Mass Surfaces

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We examine in the light of recent experimental data smooth mass surfaces which may be placed in the form

$$M-A=\Delta_m(A)+J(A)[D-D_m(A)]^2,$$

where $\Delta_m(A)$, $D_m(A)$, and $J(A)$ are key functions which characterize the mass surface and $D=N-Z$ is the neutron excess. We here attempt to find an optimum set of key functions and to evaluate various semi-empirical key functions now in use.

In this study we introduce the reference key functions

$$\Delta_m(A)=(A-100)^2/100-64, \text{ (mMU)}$$
$$J(A)=25/A, \text{ (mMU)}$$

and

$$D_m(A)=0.4A^2/(A+200).$$

We use these reference functions in such a way as to effectively subject the data and various semi-empirical functions to microscopic examination, so that "fit" becomes immediately apparent. We note that most of the semi-empirical mass surfaces in current use give rise to large systematic errors in nuclear masses. The large errors are not inherent properties of the semi-empirical equation since a set of constants can be found which reduce these errors to within the range of uncertainty caused by shell effects.

1. A REFERENCE MASS SURFACE

NUCLEAR mass surfaces have been used in pure and applied nuclear physics for the following reasons: (1) They serve to systematize existing experimental information concerning nuclear masses and energies and thereby provide a succinct summary of a large quantity of information. (2) They provide a reasonable basis for predicting unknown nuclear masses and energies. (3) They serve as smooth base surfaces from which irregularities of the experimental mass surface may be charted permitting thereby a careful study of these irregularities. (4) They serve to test various statistical theories of the nucleus. In all but the last application the theoretical justification of the expression used for the nuclear surface is unimportant as compared to the accuracy and simplicity of the mathematical representation. Since the mass surfaces which have some direct theoretical foundations are rathe cumbersome to use and, as we shall see, quite inaccurate we have developed a simpler and more accurate reference surface. This reference surface is a member of the class of surfaces in which the mass decrement, $\Delta=M-A$, may be expressed in the form

$$\Delta=\Delta_m(A)+J(A)[D-D_m(A)]^2, \tag{1}$$

where $D=N-Z$ is the neutron excess and $\Delta_m(A)$, $D_m(A)$, and $J(A)$ are functions of the mass number. Mass data, beta-decay data, and various theoretical models of the nucleus suggest that, apart from shell and pairing discontinuities, a surface of this general form may be used as an approximate representation of nuclear mass decrements. The function $\Delta_m(A)$ fixes the depth of the valley of the mass surface. The term $J(A)[D-D_m(A)]^2$ is based upon the assumption that apart from the shell and pairing discontinuities isobaric sections of the mass surface are parabolas. $D_m(A)$ fixes the neutron excess of the vertex and $J(A)$ characterizes the width of the parabola. The parabolic (A, Z) surface corresponding to Eq. (1) may be obtained by letting $D=A-2Z$ and $D_m(A)=A-2Z_m(A)$.

TABLE I. Key reference functions.

A	$\Delta_m{}^r(A)$ mMU	$J^r(A)$ mMU	$D_m{}^r(A)$	A	$\Delta_m{}^r(A)$ mMU	$J^r(A)$ mMU	$D_m{}^r(A)$
10	17.000	2.500	0.190	140	−48.000	0.179	23.059
20	0.000	1.250	0.727	150	−39.000	0.167	25.714
30	−15.000	0.833	1.565	160	−28.000	0.156	28.444
40	−28.000	0.625	2.667	170	−15.000	0.147	31.243
50	−39.000	0.500	4.000	180	−0.000	0.139	34.105
60	−48.000	0.417	5.538	190	17.000	0.132	37.026
70	−55.000	0.357	7.259	200	36.000	0.125	40.000
80	−60.000	0.312	9.143	210	57.000	0.119	43.024
90	−63.000	0.278	11.172	220	80.000	0.114	46.095
100	−64.000	0.250	13.333	230	105.000	0.109	49.209
110	−63.000	0.227	15.613	240	132.000	0.104	52.364
120	−60.000	0.208	18.000	250	161.000	0.100	55.555
130	−55.000	0.192	20.485				

For our reference mass surface we use for $A>10$ the simple key functions[1]

$$\Delta_m{}^r(A)=(A-100)^2/100-64, \text{ mMU} \tag{2}$$

$$J^r(A)=25/A, \text{ mMU} \tag{3}$$

and

$$D_m{}^r(A)=0.4A^2/(A+200). \tag{4}$$

Apart from accuracy, convenience and simplicity of computation were major considerations in our choice of

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[1] A. E. S. Green, Phys. Rev. **86**, 654 (1952). The tentative change of the constant in Eq. (2) from 62 to 64 was in response to a large change (8 mMU) in the heavy masses.

Eqs. (2)–(4). To show the order of magnitude of these functions, we give in Table I the values of these key functions for various mass numbers. Since $[D-D_m^r(A)]^2$ are usually numbers of the order of unity, we note if we compare $J^r(A)$ with $\Delta_m^r(A)$ that the latter is the dominant term in the expression for mass decrements. The over-all accuracy of $\Delta_m^r(A)$ will become apparent when we discuss Fig. 1 (see Sec. 3).

2. SEMI-EMPIRICAL MASS SURFACES

According to the well-known statistical theory of Weiszäcker[2] Bethe,[3] Bohr and Wheeler,[4] and others, nuclear energies may be represented by

$$E^w = -a_1A + a_2A^{2/3} + a_3(Z^2/A^{1/3}) + a_4(N-Z)^2/4A. \quad (5)$$

While the constants a_1, a_2, a_3, and a_4 may be related to physically important parameters it is usual to relax the theoretical constraints and instead to adjust these constants to fit the experimental data. For this reason Eq. (5) is referred to as a semi-empirical. The equation for mass decrements corresponding to Eq. (5) expressed as a function of the mass number and the neutron excess is

$$\Delta^w = \tfrac{1}{2}(A+D)\Delta_n + \tfrac{1}{2}(A-D)\Delta_{\mathrm{H}} - a_1A + a_2A^{2/3} + a_3(A-D)^2/4A^{1/3} + a_4D^2/4A, \quad (6)$$

where Δ_n and Δ_{H} are the mass decrements of the neutron and the hydrogen atom. We may place this expression in the form of Eq. (1) by finding the function $D_m(A)$ which gives the minimum of Δ^w for a fixed A and by substituting this expression into Eq. (6). They key functions so obtained are

$$\Delta_m^w(A) = -[a_1 - (3\Delta_n + \Delta_{\mathrm{H}})/4]A + (a_4 + \Delta_n - \Delta_{\mathrm{H}})D_m^w(A)/4 + a_2A^{2/3}, \quad (7)$$

$$D_m^w(A) = A[\rho A^{2/3} - (\Delta_n - \Delta_{\mathrm{H}})/a_4]/(1+\rho A^{2/3}), \quad (8)$$

$$J^w(A) = (a_4/4A)(1+\rho A^{2/3}), \quad (9)$$

where $\rho = a_3/a_4$. In some treatments the Z in Eq. (5) is replaced by $Z(Z-1)$. The key functions then are

$$\Delta_m^w(A) = -[a_1 - (3\Delta_n + \Delta_{\mathrm{H}})/4]A + (a_4 + \Delta_n - \Delta_{\mathrm{H}})D_m^w(A)/4 + (a_2 - a_3/4)A^{2/3} + a_3D_m^w(A)/4A^{1/3}, \quad (7')$$

$$D_m^w(A) = [\rho A^{2/3}(A-1) - (\Delta_n - \Delta_{\mathrm{H}})/a_4]/(1+\rho A^{2/3}). \quad (8')$$

$J^w(A)$ is unchanged. In Table II we list various sets of empirical constants which have appeared in the literature. Whenever they are known we have listed the proton and neutron mass decrements originally used in the adjustment of the empirical constants. The last set of constants is a tentative set arrived at in this paper. In the last column we indicate the equations to which these constants refer, and in the next to the last column we list the fission constant $2a_2/a_3$.

3. THE OPTIMUM MASS SURFACE

We shall now attempt to use the experimental data to evaluate a set of functions which within the class represented by Eq. (1) provide the optimum representation of this data. We shall denote these functions by the superscript letter o. Instead of attempting to evaluate this function directly we shall introduce a set of parameters which characterize the deviations of the optimum functions from our empirical functions. For this purpose we shall define R^o, Υ^o and r^o by

$$R^o = \Delta_m^o(A) - \Delta_m^r(A),$$

$$\Upsilon^o = D_m^o(A) - D_m^r(A), \quad (10)$$

and

$$r^o = J^o(A)/J^r(A). \quad (11)$$

In a similar way we may define three parameters R^w, Υ^w, and r^w to characterize the deviations of the Weiszäcker functions from our reference functions. These deviation functions may be computed for any set of semi-empirical constants.[5] We may now in principle use the experimental data to evaluate R^o, Υ^o, and r^o. The extents to which R^o deviates from 0, Υ^o deviates from 0, and r^o deviates from 1 measure the inaccuracy of our

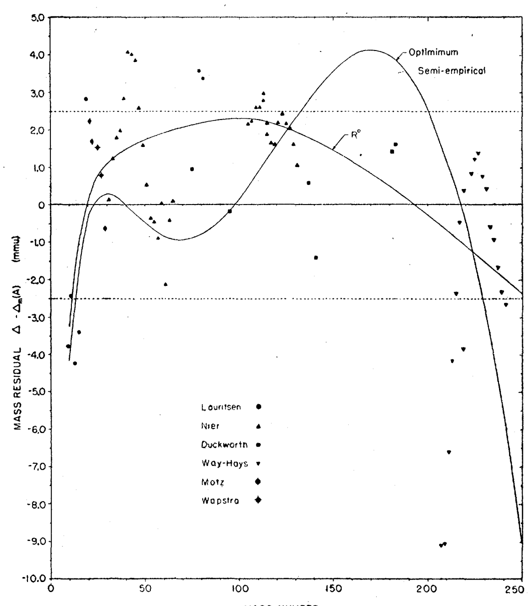

FIG. 1. Mass residuals for beta-stable odd-A nuclide *vs* mass number. R^o is our tentative, optimum, smooth residual. The optimum semi-empirical curve represents the residual for the semi-empirical equation arrived at in this paper. The base line represents our reference function $\Delta_m^r(A)$. Sources of experimental data are indicated by the symbols (see references 6–12).

[2] C. F. von Weiszäcker, Z. Physik **96**, 431 (1935).
[3] H. A. Bethe and R. F. Bacher, Revs. Modern Phys. **8**, 165 (1936).
[4] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).
[5] A preliminary report of this study of the semi-empirical equation was made at the March, 1952 meeting of the American Physical Society in Columbus, Ohio [N. Engler and A. E. S. Green, Phys. Rev. **86**, 654 (1952)].

TABLE II. Sets of empirical constants (in mMU).

Sym	Author	Ref	Δ_n	Δ_H	a_1	a_2	a_3	a_4	$2a_2/a_3$	Eqs.
I	Bethe	a	8.450	8.070	14.885	14.176	0.623	83.770	43.5	7, 8, 9
II	Fermi	b	8.930	8.123	15.04	14.0	0.627	83	44.7	7, 8, 9
III	Mattauch	c	8.945	8.131	15.74	16.5	0.647	88.24	51.0	7, 8, 9
IV	Feenberg	d	8.920	8.130	15.035	14.069	0.627	77.755	44.9	7′, 8′, 9
V	Pryce	e	8.930	8.123	15.089	15.035	0.655	84.199	45.9	7, 8, 9
VI	Metropolis	f	8.982	8.142	15.0825	14.0	0.627	82.970	44.7	7, 8, 9
VII	Fowler	g	8.930	8.132	16.432	17.989	0.741	96.872	48.6	7′, 8′, 9
VIII	This paper		8.982	8.142	16.720	18.500	0.750	100.00	49.3	7, 8, 9

[a] See reference 3.
[b] C. Goodman, *The Science and Engineering of Nuclear Power* (Addison Wesley Press, Cambridge, 1947), Chap. 2 by M. Deutsch.
[c] J. Mattauch and S. Flugge, *Introduction to Nuclear Physics* (Interscience Publishers, Inc., New York, 1946).
[d] E. Feenberg, Revs. Modern Phys. **19**, 239 (1947). This comprehensive article lists several sets of constants and goes deeply into the question of the variation of semi-empirical constants. The particular set referred to here is quoted by J. M. Blatt and V. F. Weisskopf, in *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).
[e] M. H. L. Pryce, Proc. Phys. Soc. (London) **63**, 692 (1950). The neutron and proton masses for 1950 were assumed here since they were not given in the paper.
[f] N. Metropolis and G. Reitweiser, *Table of Atomic Masses*, U. S. Atomic Energy Commission Report NP 1980, March, 1950 (unpublished).
[g] W. A. Fowler (unpublished) quoted on p. 11 of W. E. Siri, *Isotopic Tracers and Nuclear Radiations* (McGraw-Hill Book Company, Inc., New York, 1949).

reference functions. The extents to which R^o deviates from R^w, Υ^o deviates from Υ^w, and r^o deviates from r^w measure the inaccuracy of the semi-empirical function. By this procedure we shall in effect make microscopic examination of the data in relation to the analytical expressions which we are investigating.

4. THE DEVIATION FUNCTION R^o

If the experimental masses conform to Eq. (1), then $D-D_m{}^o(A)$ for the beta-stable odd mass nuclides should take on random values between ± 1 since if the neutron excess were more than one unit away from the valley for a particular set of isobars, beta-decay would produce a nuclide with less mass. Thus on the average the parabolic term contributes approximately

$$25\langle[D-D_m{}^o(A)]^2\rangle/A \approx 12.5/A, \qquad (12)$$

a quantity which may be ignored for $A>10$. We may therefore use the mass decrement values for the beta-stable odd nuclides to represent rather accurately the variation with mass number of the valley points of the mass surface. In Fig. 1 we plot the residuals of the experimental mass decrements for beta-stable odd nuclides *vs* A.

The experimental data used in Fig. 1 has for the most part been compiled from mass determinations reported since 1950 by Nier,[6] Duckworth,[7] Lauritsen,[8] Motz,[9] and their co-workers, and by Wapstra.[10] The sources of the data are indicated on the figure. For the very heavy nuclides we used a new mass table based upon the mass value for Pb^{208} reported by Hays, Richards, and Goudsmit[11] and the neutron and proton binding energies compiled by Way.[12] This set of masses runs about 8 mMU below the masses compiled by Stern.[13]

Examining Fig. 1, we see that the experimental residuals tend to fluctuate about the zero axis rather erratically. The smooth dotted curve indicated in Fig. 1 represents our tentative choice of R^o. Our effective scale (the distance between the two dashed lines is 5 mMU) is so large that we are here in a realm in which smooth curves may be chosen with considerable latitude. However, for our purposes here this latitude is essentially negligible. We note, first of all, that the deviations $R^o = \Delta - \Delta_m{}^r(A)$ are only of the order of a few millimass units and are quite small compared to the magnitude of the variation of decrements. (See column 2 Table I). Accordingly we may conclude that our reference function $\Delta_m{}^r(A)$ "fits" the experimental data quite well. In Fig. 2 we represent the residuals R^o by a series of dark circles and the residual $R^w = \Delta_m{}^w(A) - \Delta_m{}^r(A)$ for various semi-empirical equations by curved lines. To

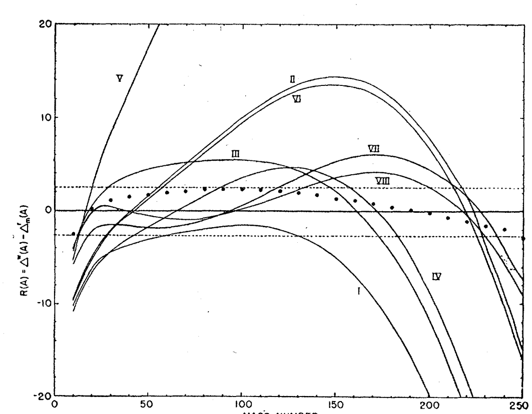

FIG. 2. Semi-empirical residuals *vs* mass number for various sets of semi-empirical constants. (See references in Table II.) I—Bethe; II—Fermi; III—Mattauch; IV—Feenberg; V—Pryce; VI—Metropolis; VII—Fowler; VIII—This paper. The small solid circles represent the estimated R^o from Fig. 1.

[6] Collins, Nier, and Johnson, Phys. Rev. **86**, 408 (1952); R. E. Halsted, Phys. Rev. **85**, 726 (1952); **88**, 666 (1952).
[7] Duckworth, Johnson, Kegley, Olson, Presont, Stanford, and Woodcock, Phys. Rev. **78**, 179, 479 (1950); **79**, 402 (1950); **81**, 286 (1951); **82**, 468 (1951); **83**, 1114 (1951); Nature **167**, 1025 (1951).
[8] Li, Whaling, Fowler, and Lauritsen, Phys. Rev. **83**, 512 (1951).
[9] H. T. Motz, Phys. Rev. **81**, 1061 (1951).
[10] A. H. Wapstra (private communications).
[11] Hays, Richards, and Goudsmit, Phys. Rev. **84**, 824 (1951); **85**, 1065 (1952).
[12] K. Way and M. Wood (private communication, 1951).
[13] M. O. Stern, Revs. Modern Phys. **21**, 316 (1949).

accommodate the computed deviations a much smaller scale is used (the distance between the two dashed lines again is 5 mMU). The large magnitude of some of the deviations may come as a shock to the reader. Since nuclear masses have not changed to this extent in recent years we must either conclude that the methods of adjustment used in these earlier studies were very sensitive to these changes in nuclear masses (particularly those of the neutron and proton) or else the adjustments were made to a limited portion of the mass surface. Of the surfaces which have appeared earlier in the literature Fowler's surface is the most accurate.

Our curve which is closest to R^o was obtained by an iterative process which started from Fermi's constants. We essentially retained Fermi's $D_m(A)$ but varied the constants by discrete steps so as to reduce to zero the departures at widely spaced mass numbers. The $\Delta_m^w(A)$ function so obtained was then plotted and the process repeated until R^w was in good agreement with R^o. Since unassessed shell effects make us somewhat uncertain as to the validity of our tentative R^o we did not go as far as is possible with our attempt to match R^w to R^o. However, we have already gone far enough to show that large systematic errors in absolute masses are not intrinsic to the semi-empirical equation but instead these errors can be reduced substantially by an adjustment of the constants. We note, however, that the constants we obtained are quite larger than those quoted earlier in the literature and represent further steps in the direction already taken by Fowler. Fowler's constants, however, are not strictly comparable with ours since he has used Eq. (7′).

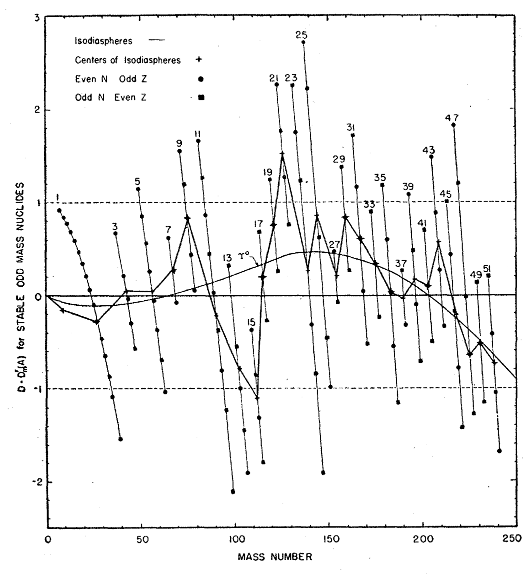

FIG. 3. $D-D_m^r(A)$ for beta-stable odd-A nuclides *vs* mass number. Υ^o locates our tentative optimum smooth line of beta-stability relative to our reference function. The base line represents our reference $D_m^r(A)$. Isodiaspheres are nuclides with equal neutron excess.

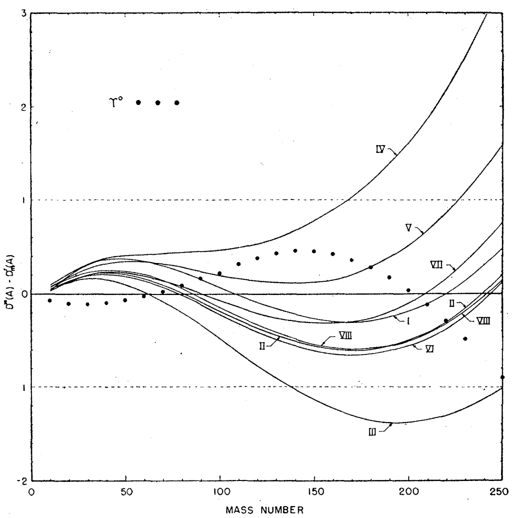

FIG. 4. Semi-empirical lines of beta-stability for various sets of semi-empirical constants. The solid circles are taken from the optimum smooth curve in Fig. 3.

5. THE FUNCTIONS $\Upsilon(A)$ AND $r(A)$

To determine the optimum $D_m^o(A)$ or $\Upsilon^o(A)$ we made use of the fact that according to Eq. (1) beta-stable odd mass nuclides should all have $D-D_m^o(A)$ values within ± 1. In Fig. 3 we have plotted $D-D_m^r(A)$ for all beta-stable odd nuclides. The jagged line joins the centers of the stable limits for various isodiaspheres. The larger discontinuities in this line undoubtedly are due to shell effects. Until these shell effects can be assessed quantitatively it is impossible to determine a precise Υ^o which represents the smooth line of beta-stability. However we have drawn a smooth curve which represents our tentative estimate of $\Upsilon^o(A)$. We could vary this curve in some places by as much as ± 0.2 unit without fear of contradicting the data. Unfortunately this uncertainty is a large fraction of the change in D (2 units) involved in beta-decay. Accordingly we must eventually fix Υ^o more precisely if we hope to make reliable estimates of the parabolic energy effect in beta-decay.

In Fig. 4 we show the Υ^w corresponding to various semi-empirical equations. We note that beyond the light nuclides these lines of least mass disagree with each other by distances which are much greater than 0.2 unit so that we can draw some conclusions from our tentative Υ^o despite its uncertainty. It would appear that Fermi's, Bethe's, Fowler's, and Metropolis' lines of least mass are more accurate than the others and are just about as good as our reference line. It also appears that all of the semi-empirical lines have the wrong general shape particularly in the very heavy region. Thus to match Υ^o it is necessary to allow ρ to vary rather than simply to make an adjustment of ρ. Using Fig. 4 the necessary variation of ρ can readily be determined. However, shell effects should first be precisely assessed before $\rho(A)$ is evaluated and interpreted.

The parabolic width ratio $r(A)$ may be evaluated for those odd mass isobars for which three or more mass values or two or more beta decay energies are known. For example, if M^{-}, M, and M^{+} are the masses of three adjacent isobars which are β^{-} unstable, beta-stable, and β^{+} or K capture unstable, it can readily be shown using Eq. (1) and Eq. (4) that

$$r=(M^{-}+M^{+}-2M_s)A/200. \qquad (13)$$

In the very heavy region there are many instances in which more than three masses are available. For these cases a graphical method was found the most convenient to evaluate the parabolic width and hence r. The results of these calculations are shown in Fig. 5. The r values evaluated from mass data are represented by solid circles whereas the r values from beta-decay energies are represented by solid squares. Because these points are quite scattered we have not attempted to draw a smooth r^o curve. Also shown on this same graph are the r^w values corresponding to various semi-empirical equations. In view of the scattered nature of the experimental points we do not feel that any one of the analytical r^w including our $r=1$ has particular merit relative to the others. It is probable that experimental error and pairing effects[14] contributed to the scattering of r values.

6. OTHER APPLICATIONS OF THE REFERENCE FUNCTIONS

Another important application of our reference surface is to the organization and interpretation of the vast quantity of data concerning nuclear Q values. Let us consider a bombardment type reaction represented by the transformation

$$x(a,d)+X(A,D)\rightarrow X'(A',D')+x'(a',d'), \qquad (14)$$

where x, X, X', and x' symbolize the incident, target, product and ejected nuclides, respectively. The Q value for this reaction in terms of the mass decrements of these nuclides is

$$Q=\delta+\Delta-\Delta'+\delta'. \qquad (15)$$

Using Eq. (1) for the target and product nuclides and the experimental decrements for the very light nuclides, Q may be placed in the form

$$Q=Q_m(A)+Q_q \qquad (16)$$

where

$$Q_m(A)=\delta-\delta'+\Delta_m(A)-\Delta_m(A'), \qquad (17)$$

and

$$Q_q=J(A)[D-D_m(A)]^2-J(A')[D'-D_m(A')]^2. \qquad (18)$$

Since $A'=A+a-a'$ we find, using Eq. (2), that for our reference function Eq. (17) becomes

$$Q_m{}^r(A)=\delta-\delta'+2(a-a') \\ -(a-a')^2/100-2(a-a')A/100. \qquad (19)$$

For reactions which produce a change in mass number from the target to the product nuclide this term usually makes the major contribution to the Q value. The great advantage of our reference surface with respect to the closely related semi-empirical surfaces is the simplicity of this expression for Q_m. The reader may convince himself of this advantage by deriving the general Q_m term corresponding to Eq. (7) or Eq. (7′). In Table III we tabulate the functions Q_m for most of the interesting bombardment reactions.[15] The table is so arranged that the function in each box is for a process in which the particle labeling the *c*olumn is *c*aptured and the particle labeling the *r*ow is *r*eleased. The constant term in each box is based upon the mass decrements for neutral light nuclides given by Li *et al.*[8] If these mass decrements change, the constants in Table III should be changed accordingly. Since Q_m is generally dominant, Table III may be used as a rough survey of nuclear reaction energies. For example, we note in accord with experience that reactions induced by deuterons, tritons, and helium-3 particles when gamma-rays, neutrons, protons, and alpha-particles are released are markedly exoergic.

For every experimental Q value let us define the residual Q value,

$$R(Q)=Q-Q_m{}^r(A). \qquad (20)$$

These differences may be attributed primarily to discontinuities in the experimental mass surface and to the parabolic effect. Since the parabolic effect is usually small or else it can be estimated the Q value residuals, which can readily be computed with the aid of Table III, provide an excellent set of data for the study of discontinuities in the mass surface.

Residual Q values may also be used to facilitate the construction of a mass table based upon the latest Q values and beta-decay energies. Using Eq. (20) we note that

$$R\{Q[X(x,x')X']\}=R(X)-R(X'), \qquad (21)$$

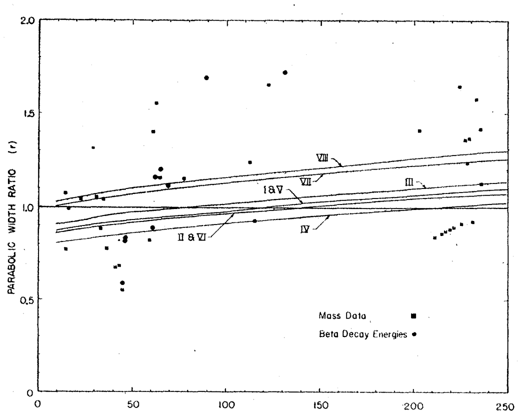

FIG. 5. Experimental and semi-empirical parabolic width ratios *vs* mass number. The horizontal line $r=1$ corresponds to our reference parabolic width function $J^r(A)$.

[14] C. D. Coryell and H. E. Suess, Phys. Rev. **86**, 609 (1952).

[15] We use the symbol χ to denote He^3.

TABLE III. Q_m^r values in mMU. (The symbol χ denotes He^3.)

	γ	n	p	d	t	χ	α
γ	0	$10.972-0.02A$	$10.132-0.02A$	$18.695-0.04A$	$22.907-0.06A$	$22.887-0.06A$	$11.713-0.08A$
n	$-10.992+0.02A$	0	-0.840	$7.743-0.02A$	$11.975-0.04A$	$11.955-0.04A$	$0.801-0.06A$
p	$-10.152+0.02A$	0.840	0	$8.583-0.02A$	$12.815-0.04A$	$12.795-0.04A$	$1.641-0.06A$
d	$-18.775+0.04A$	$-7.763+0.02A$	$-8.603+0.02A$	0.000	$4.252-0.02A$	$4.232-0.02A$	$-6.902-0.04A$
t	$-23.087+0.06A$	$-12.055+0.04A$	$-12.895+0.04A$	$-4.272+0.02A$	0	-0.020	$-11.134-0.02A$
χ	$-23.067+0.06A$	$-12.035+0.04A$	$-12.875+0.04A$	$-4.252+0.02A$	0.020	0	$-11.114-0.02A$
α	$-12.033+0.08A$	$-0.981+0.06A$	$-1.821+0.06A$	$+6.822+0.04A$	$+11.114+0.02A$	$11.094+0.02A$	0
np	$-21.164+0.04A$	$-10.152+0.02A$	$-10.992+0.02A$	-2.389	$+1.863-0.02A$	$+1.843-0.02A$	$-9.291-0.04A$
nn	$-22.004+0.04A$	$-10.992+0.02A$	$-11.832+0.02A$	-3.229	$+1.023-0.02A$	$+1.003-0.02A$	$-10.131-0.04A$

where by definition

$$R(X)=\Delta-\Delta_m{}^r(A). \qquad (22)$$

Thus we see that the residual Q values are related only to the masses of the target and product nuclides and not to the very light particles involved in the reaction. The residual Q-value data thus have the same significance as beta-decay energies, since for beta-decay energies

$$E_{\beta^-}=R(X)-R(X'), \qquad (23)$$

and

$$E_{\beta^+}+2m_ec^2=R(X)-R(X'), \qquad (24)$$

where E_{β^-} and E_{β^+} are the end point energies in β^- and β^+ decay. When sufficient Q value residuals and beta decay energies are known in a region we may solve for the mass residuals in terms of one known mass residual (say from a mass spectrographic determination). Using Eq. (22) we may readily convert these mass residuals into mass decrements. This was the procedure used in the construction of a new table of heavy masses[16] and it worked out quite well.

Applications of the reference functions to the study of fission, radioactive decay and other nuclear transformations are quite straightforward and will not be discussed here.

7. CONCLUSION

We have here made several applications of a set of reference functions [Eqs. (2)–(4)] for the purpose of examining semi-empirical mass surfaces in relation to the experimental data. We see from Fig. 2, Fig. 4, and Fig. 5 that each of these reference functions represent some sort of average of the semi-empirical functions, an average which is apparently more accurate than any of the semi-empirical functions. We have shown that most of the sets of semi-empirical constants quoted in the literature give rise to rather large systematic errors in nuclear masses. These large errors, however, are not inherent properties of the semi-empirical equation, since we have found a set which reduces these errors to within the range of uncertainty caused by shell effects. Our constants are substantially larger than those previously appearing in the literature, with the exception of the constants obtained by Fowler which are only slightly smaller than ours. It is interesting to note that of all the sets, only Fowler's and ours correspond to a fissionability constant close to the value 47.8 used by Bohr and Wheeler to predict photofission thresholds.

There is still good evidence which indicates that to fit the absolute mass surface and the line of beta-stability we must allow the semi-empirical constants to vary. However, until shell and pairing effects are quantitatively assessed such a study would probably not yield significant conclusions.

We have illustrated some applications of our reference functions to the organization and study of nuclear data.

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[16] J. S. Nader, Master's thesis, University of Cincinnati, 1952 (unpublished).