state is due not to the gross laminations but to the fine structure which Meshkovski and Shalnikov[10] found to be interspersed within the normal laminations in their measurements on large spheres. It is possible that the mean spacing of this fine structure is determined more by the local variations of impurities and strains than by the external geometry of the specimen.

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Electric Field and Energy in Dipole Lattices*

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In theories of ferroelectric crystals, the evaluation of the electric field produced by dipole lattices in every lattice point is needed. A very simple method, which generalizes Ewald's[1] ideas for dipoles, is presented here.

Consider a dipole $\mathbf{p}$ and a spherically symmetrical function $f(r)$ defined in such a manner that

$$\int f(r)dv(r)=1, \tag{1}$$

and

$$f(r)=0 \quad \text{for} \quad r\geq R. \tag{2}$$

Then it can be shown that

$$\sigma(r)=-\mathbf{p}\cdot\text{grad}f(r) \tag{3}$$

is the "equivalent electric density of the dipole p;" in other words, the potential V at a point $\mathbf{R}$ is given by the classical Coulombian expression,

$$V(\mathbf{R})=\int\frac{\sigma(r)}{|\mathbf{R}-\mathbf{r}|}dv(r)=\mathbf{p}\cdot\mathbf{R}/R^3. \tag{4}$$

The author has previously developed a method for the calculation of fields, potentials, and energies[2] in ionic lattices when the charge density around every point of the lattice is given. When this method is applied to the case of periodical lattices of dipoles $\mathbf{p}_j$ in points $\mathbf{r}_j$ of the unit cell with charge densities of the form,[3] then the following relations are found to hold for the interaction energy W and the field $\mathbf{E}_j$ in a lattice point $\mathbf{r}_j$ with moment $\mathbf{p}_j$:

$$W=(2\pi V^{-1})\Sigma_h[\mathbf{h}\cdot\mathbf{S}(h)\varphi(h)/h]^2-(2\pi/3)\Sigma_j p_j^2\int f^2(r)dv(r), \tag{5}$$

$$\mathbf{E}_j=-\partial W/\partial p_j=-(4\pi V^{-1})\Sigma_h[\varphi(h)/h]^2[\mathbf{h}\cdot\mathbf{S}(h)]\mathbf{h}\times\exp(-2\pi i\mathbf{h}\cdot\mathbf{r}_j)-(4\pi/3)\mathbf{p}_j\int f^2(r)dv(r). \tag{6}$$

The notation is as follows: the first summation Σ_h in Eq. (5) is over all the points $\mathbf{h}$ of the reciprocal space. The second summation Σ_j is over the unit cell of volume V. $\varphi(h)$ is the Fourier transform of $f(r)$.

The method allows one readily to take into account the symmetry of the particular space-group by the use of the "dipole structure factor":

$$\mathbf{S}(h)=\Sigma_j\mathbf{p}_j\exp(2\pi i\mathbf{h}\cdot\mathbf{r}_j). \tag{7}$$

$f(r)$ and its Fourier transform $\varphi(h)$ can be chosen in such a way as to yield rapidly convergent series.

The field $\mathbf{E}_j$ is a linear function of the moments $\mathbf{p}_j$, the coefficients of which are the Lorentz factors. The field in points $\mathbf{r}_j$ exterior to atoms has the same form as Eq. (6) but without the last term. It can be shown that in the modification of the Ewald method, as used for instance by Schweinler[3] in the case of dipoles, $f(r)$ is a Gaussian function; but then two series are needed. The physical interpretation is the same as in the ionic case of charges only and has been given first by Ewald[4] and by the author[2] in a slightly different form. The second series is merely a correction due to the overlapping of Gaussian charges. The idea of the "equivalent electric density" can be generalized for multipoles. A more detailed paper on the entire subject will appear later.

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Structure and Mass Spectrum of Elementary Particles. I. General Considerations

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As discussed in previous papers,[1] the nonlocal field was introduced in order to describe relativistically a system which was elementary in the sense that it could no longer be decomposed into more elementary constituents, but was so substantial, nevertheless, as to be able to contain implicitly a great variety of particles with different masses, spins, and other intrinsic properties. However, the conclusions reached so far were very unsatisfactory in many respects.[2] Among other things, the masses of the particles associated with the irreducible nonlocal fields remained completely arbitrary and simple and plausible assumptions concerning the interaction between fields did not result in the expected convergence of self-energies. It seems to the author that these disappointing consequences are not inherent in nonlocal field theory, in general, but are rather related to the particular type of field to which the author restricted himself. Instead, if we start anew from less restricted nonlocal fields, a more promising aspect of possible nonlocal theories is revealed, as shown in the following.

Let us take a scalar (or pseudoscalar) nonlocal field,

$$(x_\mu'|\,\varphi|\,x_\mu'')\equiv\varphi(X_\mu,r_\mu),$$

where x_μ', x_μ'' ($\mu=1, 2, 3, 4$) stand for two sets of space-time parameters and

$$X_\mu=(x_\mu'+x_\mu'')/2, \quad r_\mu=x_\mu'-x_\mu''.$$

The free field equation is supposed to have a general form

$$F(\partial/\partial X_\mu, r_\mu, \partial/\partial r_\mu)\varphi(X_\mu,r_\mu), \tag{1}$$

where the operator F is a certain invariant function of $\partial/\partial X_\mu$, r_μ, and $\partial/\partial r_\mu$ and is independent of X_μ so that it is invariant under any inhomogeneous Lorentz transformation. In particular, if we assume that F is linear in $\partial^2/\partial X_\mu\partial X_\mu$ and separable, i.e.,

$$F\equiv\frac{\partial^2}{\partial X_\mu\partial X_\mu}+F^{(r)}\left(r_\mu r_\mu, \frac{\partial^2}{\partial r_\mu\partial r_\mu}, r_\mu\frac{\partial}{\partial r_\mu}\right), \tag{2}$$

we have eigensolutions of the form $\varphi\equiv u(X)\chi(r)$, where u and χ satisfy

$$(\partial^2/\partial X_\mu\partial X_\mu-\mu)u(X)=0, \tag{3}$$

$$(F^{(r)}-\mu)\chi(r)=0, \tag{4}$$

μ being the separation constant. Thus, the masses of the free particles associated with the nonlocal field φ are given as the eigenvalues of $\mu^{\frac{1}{2}}$ in Eq. (4) for the internal eigenfunction χ If one chooses the operator $F^{(r)}$ such that the eigenvalues $\mu_n\equiv m_n^2$ are all positive and discrete, one can expand an arbitrary nonlocal field φ into a series of internal eigenfunctions, $\chi_n(r)$:

$$\varphi(X,r)=\Sigma_n u_n(X)\chi_n(r). \tag{5}$$

Now, the field equations for a scalar nonlocal field $(x'|\,\varphi|\,x'')$ interacting with a local spinor field $\psi(x')$, for instance, can be

deduced from an appropriate Lagrangian and are

$$\left\{-\frac{\partial^2}{\partial X_\mu \partial X_\mu}+F^{(r)}\right\}\varphi(X,r)=-g\Sigma_\alpha\bar{\psi}_\alpha(X+\tfrac{1}{2}r)\psi_\alpha(X-\tfrac{1}{2}r), \quad (6)$$

$$\gamma_\mu\frac{\partial\psi(x')}{\partial x_\mu'}+M\psi(x')=-g\int\psi(x'')(x''|\,\varphi|\,x')dx''. \quad (7)$$

We insert (5) in (6), multiply both sides by the complex conjugate $\chi_n^*(r)$, and integrate over the four-dimensional space of $r_1, r_2, r_3, r_0=-ir_4$. The result is

$$\left(\frac{\partial^2}{\partial x_\mu''\partial x_\mu''}-m_n^2\right)u_n(x'')=\int\Phi_n(x',x'',x''')\Sigma_\alpha\bar{\psi}_\alpha(x')\psi_\alpha(x''')dx'dx''', \quad (8)$$

where

$$\Phi_n(x',x'',x''')\equiv g\chi_n^*(x'-x''')\delta(\tfrac{1}{2}(x'+x''')-x''). \quad (9)$$

Similarly, we obtain from (7) the equation

$$\gamma_\mu\frac{\partial\psi(x')}{\partial x_\mu'}+M\psi(x')=-\Sigma_n\int\Phi_n(x',x'',x''')\times u_n(x'')\psi(x''')dx''dx'''. \quad (10)$$

If we compare these equations with the corresponding equations (19) of Møller and Kristensen[3] in the theory of nonlocal interaction between a local scalar (or pseudoscalar) field and a local spinor field, we notice that the internal eigenfunction $\chi_n(r)$ plays the role of a convergence factor. There is, however, an essential difference between their equations and ours. Namely, in our theory, we are obliged to take into account simultaneously all the particles with different masses m_n which were derived from an eigenvalue problem. Furthermore, the form function for each of these particles is uniquely determined by the same eigenvalue problem.

In the following letter, the above general considerations will be illustrated and further details will be examined.

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Structure and Mass Spectrum of Elementary Particles. II. Oscillator Model

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AS an illustration of the general considerations on nonlocal fields in the preceding letter, let us assume that the operator F has a very simple form

$$F\equiv-\frac{\partial^2}{\partial X_\mu\partial X_\mu}+\frac{\lambda^2}{2}\left(-\frac{\partial^2}{\partial r_\mu\partial r_\mu}+\frac{1}{\lambda^4}r_\mu r_\mu\right)^2, \quad (1)$$

where λ is a small constant with the dimension of length. One may call this the four-dimensional oscillator model for the elementary particle, which was considered first by Born[1] in connection with his idea of self-reciprocity. However, our model differs from his model in that we have introduced internal degrees of freedom of the particles which are related to the nonlocalizability of the field itself. The internal eigenfunctions in our case are

$$\chi_{n_1n_2n_3n_0}(r)=H_{n_1}(r_1/\lambda)H_{n_2}(r_2/\lambda)H_{n_3}(r_3/\lambda)\times H_{n_0}(r_0/\lambda)\exp\{-(r_1^2+r_2^2+r_3^2+r_0^2)/2\lambda^2\}, \quad (2)$$

and the corresponding eigenvalues for the mass become

$$m_{n_1n_2n_3n_0}=(\sqrt{2}/\lambda)\,|\,n_1+n_2+n_3-n_0+1\,|, \quad (3)$$

where $r_0=-ir_4$ is a real variable and n_1, n_2, n_3, n_0 are quantum numbers which can take only zero or positive integer values. $H_n(x)$ denotes the Hermite polynomial of x of degree n. All these eigenfunctions (2) decrease rapidly in any direction whatsoever in the four-dimensional r space. Furthermore, the Fourier transform of each of these eigenfunctions has exactly the same form as the original function due to the self-reciprocity. Thus, the form function (9) in the preceding letter seems to be sufficient to cut off high energy-momentum intermediate states in such a way that each term corresponding to each Feynman diagram in the expansion of the nonlocal S-matrix according to the Bloch-Kristensen-Møller formulation is convergent. However, since we have to take into account all of infinitely many of different mass states of the nonlocal system, the number of terms in the S matrix increases very rapidly with the increasing power of the coupling constant, so that we can claim nothing for the moment concerning the convergence or divergence of the S matrix as a whole.

The totality of the internal eigenfunctions (2) constitutes a complete set of orthogonal and quadratically integrable functions in the four-dimensional r space and can be regarded as the eigenvectors for an infinite-dimensional unitary representation of the Lorentz group. The eigenvalues (3) for the mass are all infinitely degenerate. For instance, all those values of n's which satisfy $n_1+n_2+n_3-n_0=0$ give the same mass, $m_0=\sqrt{2}/\lambda$. This is not a peculiar feature of the oscillator model; it is common to all those models for which the operator F is separable, because there can be no unitary representation of finite dimensions for the Lorentz group. Presumably, such an undesired degeneracy could be removed either by introducing interaction with other fields or by first introducing the coupling between the external and internal degrees of freedom. The latter possibility can be illustrated by the addition of the coupling term,

$$-\beta^2\lambda^2\left\{-\left(\frac{\partial^2}{\partial X_\mu\partial X_\mu}\right)^2+\frac{1}{\lambda^4}\left(r_\mu\frac{\partial}{\partial X_\mu}\right)^2\right\}, \quad (4)$$

to the expression (1) for F, where β is a dimensionless real constant. The free field equation becomes

$$\left[k_\mu k_\mu+\frac{\lambda^2}{2}\left(-\frac{\partial^2}{\partial r_\mu\partial r_\mu}+\frac{1}{\lambda^4}r_\mu r_\mu\right)^2+\beta^2\lambda^2\left\{-\left(k_\mu\frac{\partial}{\partial r_\mu}\right)^2+\frac{1}{\lambda^4}(k_\mu r_\mu)^2\right\}\right]\chi(k_\mu,r_\mu)=0, \quad (5)$$

in the eight-dimensional space of k_μ and r_μ, where $\chi(k_\mu,r_\mu)$ is the Fourier transform of $\varphi(X_\mu,r_\mu)$ as defined by

$$\varphi(X_\mu,r_\mu)=\int\exp(ik_\mu X_\mu)\chi(k_\mu,r_\mu)(dk_\mu).^4 \quad (6)$$

One can solve Eq. (5) in the coordinate system in which only one component of the wave vector is different from zero.[2] Thus, one obtains the mass spectrum

$$m_{n_1n_2n_3n_0}=\frac{\sqrt{2}}{\lambda}\frac{|\,n_1+n_2+n_3-n_0+1\,|}{[1-2\beta^2(n_0+\frac{1}{2})]^{\frac{1}{2}}}, \quad (7)$$

where n_0 is restricted by the condition

$$n_0<\tfrac{1}{2}(1/\beta^2-1). \quad (8)$$

If we take, for instance, $\beta=1/\sqrt{2}$, only $n_0=0$ is allowed and the mass spectrum reduces to

$$m_{n_1n_2n_3n_0}=(2/\lambda)(n_1+n_2+n_3+1), \quad (9)$$

and the degree of degeneracy of the mass eigenvalues is now finite. In particular, the lowest mass, $m_0=2/\lambda$, is free from degeneracy and the corresponding solution of (5) is given by

$$\chi_{0000}(k_\mu,r_\mu)=\exp\left\{-\frac{1}{2\lambda^2}\left(r_\mu r_\mu+\frac{2(k_\mu r_\mu)^2}{m_0^2}\right)\right\} \quad (10)$$

in an arbitrary coordinate system, where $k_\mu k_\mu=-m_0^2$.

The above advantage of introducing the coupling between external and internal degrees of freedom is offset, however, by a