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Reduction of Relativistic Two-Particle Wave Equations to Approximate Forms. I

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An extension of the Foldy-Wouthuysen method to two-particle equations is developed. The relativistic Hamiltonian contains even-even, even-odd, odd-even, and odd-odd terms. To remove terms of the three latter types and thereby to convert the Hamiltonian into an even-even operator, canonical transformations are applied. A prescription for the proper choice of generating functions for such transformations is given, and an approximate expression (to the order c^{-2}) for the transformed Hamiltonian is obtained. The even-even character of the transformed Hamiltonian makes it possible to separate out the quadruple of equations for the four ψ-components describing both of the particles in positive energy states only. As an example the Breit equation is taken and its reduced form, obtained by the proposed procedure, is compared with its approximate form as obtained (by Breit himself) using the method of large components. A discussion leads to the result that the case $m_{\mathrm{I}}=m_{\mathrm{II}}$ is singular and requires a further development of the present scheme.

INTRODUCTION

A RELATIVISTIC wave equation of the Dirac type is still our most convenient means to describe the behavior of a fermion in an external field, since, according to a suggestion of Pauli,[1] the original equation of Dirac may be amplified so as to phenomenologically include electromagnetic effects accounted for by quantum electrodynamics; moreover, by a suitable amplification it may be adapted for the description of interactions with fields of *any* kind, not necessarily electromagnetic.[2] For purposes of physical interpretation and practical application it is often found necessary to reduce a four-component equation of the Dirac type to an approximate (nonrelativistic) two-component equation of the Pauli type. This used to be effected by the well-known procedure of expressing the small components of the spinor ψ in terms of its large components. The method of Foldy and Wouthuysen,[3] serving the same purpose, is superior to that of large components, as it leaves to the Hamiltonian its Hermitian character and excludes from consideration the negative energy states.

The Foldy-Wouthuysen method applies to one-body equations; we propose to develop a corresponding method for two-body equations. We have, in particular, in mind equations of the Breit type. Although it has some defects, the Breit equation has proved useful in many cases and, like the Dirac equation, it may be amplified to include various effects and interactions.

SUMMARY OF THE FOLDY-WOUTHUYSEN METHOD

We distinguish between upper components ψ_u ($u=1, 2$) and lower components ψ_l ($l=3, 4$) of the ψ-spinor. Upon multiplication by a four-by-four matrix ω we have

$$(\omega\psi)_u=\sum_{u'}\omega_{uu'}\psi_{u'}+\sum_{l'}\omega_{ul'}\psi_{l'},\qquad (u, u'=1, 2; l, l'=3, 4),\quad (1)$$
$$(\omega\psi)_l=\sum_{u'}\omega_{lu'}\psi_{u'}+\sum_{l'}\omega_{ll'}\psi_{l'}.$$

A matrix ω is called even, if $\omega_{ul'}=\omega_{lu'}=0$, and odd, if $\omega_{uu'}=\omega_{ll'}=0$. The diagonal matrices δ (unit matrix) and β, and the spin matrices σ_x, σ_y, σ_z are even; α_x, α_y, α_z are odd.

If the Hamiltonian contained even matrices only, the quadruple of equations,[4] $\mathcal{H}\psi=E\psi$, would decompose into two pairs, not interlinked with each other, one pair involving the ψ_u, the other the ψ_l only, and describing the positive and negative energy states, respectively. Actually, a relativistic Dirac Hamiltonian is essentially not an even operator; but it can be converted into such, to any desired degree of approximation, by canonical transformations of the type

$$\mathcal{H}'=\mathcal{H}+[iS,\mathcal{H}]+[\tfrac{1}{2}iS,[iS,\mathcal{H}]]+[\tfrac{1}{3}iS,[\tfrac{1}{2}iS,[iS,\mathcal{H}]]]+\cdots,\quad (2)$$

provided the Hermitian generating functions S are suitably chosen. The prescription for that choice is: to remove an odd term t_0 from the Hamiltonian, we must include a term

$$s=(\beta/2mc^2)t_0 \quad (2a)$$

into iS. The transformation gives rise to new terms, both acceptable (even) and undesirable (odd), but the latter are of a lower order of magnitude than the term just removed.

If we write the original Hamiltonian as

$$\mathcal{H}=\beta mc^2+\mathcal{E}+\mathcal{O},\quad (3)$$

where $\mathcal{E}$ is the even and $\mathcal{O}$ the odd part, then the transformed (even) Hamiltonian appears in the form

[1] W. Pauli, Revs. Modern Phys. **13**, 203 (1941).
[2] See G. Petiau, J. phys. et radium **10**, 264 (1949), where more references are cited. More recently, L. L. Foldy, Phys. Rev. **87**, 688 and 693 (1952); W. A. Barker and Z. V. Chraplyvy, Phys. Rev. **89**, 446 (1953).
[3] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950); R. K. Osborn, Phys. Rev. **86**, 340 (1952).
[4] In this paper we restrict our considerations to time-independent Hamiltonians.

of a series

$$\mathcal{H}_{\mathrm{tr}}=\beta mc^2+\mathcal{E}+\frac{\beta}{2mc^2}\mathcal{O}^2-\frac{1}{8m^2c^4}[[\mathcal{O},\mathcal{E}],\mathcal{O}]-\frac{\beta}{8m^3c^6}\mathcal{O}^4+\cdots. \quad (4)$$

THE TWO-PARTICLE PROBLEM: TERMINOLOGY AND NOTATION

Quantities describing each of the two particles are labeled by Roman numbers I and II, respectively, e.g., masses m_{I} and m_{II}, position vectors $\mathbf{x}_{\mathrm{I}}$ and $\mathbf{x}_{\mathrm{II}}$. Also, lower case indices refer to the first, capital indices to the second particle; thus the sixteen components of the ψ-spinor are denoted by ψ_{kK} ($k, K=1, 2, 3, 4$), k indicating the state of the first, K that of the second particle.[5] We classify these components as upper-upper ψ_{uU}, upper-lower ψ_{uL}, lower-upper ψ_{lU}, and lower-lower ψ_{lL}, with $u, U=1, 2$; $l, L=3, 4$.

The original Hamiltonian contains sixteen-by-sixteen matrices, which are direct products of the familiar Dirac matrices, in this manner:

$$\Omega_{jkJK}=\eta_{jk}\omega_{JK}.$$

If Ω is multiplied into ψ, the first index of ψ is affected by η, the second by ω:

$$(\Omega\psi)_{jkJK}=\sum_{n,N}\eta_{jn}\omega_{JN}\psi_{nN}.$$

It follows that matrices with elements $\beta_{jk}\delta_{JK}$, $(\alpha_x)_{jk}\delta_{JK}$, $(\sigma_y)_{jk}\delta_{JK}$, etc., affect only the first-particle subscript of ψ; they are consistently denoted by β^{I}, α_x^{I}, σ_y^{I}, respectively. The meaning of β^{II}, α_x^{II}, etc. is likewise clear. Any of the matrices labeled by I commutes with any of those labeled by II.

By a straightforward generalization of the Foldy-Wouthuysen terminology we distinguish

- even-even matrices, such as: I (the unit matrix), β^{I}, β^{II}, σ_x^{I}, etc.;
- even-odd[6] matrices: α_x^{II}, α_y^{II}, α_z^{II};
- odd-even matrices: α_x^{I}, α_y^{I}, α_z^{I};
- odd-odd matrices, such as the direct product of two Dirac α-matrices.

Applying a rule given by Foldy and Wouthuysen to direct products of four-by-four matrices, we find that multiplication of a, say, even-odd by an odd-odd matrix yields an odd-even matrix. Furthermore, from a property of the Dirac β (reference 3, footnote 3) it is easily derived that β^{I} commutes (anticommutes) with all even-even and even-odd (with all odd-even and odd-odd) matrices, whereas β^{II} commutes with even-even and odd-even matrices and anticommutes with the other two kinds.

TRANSFORMATION OF THE TWO-PARTICLE HAMILTONIAN

The two-body Hamiltonian is a sum of the two "large" terms $\beta^{\mathrm{I}}m_{\mathrm{I}}c^2+\beta^{\mathrm{II}}m_{\mathrm{II}}c^2$ and other terms, containing matrices of all four types. Therefore the sixteen equations of $\mathcal{H}\psi=E\psi$ (or $\mathcal{H}\psi=-E\psi$) are interlocked, each of them involving ψ-components of all four types. The scope is to transform $\mathcal{H}$ into an even-even operator and thereby to disentangle the ψ-components in such fashion, that components of the uU-, uL-, lU-, lL-type appear in separate quadruples of equations. The transformations are again of the general form (2), but iS must now include a term of the form

$$s_{oe}=\frac{\beta^{\mathrm{I}}}{2m_{\mathrm{I}}c^2}t_{oe}, \quad (5a)$$

$$s_{eo}=\frac{\beta^{\mathrm{II}}}{2m_{\mathrm{II}}c^2}t_{eo}, \quad (5b)$$

$$s_{oo}=\frac{\beta^{\mathrm{I}}m_{\mathrm{I}}-\beta^{\mathrm{II}}m_{\mathrm{II}}}{2(m_{\mathrm{I}}^2-m_{\mathrm{II}}^2)c^2}t_{oo}, \quad (5c)$$

in order to remove from the Hamiltonian an odd-even term t_{oe}, or an even-odd term t_{eo}, or an odd-odd term t_{oo}, respectively. In fact, expressions (5a) and (5b) resemble (2a), and it is seen that the expression (5c), on meeting the large terms of $\mathcal{H}$ in the first commutator of (2), yields

$$[s_{oo},\beta^{\mathrm{I}}m_{\mathrm{I}}c^2]+[s_{oo},\beta^{\mathrm{II}}m_{\mathrm{II}}c^2]$$
$$=-\frac{m_{\mathrm{I}}^2-\beta^{\mathrm{I}}\beta^{\mathrm{II}}m_{\mathrm{I}}m_{\mathrm{II}}}{m_{\mathrm{I}}^2-m_{\mathrm{II}}^2}t_{oo}-\frac{\beta^{\mathrm{I}}\beta^{\mathrm{II}}m_{\mathrm{I}}m_{\mathrm{II}}-m_{\mathrm{II}}^2}{m_{\mathrm{I}}^2-m_{\mathrm{II}}^2}t_{oo}=-t_{oo},$$

just sufficient to cancel the undesired t_{oo} in $\mathcal{H}$. While removing an undesirable term, the transformation produces new terms instead. Some of them will be acceptable (even-even), others again undesired, but of a lower order than the removed one. They may be removed, in turn, by a similar procedure, and so on. Thus we can obtain a Hamiltonian consisting of even-even terms only, to any needed degree of approximation.

In particular, starting with the original Hamiltonian written in the form

$$\mathcal{H}=\beta^{\mathrm{I}}m_{\mathrm{I}}c^2+\beta^{\mathrm{II}}m_{\mathrm{II}}c^2+(\mathcal{E}\mathcal{E})+(\mathcal{E}\mathcal{O})+(\mathcal{O}\mathcal{E})+(\mathcal{O}\mathcal{O}), \quad (6)$$

where $(\mathcal{E}\mathcal{E})$ stands for all even-even, $(\mathcal{E}\mathcal{O})$ for all even-odd terms, etc., we have found (after laborious

[5] In the limit of vanishing mutual interaction, the component ψ_{kK} becomes a product of two functions, each describing some state of one particle:

$$\psi_{kK}(\mathbf{x}_{\mathrm{I}},\mathbf{x}_{\mathrm{II}})=\psi_k(\mathbf{x}_{\mathrm{I}})\psi_K(\mathbf{x}_{\mathrm{II}}).$$

[6] That is, even with respect to the first, odd with respect to the second particle.

calculations) for the transformed Hamiltonian:

$$\mathcal{H}_{\mathrm{tr}}=\beta^{\mathrm{I}}m_{\mathrm{I}}c^2+\beta^{\mathrm{II}}m_{\mathrm{II}}c^2+(\mathcal{E}\mathcal{E}) \tag{7a}$$

$$+\frac{\beta^{\mathrm{I}}}{2m_{\mathrm{I}}c^2}(\mathcal{O}\mathcal{E})^2+\frac{\beta^{\mathrm{II}}}{2m_{\mathrm{II}}c^2}(\mathcal{E}\mathcal{O})^2 \tag{7b}$$

$$+\frac{1}{8m_{\mathrm{I}}^2c^4}[[(\mathcal{O}\mathcal{E}),(\mathcal{E}\mathcal{E})],(\mathcal{O}\mathcal{E})]+\frac{1}{8m_{\mathrm{II}}^2c^4}[[(\mathcal{E}\mathcal{O}),(\mathcal{E}\mathcal{E})],(\mathcal{E}\mathcal{O})] \tag{7c}$$

$$-\frac{\beta^{\mathrm{I}}}{8m_{\mathrm{I}}^3c^6}(\mathcal{O}\mathcal{E})^4-\frac{\beta^{\mathrm{II}}}{8m_{\mathrm{II}}^3c^6}(\mathcal{E}\mathcal{O})^4 \tag{7d}$$

$$+\frac{\beta^{\mathrm{I}}\beta^{\mathrm{II}}}{8m_{\mathrm{I}}m_{\mathrm{II}}c^4}\{[[(\mathcal{O}\mathcal{E}),(\mathcal{O}\mathcal{O})]_+,(\mathcal{E}\mathcal{O})]_+ +[[(\mathcal{E}\mathcal{O}),(\mathcal{O}\mathcal{O})]_+,(\mathcal{O}\mathcal{E})]_+\} \tag{7e}$$

$$+\frac{\beta^{\mathrm{I}}m_{\mathrm{I}}-\beta^{\mathrm{II}}m_{\mathrm{II}}}{2(m_{\mathrm{I}}^2-m_{\mathrm{II}}^2)c^2}(\mathcal{O}\mathcal{O})^2 \tag{7f}$$

$$+\frac{\beta^{\mathrm{II}}m_{\mathrm{I}}-\beta^{\mathrm{I}}m_{\mathrm{II}}}{8m_{\mathrm{I}}m_{\mathrm{II}}(m_{\mathrm{I}}^2-m_{\mathrm{II}}^2)c^6}[(\mathcal{O}\mathcal{E}),(\mathcal{E}\mathcal{O})]^2 \tag{7g}$$

$$-\frac{\beta^{\mathrm{I}}m_{\mathrm{I}}+\beta^{\mathrm{II}}m_{\mathrm{II}}}{16m_{\mathrm{I}}^2m_{\mathrm{II}}^2c^6}[(\mathcal{O}\mathcal{E})^2,(\mathcal{E}\mathcal{O})^2]_+ \tag{7h}$$

$$+\frac{\beta^{\mathrm{I}}}{8m_{\mathrm{I}}m_{\mathrm{II}}^2c^6}(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{E})^2(\mathcal{E}\mathcal{O})+\frac{\beta^{\mathrm{II}}}{8m_{\mathrm{I}}^2m_{\mathrm{II}}c^6}(\mathcal{O}\mathcal{E})(\mathcal{E}\mathcal{O})^2(\mathcal{O}\mathcal{E}) \tag{7i}$$

$$\frac{3}{16}\,\frac{\beta^{\mathrm{I}}\beta^{\mathrm{II}}(m_{\mathrm{I}}^2+m_{\mathrm{II}}^2)-2m_{\mathrm{I}}m_{\mathrm{II}}}{m_{\mathrm{I}}m_{\mathrm{II}}(m_{\mathrm{I}}^2-m_{\mathrm{II}}^2)c^4}\times[[(\mathcal{E}\mathcal{O}),(\mathcal{O}\mathcal{E})],(\mathcal{O}\mathcal{O})] \tag{7j}$$

$$+\cdots.$$

The reciprocal speed of light has been chosen as expansion parameter of the series, and in the above expression we have retained terms out to the order $(1/c)^2$, under the assumption that $(\mathcal{E}\mathcal{E})$ and $(\mathcal{O}\mathcal{O})$ are of the order c^0, and $(\mathcal{O}\mathcal{E})$ and $(\mathcal{E}\mathcal{O})$ of the order c^1. Expression (7) compares with (4) into which it goes over if the mass of the one particle is assumed to be so great that it becomes an immovable supplier of field for the other particle. A considerable simplification arises if $(\mathcal{O}\mathcal{E})$ and $(\mathcal{E}\mathcal{O})$ commute: the two terms (7e) become indentical, and the terms (7g, h, i, j) vanish or cancel each other.

THE REDUCED WAVE EQUATION

The transformed Hamiltonian does still yield a sixteen-component wave equation; but now it is possible to separate out a four-component wave equation describing such states only, in which both particles possess positive energy. This reduced wave equation will involve only the upper-upper or only the lower-lower components of the full ψ-spinor. The Hamiltonian ($\mathcal{H}_{\mathrm{red}}$) to operate on the ψ_{uU} is obtained from $\mathcal{H}_{\mathrm{tr}}$ in the following way: represent the matrix part of every term of $\mathcal{H}_{\mathrm{tr}}$ as a direct product of two even four-by-four matrices and retain only their top-left quarters, omitting the rest. $\mathcal{H}_{\mathrm{red}}$ for the ψ_{lL}-components is to be assembled in a similar way from the bottom-right portions of the matrices.

This rule implies the reinterpretation of the σ's as the Pauli two-by-two spin matrices (rather than the Dirac spin matrices); also, if β^{I} or β^{II} (or some combination of them) appears in a term as a *left* multiplier, one has to put $\beta^{\mathrm{I}}=\beta^{\mathrm{II}}=1$, when separating out the ψ_{uU}, and $\beta^{\mathrm{I}}=\beta^{\mathrm{II}}=-1$, if the ψ_{lL} are selected. The choice of the first or the second possibility is evidently dictated by whether the original Hamiltonian is meant to be the energy operator or the minus energy operator.

APPLICATION TO THE BREIT EQUATION

If ϵ_{I}, ϵ_{II} denote the algebraic charges of the two particles, and the (stationary) external field is described by the electromagnetic potentials ϕ, $\mathbf{A}$, we have in the Breit Hamiltonian:

$$(\mathcal{O}\mathcal{E})=c\boldsymbol{\alpha}^{\mathrm{I}}\cdot\mathbf{p}^{\mathrm{I}}-\epsilon_{\mathrm{I}}\boldsymbol{\alpha}^{\mathrm{I}}\cdot\mathbf{A}^{\mathrm{I}},\quad (\mathcal{E}\mathcal{O})=c\boldsymbol{\alpha}^{\mathrm{II}}\cdot\mathbf{p}^{\mathrm{II}}-\epsilon_{\mathrm{II}}\boldsymbol{\alpha}^{\mathrm{II}}\cdot\mathbf{A}^{\mathrm{II}},$$

and these two expressions are seen to commute with each other; furthermore

$$(\mathcal{E}\mathcal{E})=-\epsilon_{\mathrm{I}}\phi^{\mathrm{I}}-\epsilon_{\mathrm{II}}\phi^{\mathrm{II}}-\epsilon_{\mathrm{I}}\epsilon_{\mathrm{II}}/r,$$

where $\mathbf{r}=\mathbf{x}_{\mathrm{I}}-\mathbf{x}_{\mathrm{II}}$, contains the minus electrostatic energy due to the external field and mutual interaction; and

$$(\mathcal{O}\mathcal{O})=\frac{\epsilon^{\mathrm{I}}\epsilon^{\mathrm{II}}}{2r}(\boldsymbol{\alpha}^{\mathrm{I}}\cdot\boldsymbol{\alpha}^{\mathrm{II}})+\frac{\epsilon^{\mathrm{I}}\epsilon^{\mathrm{II}}}{2r^3}(\boldsymbol{\alpha}^{\mathrm{I}}\cdot\mathbf{r})(\boldsymbol{\alpha}^{\mathrm{II}}\cdot\mathbf{r})$$

represents the magnetic interaction and the Breit interaction (a retardation effect).

Hence the reduced Breit equation is found to be $\mathcal{H}_{\mathrm{red}}\psi=-E\psi$, with

$$\mathcal{H}_{\mathrm{red}}=-m_{\mathrm{I}}c^2-\epsilon_{\mathrm{I}}\phi_{\mathrm{I}}-\tfrac{1}{2}(\epsilon_{\mathrm{I}}\epsilon_{\mathrm{II}}/r) \tag{8a}$$

$$-(\mathbf{p}_{\mathrm{I}}^2/2m_{\mathrm{I}})+(\epsilon_{\mathrm{I}}/m_{\mathrm{I}}c)(\mathbf{A}_{\mathrm{I}}\cdot\mathbf{p}_{\mathrm{I}})-(\epsilon_{\mathrm{I}}^2/2m_{\mathrm{I}}c^2)\mathbf{A}_{\mathrm{I}}^2+(\epsilon_{\mathrm{I}}\hbar/2m_{\mathrm{I}}c^2)(\boldsymbol{\sigma}^{\mathrm{I}}\cdot\mathbf{H}_{\mathrm{I}}) \tag{8b}$$

$$+(\mathbf{p}_{\mathrm{I}}^4/8m_{\mathrm{I}}^3c^2) \tag{8d}$$

$$+\frac{\epsilon_{\mathrm{I}}\hbar}{4m_{\mathrm{I}}^2c^2}\boldsymbol{\sigma}^{\mathrm{I}}\cdot\left\{(\mathbf{E}_{\mathrm{I}}\times\mathbf{p}_{\mathrm{I}})+\left(\frac{\epsilon_{\mathrm{II}}\mathbf{r}}{r^3}\times\mathbf{p}_{\mathrm{I}}\right)\right\}+\frac{\epsilon_{\mathrm{I}}\hbar^2}{8m_{\mathrm{I}}^2c^2}\{\mathrm{div}\mathbf{E}_{\mathrm{I}}+4\pi\epsilon_{\mathrm{II}}\delta(r)\} \tag{8c}$$

$$+\frac{\epsilon_{\mathrm{I}}\epsilon_{\mathrm{II}}}{4m_{\mathrm{I}}m_{\mathrm{II}}c^2}\left\{\frac{(\mathbf{p}_{\mathrm{I}}\cdot\mathbf{p}_{\mathrm{II}})}{r}-\frac{(\mathbf{r}\cdot\mathbf{p}_{\mathrm{I}})(\mathbf{r}\cdot\mathbf{p}_{\mathrm{II}})}{r^3}\right\}$$

$$+\frac{\epsilon_{\mathrm{I}}\epsilon_{\mathrm{II}}\hbar}{4m_{\mathrm{I}}m_{\mathrm{II}}c^2}\left\{-i\frac{(\mathbf{r}\cdot\mathbf{p}_{\mathrm{I}})}{r^3}-2\frac{\boldsymbol{\sigma}^{\mathrm{II}}\cdot(\mathbf{r}\times\mathbf{p}_{\mathrm{I}})}{r^3}\right\}$$

$$+\frac{\epsilon_{\mathrm{I}}\epsilon_{\mathrm{II}}\hbar^2}{8m_{\mathrm{I}}m_{\mathrm{II}}c^2}\left\{3\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\cdot\mathbf{r})(\boldsymbol{\sigma}^{\mathrm{II}}\cdot\mathbf{r})}{r^5}-\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\cdot\boldsymbol{\sigma}^{\mathrm{II}})}{r^3}+4\pi(\boldsymbol{\sigma}^{\mathrm{I}}\cdot\boldsymbol{\sigma}^{\mathrm{II}})\delta(r)\right\} \quad (8e)$$

$$-\frac{\epsilon_{\mathrm{I}}^2\epsilon_{\mathrm{II}}^2}{8(m_{\mathrm{I}}+m_{\mathrm{II}})c^2}\left\{\frac{3}{r^3}-2\cdot\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\cdot\boldsymbol{\sigma}^{\mathrm{II}})}{r^3}-\frac{(\boldsymbol{\sigma}^{\mathrm{I}}\cdot\mathbf{r})(\boldsymbol{\sigma}^{\mathrm{II}}\cdot\mathbf{r})}{r^5}\right\}, \quad (8f)$$

plus symmetrical terms, obtained from the above by interchanging the indices I, II.

We shall compare Eq. (8) with the Breit equation as reduced to an approximate four-component form by means of the method of large components.[7] First, we notice that the following terms, found there, do not appear in (8):

$$\frac{\epsilon_{\mathrm{I}}\hbar i}{4m_{\mathrm{I}}c^2}\left\{\mathbf{E}_{\mathrm{I}}+\frac{\epsilon_{\mathrm{II}}\mathbf{r}}{r^3}\right\}\cdot\mathbf{p}_{\mathrm{I}}+\text{symm. term.} \quad (9)$$

As the expression in the bracket is just the total field at the location of particle I (i.e., both the external field and that produced by particle II), we recognize (9) as a "Darwin term." A term of this type is also absent from the one-body equation treated by the Foldy-Wouthuysen method (reference 3). Its contribution to the S level of the hydrogen spectrum is made up by a "divergence term."[8] The importance of terms of this kind has been pointed out by Foldy (reference 2): they compensate for the idealization of treating the particles, actually small volume charges, like point charges. Accordingly, divergence terms have been carefully worked out in (8); in addition to the last but one term in (8c) and its symmetrical counterpart which are due to the external field and are different from zero only at its sources, we also have the last terms in (8c) and (8e), which represent the direct contact interactions between the two particles.

Otherwise, our expression (8) agrees with Breit's reduced Hamiltonian in all terms; it also contains (8f), which is the e^4 term, responsible for the discrepancy of the fine structure of the helium spectrum, as calculated from the Breit equation, with the spectrum found experimentally. Because of this term, Breit had to impose certain restrictions on the application of his equation.[9] It has been pointed out[10] that the term in question stems from an admixture of negative energy states. But then its appearance is surprising in our expression (8), after a treatment that cuts off such states.

THE SINGULAR CASE OF EQUAL MASSES

It should be emphasized, however, that no positive conclusions on the spectra of helium-like atoms can be drawn from Eq. (8). Because of the form of (5c) our procedure applies to particles of different masses only; and even although we see no mass differences any longer in the denominators of $\mathcal{H}_{\mathrm{red}}$, we have to remember that (8) represents merely one fourth part of the total transformed Hamiltonian, and that in other portions of it (those characterized by $\beta^{\mathrm{I}}=1$, $\beta^{\mathrm{II}}=-1$, and $\beta^{\mathrm{I}}=-1$, $\beta^{\mathrm{II}}=1$) infinite terms will appear if we simply put $m_{\mathrm{I}}=m_{\mathrm{II}}$. (Notice that just e^4 terms will be affected.)

It seems that the case of equal masses cannot be handled as a limiting case within our present scheme. Rather we should try to treat it separately, and ask what term s in the function iS would destroy an odd-odd term t_{oo} of the equal-mass Hamiltonian. We should choose s so that

$$[s,\ (\beta^{\mathrm{I}}+\beta^{\mathrm{II}})mc^2]=-t_{oo}.$$

Let $t_{oo}=\tau q$, where q is a nonmatrix factor and τ an odd-odd matrix, and denote by $\mathfrak{z}$ the matrix similarly contained in s; then our problem amounts to finding a 16-by-16 matrix $\mathfrak{z}$ which would satisfy

$$[\mathfrak{z},\ \beta^{\mathrm{I}}+\beta^{\mathrm{II}}]=-\tau. \quad (10)$$

Recalling the definition of β^{I} and β^{II}, we see that this is equivalent to the set of equations:

$$(\beta_{kk}+\beta_{KK}-\beta_{jj}-\beta_{JJ})\mathfrak{z}_{jkJK}=\tau_{jkJK}.$$

Specializing j, k, J, K to u, l, L, U, and then to l, u, U, L, and noticing that $\beta_{uu}=1$, $\beta_{ll}=-1$, we would have (in order to keep $\mathfrak{z}_{jkJK}$ finite) to assume

$$\tau_{ulLU}=\tau_{luUL}=0 \quad (u,\ U=1,\ 2;\ l,\ L=3,\ 4),$$

a condition which, in general, is not fulfilled either by direct products of Dirac's α_x, α_y, α_z, or linear combinations of such products. Thus, Eq. (10) cannot be satisfied by a finite $\mathfrak{z}$, which means that *in the case of particles of equal masses there exists no finite transformation that would convert the Hamiltonian into an even-even operator* by removing terms of other types from it. Whether there exists some other way to separate out wave functions corresponding to positive energy states is a matter of further investigation.

The writer wishes to acknowledge many helpful discussions with Dr. W. A. Barker.

[7] G. Breit, Phys. Rev. **34**, 553 (1929), Eq. (48′).

[8] We adopt the terminology used by Barker and Chraplyvy, reference 2.

[9] For a discussion of this point see H. Bethe in *Handbuch der Physik* (Verlag. Julius Springer, Berlin, 1933), Vol. XXIV, 1, p. 375.

[10] Recently by G. E. Brown and D. G. Ravenhall, Proc. Roy. Soc. (London) **A208**, 552 (1951).