the whole experiment, are of importance. If one can be reasonably sure that systematic errors are eliminated, a value of ϵ much greater than unity would presumably imply that the function used in the least squares fit is incorrect. A possible procedure would be to increase ν_m by one unit, or by two units for the correlation experiment, and reanalyze the results. When $\alpha_{\nu_m} \approx 0$, that is, when $\alpha_{\nu_m}^2 / \sigma^2(\alpha_{\nu_m}) \approx 1$ with the assumed ν_m, one can take the maximum ν to be this value of ν_m decreased by one (or two) units. Of course, for most practical cases of angular correlation it is necessary to distinguish only between $\nu_m = 2$ and 4.

V. MEASUREMENTS WITH A DECAYING SOURCE

In the foregoing it was assumed that for an angular correlation, the lifetime of the source is large compared to the time of measurement. This assumption is well-fulfilled, for example, in the cases studied by Klema and McGowan.[2] If this is not the case corrections must be made for depletion of source strength during the course of the measurements. The explicit problem considered in this section is the optimal procedure for determining the lifetime of the source.

Consider a source which initially has N radioactive nuclei. In most cases N is essentially unknown and is eventually eliminated from the final results. The procedure considered is that one records the time intervals at which $n, 2n, \cdots rn$ counts are obtained. For a source with N' radioactive nuclei, the probability for a decay between time t to $t + dt$ is

$$p dt = N' \mu_0 e^{-N' \mu_0 t} dt. \tag{31}$$

For the moment we assume detector efficiency $\eta = 1$. The probability $P(n, N; t)$ that the nth count occurs in the interval dt at t is obtained from the integro-difference equation (11) with the kernel

$$K(t - t') = (N - n + 1) \mu_0 e^{-(N-n+1)\mu_0 (t - t')}.$$

The solution of (11) is subject to $P(1, N; t) = N \mu_0 e^{-N \mu_0 t}$.

Then the required probability is[6]

$$P(n, N; t) = \frac{N!\mu_0}{(n-1)!(N-n)!} \times [1-e^{-\mu_0 t}]^{n-1} e^{-(N-n+1)\mu_0 t}. \quad (32)$$

For $n\mu_0 t \ll 1$ and $n \ll N$ this reduces to (13) with $\mu = N\mu_0$.

For the sequence of events that n counts are collected in τ_s to $\tau_s + d\tau_s$, $s = 1 \cdots r$, so that nr is the total number of counts recorded, the probability is $P(\tau_1, \tau_2 \cdots \tau_r) d\tau_1 d\tau_2 \cdots d\tau_r$ and

$$P(\tau_1, \tau_2 \cdots \tau_r) = \prod_{s=1}^{r} P(n, N-sn+n; \tau_s)$$

$$= \frac{N!\mu_0{}^r}{(n-1)!^r (N-nr)!} \exp[-\sum_{s=1}^{r} (N-ns+1)\mu_0 \tau_s] \times \prod_{s=1}^{r} [1-e^{-\mu_0 \tau_s}]^{n-1}. \quad (33)$$

The most probable counting rate μ_0 is obtained by eliminating N from the equations

$$\partial P/\partial N = 0, \quad \partial P/\partial \mu_0 = 0.$$

These give

$$\sum_{\nu=1}^{nr} \frac{1}{N-\nu+1} = \mu_0 \sum_{s=1}^{r} \tau_s = \mu_0 T, \quad (34a)$$

and

$$\Phi \equiv \sum_{s=1}^{r} \frac{\mu_0 \tau_s}{e^{\mu_0 \tau_s}-1} = \frac{1}{n-1}\{-r + \sum_{s=1}^{r} (N-ns+1)\mu_0 \tau_s\}. \quad (34b)$$

Introducing

$$\langle r \rangle = \frac{1}{T} \sum_{s=1}^{r} s\tau_s,$$

the resulting equation, after elimination of N is,

$$\sum_{\nu=1}^{nr} [n\mu_0 T\langle r \rangle + r + (n-1)\Phi - \nu\mu_0 T]^{-1} = 1. \quad (35)$$

The given data fix T, $\langle r \rangle$ and Φ as a function of μ_0. Then the root of (35) fixes the most probable counting rate μ_0.

The result (35) may be considered in the case $n=1$. Then we obtain from (34b)

$$(r/\mu_0 T) - (N+1) = -\langle r \rangle. \quad (36a)$$

If in (34a) the sum is replaced by an integral (corresponding to replacing $N!$ by $N \log N - N$), one has

$$(N-nr)/N \cong e^{-\mu_0 T}.$$

Using this result and $n=1$ in (36a) there results

$$\frac{1}{\mu_0 T} - \frac{1}{e^{\mu_0 T}-1} = \frac{1}{r}(r+1-\langle r \rangle). \quad (36b)$$

If we introduce

$$t_s = \sum_{1}^{s} \tau_i,$$

this becomes

$$\frac{1}{\mu_0 T} - \frac{1}{e^{\mu_0 T}-1} = \frac{1}{rT} \sum_{s=1}^{r} t_s. \quad (36c)$$

Equation (36c) is in the form given for this case by Hole[6] and Peierls.[7]

Since nr is usually a very large number compared to unity, the sum involved in (35) is rather difficult to evaluate. Replacement of the sum by an integral is permissible, leading to a comparatively simple result for μ_0, only if the summand never becomes large. This corresponds to values of $N \gg nr$. Then, with $nr \gg 1$,

$$(n-1)\Phi + r + n\mu_0 T\langle r \rangle \cong \mu_0 T nr e^{\mu_0 T}/(e^{\mu_0 T}-1). \quad (37)$$

For counter efficiency $\eta \neq 1$ one must replace the probability function (32) by[6]

$$P(n, N, \eta; t) = \frac{\mu_0 N! \eta^n}{(n-1)!(N-n)!} e^{-\mu_0 t}(1-e^{-\mu_0 t})^{n-1} \times [1-\eta(1-e^{-\mu_0 t})]^{N-n}. \quad (38)$$

For $n \ll N$ and $n\mu_0 t \ll 1$ this reduces to (13) with $\mu = N\mu_0 \eta$. The probability of the sequence of events: n counts in τ_1 to $\tau_1 + d\tau_1 \cdots n$ counts in τ_r to $\tau_r + d\tau_r$ is (per $d\tau_1 d\tau_2 \cdots d\tau_r$)

$$P(\tau_1 \tau_2 \cdots \tau_r) = \prod_{s=1}^{r} P(n, N-sn+n, \eta; \tau_s)$$

$$= \frac{\mu_0{}^r \eta^{nr} N!}{(n-1)!^r (N-nr)!} e^{-\mu_0 t} \prod_{s=1}^{r} (1-e^{-\mu_0 \tau_s})^{n-1} \times \prod_{s=1}^{r} [1-\eta(1-e^{-\mu_0 \tau_s})]^{N-ns}. \quad (39)$$

Application of the conditions $\partial P/\partial N = 0$ and $\partial P/\partial \mu_0 = 0$ and elimination of N again gives a determining equation for μ_0. While the procedure for obtaining a numerical value for μ_0 from the ensuing result is essentially no more difficult than in the case of unit efficiency, the analytical form is even more cumbersome and we need give no further details. Once μ_0 is found, the source depletion in a time t could be obtained from $1-e^{-\mu_0 t}$.

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[6] This result has previously been given by N. Hole, Arkiv. Mat. Astron. Fysik 34B, No. 12 (1948).

[7] R. Peierls, Proc. Roy. Soc. (London) 149, 467 (1935).