heat, since the heat vibrations involve all of the atoms in the lattice, the impurity content relatively few.

SUMMARY

We can sum up our results as follows:

(1) The high secondary yield from large single crystals of MgO is in accord with current band theory. Specifically, it implies that the high yields observed from thin film MgO targets that are used commercially, are not merely a consequence of thin film behavior but are related to body properties of the MgO.

(2) The temperature dependence of the yield can be used to obtain data on the interaction between the lattice and the internal secondaries which have several electron volts of energy.

(3) Field-enhanced emission is not observed in these experiments.

(4) Neutralization of surface charge by bombardment must be approached with care because a lack of uniformity may easily produce highly erroneous results.

(5) No correlation was found between conductivity and yield. However, the causes of the observed conductivity have not been determined.

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Theory of Alpha Decay. I*

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A preliminary report on the application of the perturbation of boundary conditions theory of Feshbach, Weisskopf, and Peaslee to the problem of the decay of a virtual state is presented. In particular, the formulas for alpha decay are derived. The derivation assumes no model of the nucleus nor of nuclear forces. It contains the slope of the logarithmic derivative of the wave function *versus* energy as the only factor related to the nuclear structure; therefore, it includes earlier theories of alpha decay as special cases.

1. INTRODUCTION

IN 1928 Gamow and, independently, Condon and Gurney[1] made use of quantum mechanics to explain the tremendous variation of alpha-decay half-lives with small variations in alpha-particle energy (from 1.4×10^{10} years to 3×10^{-7} second, for alpha-particle energies of 4 to 9 Mev, respectively). The foregoing theories correctly accounted for the exponential-type behavior of the half-life as a function of energy, but solved only the Coulomb barrier penetration problem. This paper presents a preliminary attack upon internal alpha-decay theory. The most frequent assumption of older theories for the internal problem was that of a single particle in a square well ground state. In fact, the spontaneous decay of an alpha particle from a square well bounded by a Coulomb barrier has been rigorously solved as an eigenvalue problem by Preston.[2] Another,[3] admittedly approximate, attack postulated that the intrinsic alpha-decay probability (the decay probability without Coulomb barrier) is roughly the same as that of a neutron of the same energy. The application of the WKJB approximation to the model of a single particle in a square well surrounded by a Coulomb barrier is widely used, but sometimes incorrectly. Sufficient care is not always taken at the joining point R (defined as the edge of the square well and considered in actuality to include the "naked" nuclear radius, the range of alpha-nucleus forces, and the "naked" alpha-particle radius), where Kramers type joining formulas are not applicable even when the drop in potential from Coulomb to the nuclear is gradual through a distance of the order of the nuclear force range. If this drop is taken to be vertical, the wave equation then has a singular point and no Taylor expansion whatever is possible, so that joining of the solutions in this manner completely breaks down.

Although our treatment assumes no model of the nucleus nor of nuclear forces, it does postulate that the alpha particle is an entity at the radius R and thereafter. In addition, this paper asks for a certain smoothness, to be defined below, in the nuclear wave function at the nuclear surface. Because of the generality of this

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[1] G. Gamow, Z. Physik **51**, 204 (1928); G. Gamow and F. Houtermans, Z. Physik **52**, 495 (1928); R. W. Gurney and E. U. Condon, Nature **122**, 439 (1928); and Phys. Rev. **33**, 127 (1929).

[2] M. A. Preston, Phys. Rev. **69**, 535 (1946); **71**, 865 (1947).

[3] H. A. Bethe, Revs. Modern Phys. **9**, 69 (1937). See also B. L. Cohen, Phys. Rev. **80**, 105 (1950).

assumption the present theory includes prior theories as special cases. However, we feel this treatment to be incomplete, and we present it only in the hope that it is useful even in its preliminary form.

2. THE THEORY OF CHARGED PARTICLE DECAY

Following a suggestion of Weisskopf,[4] it is possible to extend the theory of Feshbach, Peaslee, and Weisskopf[5] to the decay of virtual nuclear states. This can be done in the following way.[6]

We discuss the decay of a nucleus, atomic number Z; mass number A; through charged-particle emission (charge z) with orbital angular momentum l, z component m; in a half-life $T_{\frac{1}{2}}$; with a decay constant, $\gamma/\hbar$ sec^{-1}; and with decay energy E. Let r be the distance from the center of mass of the daughter nucleus to that of the emitted-particle. R is the largest value of r for which the potential energy V is appreciably different from $Zze^2/\epsilon_0 r$, the Coulomb potential. With the foregoing notation, then, the substitution $\psi=\Phi^{(l)}(r)r^{-1}Y_l^m(\theta,\phi)$ into Schrödinger's (center-of-mass) equation yields the following for Φ:

$$d^2\Phi^{(l)}/dr^2-U_l(r)\Phi^{(l)}=-(2\mu E/\hbar^2)\Phi^{(l)}, \quad (2.1)$$

with

$$U_l(r)\equiv l(l+1)/r^2+(2\mu/\hbar^2)V(r) \quad \text{for} \quad r>R.$$

The assumption $V=V(r)$ and the separation of variables is made only for $r\geqslant R$. The general solution of (2.1) can be written in the form

$$\Phi^{(l)}=au_l(r)+bv_l(r),$$

with $u_l\equiv v_l^*$ (the asterisk means complex conjugate), a and b constants, and with asymptotic behaviors

$$u_l \underset{r\to\infty}{\longrightarrow} \exp[-ikr+i(\pi l/2)-i\alpha\ln(2kr)+i\eta_l], \quad (2.2a)$$

$$v_l \underset{r\to\infty}{\longrightarrow} \exp[+ikr-i(\pi l/2)+i\alpha\ln(2kr)-i\eta_l], \quad (2.2b)$$

where $\alpha\equiv 2\pi ZZ'l^2/h(\text{velocity})$ and η_l are the usual Coulomb field type corrections to the asymptotic forms,[7] $k=1/\lambda\!\!\!^{-}=(2\mu E)^{\frac{1}{2}}/\hbar$. A decaying state is defined by having outgoing waves only at $r=\infty$; thus, $\Phi^{(l)}=v_l$ for $r\geqslant R$. The other boundary condition of the problem occurs at the nuclear surface, that is at $r=R$. Of interest is the ratio of derivative to value of the radial wave function at R. Define this quantity as f/R, which must equal $(1/v)(dv/dr)$ evaluated at R. We shall leave the subscript l to be understood. If, following FPW, we set $v(r=R)\equiv|v|_R\exp(i\delta)$,

$$x\equiv kR, \text{ and } v'\equiv dv/dx=(1/k)|dv/dr|_{r=R}\exp(i\delta_1); \quad (2.3)$$

then

$$f/R=[(1/v)(dv/dr)]_R$$

or

$$f=x|v'/v|\exp[i(\delta_1-\delta)] \quad (2.4)$$

is the boundary condition on the solution at $r=R$.

The Wronskian of two independent solutions of a canonical second-order differential equation is a constant; hence

$$k\begin{vmatrix} u' & v' \\ u & v \end{vmatrix}=\begin{vmatrix} du/dr & dv/dr \\ u & v \end{vmatrix}_{\substack{\text{evaluated}\\ \text{at } r=\infty}}=-2ik; \quad (2.5)$$

whence, using $u=v^*$, we get

$$|vv'|\sin(\delta_1-\delta)=1. \quad (2.6)$$

If we write

$$f=\Delta+is, \quad (2.7)$$

which is the boundary condition, with Δ and s real and defined by (2.4) and (2.6),

$$\Delta\equiv x|v'/v|\cos(\delta_1-\delta), \quad (2.8a)$$

$$s\equiv x/|v|^2. \quad (2.8b)$$

It is well known that the time variation of the wave function is of the form $\exp(-iWt/\hbar)$ and that a complex W is a description of a decaying state.

We therefore write for the decaying state s of energy E_s,

$$W_s=E_s-i(\gamma_s/2), \quad (2.9)$$

where the decay probability λ_s, of the state s, is simply $\gamma_s/\hbar$, per second, per nucleus.

f is a function of W, so we can expand f in the series

$$f(W_s)=f(E_s)-i(\gamma_s/2)\partial f/\partial E_s+\cdots, \quad (2.10)$$

and use only the linear term, since for charged particle decay $\gamma_s\ll E_s$.

The nuclear boundary condition for decay is expected to be of the same form for each decaying state s, so we are led (FPW) to expect simplification by writing f as a periodic function; $f=X\cot\mathbf{Z}$. $\mathbf{Z}=\mathbf{Z}(E)$ is a quite arbitrary function of E, and $X\equiv KR$, where K is the propagation vector magnitude of the escaping particle just within the nucleus. For example, the nuclear boundary condition for a square well is $f=X\cot[X-(e\pi/2)]$.

Now as we vary E we can expect decaying states only when (2.7) holds, or when (from 2.10)

$$\Delta(E_s)=X_s\cot\mathbf{Z}(E_s), \quad (2.11)$$

and

$$-(\gamma_s/2)\partial f/\partial E|_{E_s}=S. \quad (2.12)$$

We shall later show that for decaying states $(\Delta(E_s)/X_s)$ is a slowly varying function of E_s, so that we can with good accuracy put

$$\mathbf{Z}(E_{s+1})=\mathbf{Z}(E_s)+\pi, \quad (2.13)$$

[4] V. F. Weisskopf (private communication, 1949).
[5] Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 145 (1947), which we refer to as FPW.
[6] See also J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), which we refer to as BW.
[7] See N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, London, 1949), second edition.

since the cotangent is π-periodic, and since the slope of the cotangent is always greater than or equal to one in magnitude, thus is insensitive to small changes in $(\Delta(E_s)/X_s)$. The proof of these statements must await specific calculation of v and v'. We can, however, point out *a priori*, that in the case of charged particle decay with kinetic energies definitely below the top of the Coulomb and centripetal barriers, the magnitudes of v and of v' are large compared to one. This conclusion follows from the fact that the particle probability density in such a case is much greater within the nucleus ($\sim|v|^2$) than at infinity (1).

In the foregoing case, therefore, (2.6) yields

$$\sin(\delta_1-\delta)=|vv'|^{-1}.$$

Thus

$$\Delta=x|v'/v|(1-|vv'|^{-2})^{\frac{1}{2}}\approx x|v'/v|, \qquad (2.14)$$

thus the quantity,

$$\Delta(E_s)/X_s\approx(x/X)|v'/v|, \qquad (2.15)$$

is what we shall have to show is weakly energy dependent. [See (2.32) and thereafter.]

We shall need $\partial f/\partial E|_{E_s}$ in order to get γ_s in (2.12):

$$\partial f/\partial E|_{E_s}=X'\cot\mathbf{Z}(E)|_{E_s}-X\mathbf{Z}'(E)/\sin^2\mathbf{Z}|_{E_s}. \qquad (2.16)$$

The fundamental postulate of this treatment is that the slope of $\mathbf{Z}$ at the energy E_s at which decay occurs can be approximately given by the average slope in the region of E_s, that is, that

$$\mathbf{Z}'|_{E_s}\approx\pi/D, \qquad (2.17)$$

where D is the energy level spacing.[8] ($\mathbf{Z}$ is π-periodic.) This assumption is not meant to be more than a rough approximation.

(2.11) gives

$$(\sin\mathbf{Z})^{-2}=[(\Delta^2/X^2)+1] \text{ at } E=E_s. \qquad (2.18)$$

Substituting Eqs. (2.17), (2.18), and (2.11) into (2.16) gives

$$\left.\frac{\partial f}{\partial E}\right|_{E_s}\sim X'\frac{\Delta}{X}-\frac{X\pi}{D}\left(\frac{\Delta^2}{X^2}+1\right),\quad E=E_s. \qquad (2.18a)$$

Using (2.12), (2.8b), and (2.18a), we obtain

$$\gamma_s\approx\frac{2}{\pi}\frac{xX}{|v|^2}\cdot\frac{D}{[\Delta^2+X^2]-(1/\pi)X'\Delta D},\quad E=E_s. \qquad (2.19)$$

With the aid of (2.14), Eq. (2.19) becomes

$$\gamma_s\approx\frac{D}{2\pi}\cdot\frac{4xX}{x^2|v_l'|^2+X^2|v_l|^2-(1/\pi)xX'|v_l'v_l|D},\quad E=E_s. \qquad (2.20)$$

The last term in the denominator will be shown [see (2.34)] to be small compared to the other two, so so we can neglect it and find

$$\gamma_s\approx\frac{D}{2\pi}\cdot\frac{4xX}{x^2|v_l'|^2+X^2|v_l|^2},\quad E=E_s. \qquad (2.21)$$

The $D/2\pi$ is interpreted as $1/\hbar$ times the intrinsic (nonbarrier) decay probability of the nucleus and the other factor as the discontinuity and barrier transmission probabilities (see interpretation below). Equation (2.21) is the fundamental equation for decay of a charged particle from a nucleus through the Coulomb and centripetal barriers. Approximations (2.20) to (2.21) and (2.15) must yet be verified.

In order to estimate the correction (2.20) to (2.21) it is necessary to assign an energy dependence to K. The usual, and indeed most reasonable, premise is that the kinetic energy of the charged particle should vary linearly with the decaying state energy E. We therefore put

$$K^2=(2\mu/\hbar^2)(E_0+E), \qquad (2.22)$$

where E_0 is a constant. Therefore

$$X_s=(2\mu R^2/\hbar^2)^{\frac{1}{2}}(E_0+E_s)^{\frac{1}{2}}, \qquad (2.23)$$

and

$$X'=\mu R^2/\hbar^2X, \qquad (2.24)$$

so that

$$\gamma_s\approx\frac{D}{2\pi}\cdot\frac{4xX}{x^2|v_l'|^2+X^2|v_l|^2-(\mu R^2/\pi\hbar^2)(x/X)|v_l'v_l|D} \qquad (2.25)$$

is Eq. (2.21) corrected by substituting the assumption (2.22) into (2.20). We shall, however, find that (2.21) is sufficiently accurate for our purposes.

We next calculate v and v' for $l=0$, using the WKBJ approximation.[9] From (2.1) we must solve

$$(d^2v/dr^2)+k^2(r)v=0, \qquad (2.26)$$

where $k^2(r)\equiv(2\mu E/\hbar^2)-(2\mu Zze^2/\hbar^2 r)\equiv A-(B/r)$. This equation defines A and B; note that $k\equiv k(\infty)$. The WKBJ solution is

$$v=\frac{1}{[k(r)]^{\frac{1}{2}}}\exp\left[\pm i\int_\alpha^r k(r)dr\right],\quad \alpha=\text{constant}, \qquad (2.27)$$

which is valid when

$$\frac{d}{dr}\left(\frac{1}{k(r)}\right)\ll 1,$$

or[10] when

$$F(r)\equiv\frac{(B/r^2)}{2[(B/r)-A]^{\frac{3}{2}}}\ll 1. \qquad (2.28)$$

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[8] This is the same postulate as in FPW; see also the discussion there. We shall discuss the type levels D refers to in Sec. 3.

[9] See, for example L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949). This barrier penetration solution is well known from the very earliest attacks on the problem. We outline our method for convenience.

[10] In alpha decay B is about $0.559\times10^{13}\,Z$ cm^{-1}, with A equal to $1.96\times10^{25}\,E$ cm^{-2}, E in Mev, so that the outer turning point $r_1=B/A$ is roughly $2.85\times10^{-13}\,Z/E$ cm.

It is clear that (2.28) holds in the classically allowed region, $r>r_1$; the validity must be examined in the classically forbidden

Using the boundary condition

$$v \underset{r\to\infty}{\longrightarrow} \exp(ikr)$$

and the Kramers joining formulas[9] through the point $r=r_1$, we obtain for $r<r_1$

$$v=y^{\frac{1}{4}}(1-y)^{-\frac{1}{4}}\exp\left[-i\frac{\pi}{4}\int_R^{r_1}\kappa dr\right], \qquad (2.29)$$

where $\kappa\equiv[(B/r)-A]^{\frac{1}{2}}$, and $y\equiv ER/Zze^2=AR/B$, that is, the ratio of the Coulomb barrier height at $r=R$ to the kinetic energy at $r=\infty$.

Integration of (2.29) for $l=0$ and a Coulomb barrier yields

$$v(r)=y^{\frac{1}{4}}(1-y)^{-\frac{1}{4}}\exp\left[-i\frac{\pi}{4}+\left(\frac{2\mu}{\hbar^2}Zze^2R\right)^{\frac{1}{2}}\gamma(y)\right], \qquad (2.30)$$

where[11]

$$\gamma(y)\equiv y^{-\frac{1}{2}}(\text{arc}\cos y^{\frac{1}{2}})-(1-y)^{\frac{1}{2}}.$$

The quantity $|v|^{-2}$ gives the Coulomb penetration factor. Differentiation yields

$$v'(x)=\frac{v(x)}{2k}\left\{\frac{1}{2(1-y)R}-2\left(\frac{2\mu Zze^2}{\hbar^2R}\right)^{\frac{1}{2}}\cdot(1-y)^{\frac{1}{2}}\right\}. \qquad (2.31)$$

Equations (2.31), (2.30), and (2.21) give the computational formulas for $l=0$, charged particle decay which we shall later display for alpha decay.

Discussion of Approximations

We are now able to examine our approximations. We shall do so for the special case of alpha decay in which we are primarily interested. The first is that $(x/X)|v'/v|$ is a weak function of the decay energy. [This is necessary for (2.13); see (2.15).]

From (2.31),

$$\frac{x}{X}\left|\frac{v'}{v}\right|=\frac{1}{2X}\left\{\frac{1}{2(1-y)}-2\left(\frac{2\mu Zze^2R}{\hbar^2}\right)^{\frac{1}{2}}(1-y)^{\frac{1}{2}}\right\}, \qquad (2.32)$$

where y ranges from 0.19 to 0.35 for alpha decayers (including those in the upper middle of the periodic table).

region, however, since (2.28) fails both for small r and for r close to r_1, the turning point. In the worst case, that of $_{62}Sm^{148}$, we have

$F(r)$:	∞	$0.059E^{\frac{1}{2}}$	$0.026E^{\frac{1}{2}}$	$0.019E^{\frac{1}{2}}$	$0.024E^{\frac{1}{2}}$
$r\times10^{13}$ (cm):	$r_1=177/E$	$133/E$	$89/E$	$45/E$	$17.5/E$

with E the alpha-decay energy in Mev.

It appears then that the WKBJ approximation is applicable; however, Eq. (2.28) gives no indication of the accuracy of joining at $r=r_1$.

A check upon the accuracy of the *entire* approximation is obtained by comparison with the exact calculations of Preston and of Tyson [M. A. Preston, Phys. Rev. **71**, 865 (1947); James K. Tyson, Ph.D. thesis, Massachusetts Institute of Technology, 1948 (unpublished)]. This comparison shows that the terms of our WKBJ solution (to be derived below) are accurate for alpha decay to one part in 220, which is fantastically good since we have reason to believe that errors in D are 100 percent to 1000 percent.

11 Compare Bethe, reference 3, p. 161 ff.

X is given by (2.23), so that indeed (2.32) is weakly energy dependent.[12]

We must next show that

$$H\equiv\left[\left(\frac{\mu R^2}{\pi\hbar^2}\right)D\left(\frac{x}{X}\right)\left|\frac{v'}{v}\right|\right]\Big/\left[x^2\left|\frac{v'}{v}\right|^2+X^2\right]\ll1, \qquad (2.33)$$

in order that (2.21) be valid. For kinetic energies within the nucleus of 1 to 16 Mev, $D\leqslant1$ Mev, $R\sim10^{-12}$ cm, and for nuclei in the alpha-decay region,

$$H\lesssim0.01\ll1. \qquad (2.34)$$

Interpretation

Under the assumption of equidistant nuclear energy levels it has been shown[13] that the nuclear period is $2\pi\hbar/D$. The period of a particle in a one-dimensional square well of quantum number n is $T_n=(2\pi\hbar/D)\times(1+1/2n)$, and that of a harmonic oscillator is of course $T=2\pi\hbar/D$ (equally spaced levels). Thus the factor $D/2\pi$ of formula (2.21) may be considered as $\hbar$ multiplied by the time rate of appearance of an outgoing alpha particle at the nuclear radius. The remaining term is easily shown to be exactly the barrier penetration factor of a particle whose wavelength is large compared to the rise of the (Coulomb) barrier at $r=R$. It is therefore asserted that this theory includes the single charged particle in a square well as a special case. It should be recognized that the barrier penetration includes not only the penetration of the body of the Coulomb barrier $(1/|v|^2)$, but also the very real discontinuity barrier at $r=R$, [which may be likened to the neutron penetration, $4kK/(k+K)^2$].

3. THE THEORY OF ALPHA DECAY, CONTINUED, ORBITAL ANGULAR MOMENTUM

The general formula (2.21) is applicable to nonzero angular momenta as well; we must, however, make quite clear what we mean by the energy level spacing D, and further we must verify that the approximations of Sec. 2 are still valid in this case. That D shall be between energy levels in the parent nucleus of the same spin and parity follows immediately from the definition of D $(Z'|_{E_s}\approx\pi/D)$; we vary the energy of the system and postulate that the slope at the decay energy, E_s, will be roughly given by the average slope. No continuous variation of the parameter E from resonance s to $s+1$ can cause a discontinuous change in spin or parity. Nuclei of different spins are, so to speak, "different" nuclei. This can be seen in another way from the interpretation of the nuclear period $T\approx2\pi\hbar/D$. The period of a nucleus with spin J is determined by the level system of such a nucleus (with spin J).

12 Actually we need the value of K only to an order of magnitude, so that we can neglect even its linear variation in the square and will later assign the same wave number, K, to all outgoing alpha particles within the nucleus regardless of the nucleus or of the alpha-decay energy.

13 V. F. Weisskopf, Helv. Phys. Acta **23**, 187 (1950).

Now let us ask for the dependence of D on the orbital angular momentum l of the outgoing alpha particle. There are the following possibilities for l, if i_p is the parent spin and i_d the daughter spin:

$$|i_p+i_d|,\quad |i_p+i_d-1|,\quad \cdots,\quad |i_p-i_d|;$$

a total of the smaller of $(2i_p+1)$ and $(2i_d+1)$ in all. Parity conservation allows a total of $i+1$ or i (i is the smaller of i_p and i_d), only, of these. We assert that the boundary condition of $f_l=X\cot Z_l(E)$ differs for each value l of the outgoing orbital momenta. This assertion immediately follows from the factor $Y_{lm}(\theta,\phi)$ governing the distribution in angles of the complete wave function. Thus we have different Z's and hence separate D's for each l. D must stand for the level spacing between alpha energy levels of equal outgoing orbital angular momentum. In this paper we shall not attempt to distinguish between different D's that are between levels of the same spin and parity.[14] Indeed we shall be forced to take for D the first energy level spacing because of the paucity in data.

The effect of angular momenta in the external barrier penetration is small for small l, as Table I shows. (The centripetal barrier height at $r=R$ is $0.054l(l+1)$ Mev and the Coulomb is 25 Mev.) These values were computed from the data of $_{88}Ra^{226}$ using the method exhibited below.

It should be noted that large spins, because of the possibility of many-channel decay,[14] can lead to barrier penetration greater than a single $l=0$ penetration. For example, in Table I, spin 2 decaying to a spin-2 residual nucleus with even parity would be listed as 1.407 (>1), if one weights all possible channels equally.

TABLE I. Ratio λ_l/λ of the alpha-decay probability for angular momentum l to that for angular momentum zero.

Orbital angular momentum quantum number, l	λ_l/λ
0	1
1	0.7
2	0.37
3	0.137
4	0.0368
5	7.11×10^{-3}
8	9.95×10^{-6}

We now proceed to solve the barrier penetration for $l\neq0$. Our equation is again

$$d^2v_l/dr^2+k_l^2(r)v_l=0, \tag{3.1}$$

but now

$$\begin{aligned}k_l^2(r)&\equiv(2\mu E/\hbar^2)-(2\mu Zze^2)/\hbar^2r-(l(l+1)/r^2)\\&\equiv A-(B/r)-(l(l+1)/r^2),\end{aligned} \tag{3.2}$$

which defines the constant A and B.

The WKBJ solution is

$$v_l(R)=A^{\frac{1}{4}}\left[\frac{l(l+1)}{R^2}+\frac{B}{R}-A\right]^{-\frac{1}{4}}\left[\exp\left(-i\frac{\pi}{4}+\int_R^{r_1'}\kappa dr\right)\right]$$

with

$$\begin{aligned}\kappa&\equiv[(l(l+1)/r^2)+(B/r)-A]^{\frac{1}{2}},\\ r_1'&\equiv(B/2A)\{1+[1+(4A/B^2)l(l+1)]^{\frac{1}{2}}\}.\end{aligned} \tag{3.3}$$

Evaluation of the integral yields

$$\begin{aligned}\int_R^{r_1'}\kappa dr=&-R[l(l+1)/R^2+B/R-A]^{\frac{1}{2}}\\&+[l(l+1)]^{\frac{1}{2}}\ln\left|\frac{(2l(l+1)/R)+B+2[l(l+1)]^{\frac{1}{2}}[l(l+1)/R^2+B/R-A]^{\frac{1}{2}}}{\{B^2+4l(l+1)A+B[B^2+4Al(l+1)]^{\frac{1}{2}}\}/\{B+[B^2+4Al(l+1)]^{\frac{1}{2}}\}}\right|\\&+(B/2A^{\frac{1}{2}})\cdot\{\arcsin(1)-\arcsin[(2AR-B)/(B^2+4Al(l+1))^{\frac{1}{2}}]\}.\end{aligned} \tag{3.4}$$

Formula (3.4)[15] reduces to the exponent of (2.30) by setting $l=0$ if one remembers that

$$\arccos(2u-1)=2\arccos u^{\frac{1}{2}}.$$

Defining

$$w\equiv l(l+1)A/B^2, \tag{3.5}$$

$$\begin{aligned}v_l(x)=&y^{\frac{1}{2}}(w+y-y^2)^{-\frac{1}{4}}\\&\times[1+2y^{-1}[w+w^{\frac{1}{2}}(w+y-y^2)^{\frac{1}{2}}]/(1+4w)^{\frac{1}{2}}]^{\surd[l(l+1)]}\\&\cdot\exp[-\tfrac{1}{4}i\pi+\Gamma(y)],\end{aligned} \tag{3.6}$$

with

$$\Gamma(y)\equiv(B/A^{\frac{1}{2}})\cdot\{\tfrac{1}{2}\arccos[(2y-1)/(1+4w)^{\frac{1}{2}}]-(w+y-y^2)^{\frac{1}{2}}\}.$$

Differentiation gives

$$\begin{aligned}v_l'(x)=&[v_l(x)/x]\{[(2w+y)/4(w+y-y^2)]\\&+[[y(y-1)BA^{-\frac{1}{2}}-w^{\frac{1}{2}}(l(l+1))^{\frac{1}{2}}]/(w+y-y^2)^{\frac{1}{2}}]\}.\end{aligned} \tag{3.7}$$

Equations (3.6), (3.7), and (2.21) give the computational formula for alpha decay in the case $l\neq0$.

Validity of $l\neq0$ Calculations

The energy variation of $(x/X)|v_l'/v_l|$, which we must show to be weak in order that (2.13) may be valid, is easily shown by neglecting the small quantity w in comparison to $y-y^2$ in (3.7), so that $(x/X)|v_l'/v_l|$

[14] Use must here be made of the fact that our problem is a many-body one in order to admit the *a priori* possibility of many orbital angular momentum decay channels for the same energy, E_s. In a one-body potential well, for example, the decaying state energy is different for different decay orbital angular momenta so that for a given energy state, decay is possible for only one orbital angular momenta, l.

[15] Equation (3.4) is identical to Bethe, reference 3, Formula (631), and the first terms of Tyson's asymptotic expansion (see reference 10).

has approximately the energy dependence of $l=0$ for $l\lesssim 10$. Because of the smallness of w the arguments of (2.33) and (2.34) follow verbatim for $l\neq 0$ so that (2.20)→(2.21) is still valid. The situation is not so obvious for the external (barrier) problem, however, since one might expect $-(d/dr)(1/k(r))=k'(r)/k^2(r)$ to become of the order of, or larger than, unity for small r. A short numerical calculation shows that this is not the case and that, in fact, $F(r)$ [defined in (2.28)] is still the same as for $l=0$ unless $l\geqslant 10$. These last remarks can be directly seen by recognizing that $k_{l=0}^2(r)\approx k_{l\neq 0}^2(r)$ and comparing the derivative of $k(r)$ in the two cases. This last ratio, from (3.1), is

$$1+2l(l+1)/RB\approx 1 \text{ for alpha decay,} \tag{3.8}$$

with $B=0.559\times 10^{13}Z/\text{cm}$, R ranges from 8 to 11×10^{-13} cm. The accuracy of joining is the same in this case because the r dependence is essentially that of the Coulomb field at $r=r_1'$.

4. THE EFFECTIVE RADIUS R

In this section we use the formulas derived in the preceding sections to give a computational formula for the nuclear radius R, as defined in Sec. 2. The "effective" radius R can be thought of as including the "naked" radius of the nucleus, the "naked" alpha-particle radius, and the range of nuclear forces between them. Further, the calculations are made for zero angular momentum. Thus in the case of a single channel large angular momentum, the R thus calculated will be too small, and in the case of many channel decay with small-to-zero orbital angular momenta possible, the R will be slightly larger. The effects of angular momenta are negligible for $l=0$, 1, and 2 regardless of the number of channels open, since our accuracy, because of the assumption connected with D, is not good enough to distinguish these cases. However, for $l=3$ and above, the calculation for $l=0$ is misleading. We are forced to calculate for $l=0$, regardless of other considerations, however, because there does not exist one case where the spin of both daughter and parent nucleus in the alpha decay is known.[16] In fact, we know only very few cases where the spin of one member is known. We do not wholly leave the problem of (external) angular momentum with these remarks; we shall take up the problem again in connection with explaining anomalous radii.

The Energy Level Spacing D

We will not be able to get D between levels of equal spin and parity for the simple reason that few spins and parities of the various levels are known.[17] We must therefore estimate the energy and level spacing. It will turn out that the radius R depends weakly on D, so that our estimates may be off by large factors without appreciably altering the nuclear radius; a factor of five in D alters R by at most 5 percent and usually ~2.5 percent. We will therefore, arbitrarily, choose the lowest-lying energy level spacing (obtained from any experiment whatever) in the parent nucleus to be D, and in the case where no experimental evidence exists for the parent in question, we shall estimate the spacing from surrounding nuclei of the same type (i.e., even-odd, etc.). This choice we make for the purpose of consistency. If the experiments truly give the lowest spacing, we shall expect our D to be larger,[18] (for it is unlikely that the ground state and the first excited state are so kind to us as to be of the same spin and parity). On the other hand, there is still the possibility that the low-lying levels have not yet been detected; this leads to a smaller D than we use.

In summary, the requirements that D shall be between alpha-forming levels of the same spin and parity tends to make our choice of D too small, and uncertainty in measurement tends too give to large a D.

Kinetic Energy of an Alpha Particle in the Nucleus

We must determine the propagation constant K or, alternatively, the internal kinetic energy of an alpha particle just within the nuclear radius. A lower bound for this energy is the lowest momentum state of an Einstein-Bose particle in the alpha nuclear-volume, which we take to be given by a radius of $1.57A^{\frac{1}{3}}\times 10^{-13}$ cm. This "alpha nuclear-volume" is larger than the true nuclear volume; it is the "trial" nuclear volume of an escaping alpha particle. Thus it includes the range of nuclear forces as well as the alpha particle radius.

The kinetic energy lower bounds are then: 0.76 Mev for mass number $A=150$, 0.63 for 200, and 0.53 for 250, respectively.

An even numbered conglomerate of Fermi-Dirac particles (for example, an alpha particle) can only be considered to obey Einstein-Bose statistics when the perturbations on its constituent particles are smaller than the binding of the particles in the conglomerate. Such is *not* the case for an alpha particle in a nucleus, where the perturbations are indeed of the order of the nucleon binding energy. Therefore, we may *not* consider the conglomerate of two protons and two neutrons, each pair with opposing spins, to be a true Bose particle. We shall call it a four-particle.

This four-particle, in fact, must obey the exclusion principle to some extent because of its Fermi-Dirac constituents. (For example, two alpha particles cannot occupy the same configuration state.) If it were a

[16] Except of course for the presumption that all even-Z and even-A spins are zero and are of the same parity.

[17] Again except for the presumption of zero spin and even parity for the ground states of even-even nuclei.

[18] The mounting evidence that the first excited states of even-even nuclei are spin 2, positive parity, indicates that D is probably larger for even-even nuclei. See, for example, the paper of Gertrude Scharff-Goldhaber, Phys. Rev. **90**, 587 (1953).

complete Fermi-Dirac particle, we can use

$$N=V_cV_p/h^3, \tag{4.1}$$

where N is the number of equal particles, V_c the volume in configuration space, and V_p the volume in momentum space, to arrive at a figure of 5.02 Mev for the kinetic energy of the highest energy Fermi-Dirac four-particle (when formed) in the nucleus. This figure is independent of A. We, therefore, suggest that the true kinetic energy of an alpha particle in the nucleus lies between 5 and 0.5 Mev.

The foregoing remarks lead to a best guess of the kinetic energy of a four-particle in the nucleus of the order of 3 Mev. K shall, therefore, be chosen to correspond to 3 Mev for ground-state transitions, less for transitions to excited states, and more for transitions from excited states ("long range"). Fortunately the theory is even more weakly dependent on the choice of the internal kinetic energy of the alpha particle than on the choice of D, so that a factor of five change in this energy is not significant, (a factor of $\frac{1}{5}$ times the internal kinetic energy chosen changes the radius by less than 2 percent, a factor of five by less than 0.6 percent).

Electron Screening Correction

The electron shell about the nucleus weakens the positive potential barrier. Using the Thomas-Fermi approximation,[19] the first-order atomic electron correction to the Coulomb potential is a constant. We choose to add this correction to the decay energy as follows:

$$E_t=E_d+72.8\times10^{-6}(Z_d)^{4/3}\text{ Mev}, \tag{4.2}$$

where E_t is the total energy, which is utilized in the foregoing formalism, E_d is the measured decay energy (energy of alpha+recoil nucleus), and Z_d is the charge of the daughter nucleus.[20]

Calculation of the Radius R

We take formulas (2.21), (2.30), (2.31) together with the preceding work of this section and obtain

$$\gamma_\alpha\sim\frac{2D_0}{\pi}\left(\frac{3}{E_t}\right)^{\frac{1}{2}}\cdot\left(\frac{1-y}{y}\right)^{\frac{1}{2}}\times\left\{\exp\left[-2\left(\frac{2\mu}{\hbar^2}Zze^2R\right)^{\frac{1}{2}}\cdot\gamma(y)\right]\right\}\cdot\left\{\frac{3}{E_t}+\frac{1}{4k^2}\left[\frac{1}{2(1-y)R}-2\left(\frac{2\mu}{R\hbar^2}Zze^2\right)^{\frac{1}{2}}\cdot(1-y)^{\frac{1}{2}}\right]^2\right\}. \tag{4.3}$$

The inaccuracy in D and in K permits rough approximations for factors not of the exponential in (4.3). Thus for such terms we put $R=1.57\times10^{-13}A^{\frac{1}{3}}$ cm^2 and thus $y\approx\chi\equiv0.545E_tA^{\frac{1}{3}}Z^{-1}$. The term $1/[2(1-y)R]$ turns out not only to be negligible but to give a better approximation (according to the formulas of Tyson, reference 10) if neglected. So that (4.3) becomes

$$y^{\frac{1}{2}}\gamma(y)=\frac{E_t^{\frac{1}{2}}(1+4/A_d)^{\frac{1}{2}}}{2.520Z_d}\ln\left\{\frac{D_0}{\gamma_\alpha}\cdot0.367\frac{Y^{\frac{1}{2}}}{1+\frac{1}{3}Y}\right\} \tag{4.4}$$

with

$$Y\equiv(Z_d/0.545A_d^{\frac{1}{3}})-E_t, \tag{4.5}$$

$$E_t=E_d+72.8\times10^{-6}Z_d^{4/3}, \tag{4.2}$$

$$R=2.879\times10^{-13}yZ_d/E_t, \tag{4.6}$$

where E_t, E_d, γ_α, D_0 are in Mev. Z_d and A_d refer to the daughter nucleus.

Equations (4.4), (4.5), (4.2), and (4.6) comprise the computational formulas for alpha decay. These formulas are good only for normal alpha decay and make no pretense of giving proper radii from anomalous fine structure [*viz.*, the most energetic lines of Am^{241}; see Asaro, Reynolds, and Perlman, Phys. Rev. **87**, 277 (1952)]. We have not changed K for different energies E_t, neither among different nuclei, nor in fine structure of the same nucleus. The effect of the variations is negligible, and we do not know K well anyway.

The radii calculated on the basis of the foregoing theory will be presented and discussed critically in Part II, to be submitted shortly for publication in *The Physical Review.*

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[19] E. Thomas, Proc. Cambridge Phil. Soc. **23**, 542 (1927); E. Fermi, Z. Physik **48**, 73 (1928); E. B. Baker, Phys. Rev. **36**, 630 (1930).

[20] The importance of this correction was emphasized by G. Ambrosino and H. Piatier, Compt. rend. **232**, 400 (1951). We use Z_d because the inner shells can rearrange to the daughter Coulomb field. See P. Benoist-Guental, J. phys. et radium **13**, 486 (1952). We have made use of the work of Rasmussen, Thompson, and Ghiorso, University of California Radiation Laboratory Report UCRL-1473, 1952 (unpublished) and their references to older work.