The Energy Loss of a Fast Charged Particle by Čerenkov Radiation*

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The calculation of the Čerenkov loss in emulsion is discussed. The loss in traversing the AgBr grains is 2.2×10^{-3} Mev/g cm^{-2} which is small compared to the relativistic rise of the total ionization loss (0.12 Mev/g cm^{-2}). For gases the Čerenkov loss is somewhat larger, being 0.14 Mev/g cm^{-2} for He and 0.04 Mev/g cm^{-2} for O_2. The theory is in reasonable agreement with experiments on the grain count in emulsion and on the droplet count in a cloud chamber filled with O_2. For gases an approximate analytic expression for the Čerenkov loss is obtained. Values of the density effect for 6 additional substances are presented.

I. INTRODUCTION

THE energy loss of a fast charged particle by Čerenkov radiation was shown to be small in comparison with the relativistic rise of the ionization loss for the case of emulsion.[1] The calculation of the Čerenkov loss depends on the detailed behavior of the refractive index n in the x-ray region. In this paper the expression for n of AgBr will be more fully discussed and the Čerenkov loss will be obtained for the case of gases which have been used in measurements of the droplet count at relativistic energies. The Čerenkov loss for gases is also a small fraction ($<\frac{1}{3}$) of the relativistic rise of the ionization loss so that the theory is in agreement with the observed increase of the droplet count. An approximate analytic expression for the Čerenkov loss in gases is obtained. At the end of this paper we give values of the density effect for additional substances not covered in the previous work.[2]

The relevance of the Čerenkov loss to the relativistic rise arises from the fact that the total ionization loss includes the Čerenkov radiation, as was shown by Fermi.[3] Hence the energy deposited in the region of impact parameters $<b$ can be obtained from the total ionization loss by subtracting the Čerenkov loss for impact parameters $>b$. For emulsion, b should be taken as the mean grain radius while for cloud-chamber experiments with gases half the width of the track should be used.

The present work[1] is based on the theory of Fermi[3] who gave a general expression for the Poynting flux from a cylinder of arbitrary radius with axis along the path of the charged particle. Previous work on the Čerenkov loss is based on the remark of Bohr[4] that for a medium with no absorption and described by a single type of dispersion oscillator the relativistic rise as obtained from Fermi's theory should escape entirely as Čerenkov radiation. This result was confirmed by Messel and Ritson[5] and by Schoenberg.[6] While the present work was being completed there appeared a paper by Budini[7] on the Čerenkov loss with results very similar to those obtained here. The distribution of the ionization loss between excitation of atoms in the neighborhood of the particle and Čerenkov radiation has also been investigated recently by Huybrechts and Schoenberg.[8]

II. ČERENKOV LOSS FOR EMULSION

The energy deposited at distances $>b$ from the path of the charged particle can be written in view of Eq. (5) of I,

$$W_b=\frac{2A}{\beta^2}Rl\int_0^\infty i\nu\left(1-\beta^2-\frac{\alpha}{1+\alpha}\right)\left(\frac{k_p^*}{k_p}\right)^{\frac{1}{2}}\times\exp[-(k_p+k_p^*)b_p]d\nu, \quad (1)$$

where $A=2\pi n_0e^4/mc^2\rho$, $n_0=$electronic density, $\rho=$density in g/cc, ν is the frequency ω expressed in terms of 2π times the plasma frequency ν_p,

$$\nu=\omega(4\pi n_0e^2/m)^{-\frac{1}{2}}=\omega(2\pi\nu_p)^{-1}, \quad (2)$$

b_p is the distance b expressed in units c/ν_p, α is 4π times the polarizability, and k_p is the square root with real part $\geqq0$ of

$$k_p^2=\nu^2(\beta^{-2}-1-\alpha). \quad (3)$$

Thus $k_p=(c/2\pi\nu_p)k$ where k has been defined in I. With the use of ν_p, the expression for α [Eq. (6) of I] becomes

$$\alpha=-\sum_i\left(\frac{f_i}{\nu_i^2}\right)\frac{\ln[(\nu_i^2-\nu^2-2i\eta_i\nu)/\nu_i^2]+[(\nu^2+2i\eta_i\nu)/\nu_i^2]}{[(\nu^2+2i\eta_i\nu)/\nu_i^2]^2}, \quad (4)$$

where f_i is the oscillator strength, ν_i is the frequency of the ith absorption limit, and η_i is the half-width at half-maximum of a transition; ν_i and η_i are in units ν_p. Equation (1) represents the loss due to Čerenkov

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[1] R. M. Sternheimer, Phys. Rev. **89**, 1148 (1953). This paper will be referred to as I.

[2] R. M. Sternheimer, Phys. Rev. **88**, 851 (1952). This paper will be referred to as II.

[3] E. Fermi, Phys. Rev. **57**, 485 (1940).

[4] A. Bohr, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. **24**, No. 19 (1948).

[5] H. Messel and D. Ritson, Phil. Mag. **41**, 1129 (1950).

[6] M. Schoenberg, Nuovo cimento **8**, 159 (1951).

[7] P. Budini, Phys. Rev. **89**, 1147 (1953).

[8] M. Huybrechts and M. Schoenberg, Nuovo cimento **9**, 764 (1952).

radiation escaping to distances $>b$ from the passing particle.

The case of emulsion will be discussed first. The plasma energy $h\nu_p$ is 3.55 ry for AgBr. The ν_i were obtained as a first approximation from the ionization potentials E_i of Ag and Br given by Sommerfeld.[9] For the outer shells the data of White[10] were also used. These values of E_i and the corresponding f_i are listed in Table I. The E_i of $N_{\text{II–III}}$ of Ag and $M_{\text{IV–V}}$ of Br are so close together that these two limits were treated as a single term ($i=11$) in Eq. (4) using the combined oscillator strength $f_{11}=16/82$ and the width of the $M_{\text{IV–V}}$ transitions. Thus Eq. (4) had altogether 14 terms.

The mean excitation potential I of AgBr is 28.1 ry, as obtained from the values of Bakker and Segrè[11] for Ag (31.5 ry) and Br (24.2 ry). Equation (4) assumes that the photoelectric absorption cross section goes as[12] ν^{-3} for $\nu>\nu_i$. The absorption limit ν_i therefore corresponds to an average frequency $\nu_{i,\text{av}}$ given by

$$\ln\nu_{i,\text{av}}=\int_{\nu_i}^{\infty}\nu^{-3}\ln\nu d\nu\Big/\int_{\nu_i}^{\infty}\nu^{-3}d\nu=\ln\nu_i+\tfrac{1}{2}, \quad (5)$$

so that $\nu_{i,\text{av}}=e^{\frac{1}{2}}\nu_i$. The geometric mean of the $e^{\frac{1}{2}}E_i$ is 26.4 ry. In order to obtain agreement with $I=28.1$ ry, the E_i were multiplied by a factor $28.1/26.4=1.064$. Thus the ν_i are obtained by taking $(1.064/3.55)E_i=0.300E_i$, where the E_i are in ry units. A further correction was made in order that α gives the observed n in the optical region. The measured value of n at[13] $\lambda=5893$A ($\nu=0.044$) is 2.25, whereas the value of

$$n=Rl(1+\alpha)^{\frac{1}{2}} \quad (6)$$

obtained with α given by Eq. (4) and $\nu_i=0.300E_i$ is $n=1.73$. In Eq. (4) the lowest frequency ν_{18} primarily determines n in the optical region. Its value was decreased from 0.196 to 0.134 to bring about agreement with $n=2.25$. This correction is quite unimportant and merely ensures that the Čerenkov loss in the optical region is obtained correctly. However, the main concern of this paper is the Čerenkov loss in the x-ray region which is practically independent of ν_{14}. It may also be noted that the original value $\nu_{14}=0.196$ is only very approximate. The actual absorption from the $N_{\text{IV–V}}$ shell of Ag should in principle be characterized by several absorption regions corresponding to the energy bands in the crystal. In the approximation in which this complicated structure is replaced by a single absorption continuum with $f_{14}=11/82$, the value $\nu_{14}=0.132$ gives the observed n in the optical region.

TABLE I. Data used to calculate the Čerenkov loss in AgBr. The absorption edges E_i and the widths are given in Rydberg units.

Type	E_i	f_i	Width
K, Ag	1878	2/82	0.824
K, Br	993	2/82	0.342
L, Ag	258	8/82	0.235
L, Br	119	8/82	0.235
M_{I}, Ag	53.4	2/82	0.235
$M_{\text{II–III}}$, Ag	44.8	6/82	0.235
$M_{\text{IV–V}}$, Ag	27.6	10/82	0.235
M_{I}, Br	19.0	2/82	0.235
$M_{\text{II–III}}$, Br	13.5	6/82	0.235
N_{I}, Ag	7.3	2/82	0.074
$N_{\text{II–III}}$, Ag	5.4	6/82	0.074
$M_{\text{IV–V}}$, Br	5.2	10/82	0.235
N_{I}, Br	3.2	2/82	0.074
$N_{\text{II–III}}$, Br	1.5	5/82	0.074
$N_{\text{IV–V}}$, O, Ag	0.65	11/82	0.074

The widths η_i were obtained from the table given by Compton and Allison.[14] These authors gave 11.2 ev for the full K width at half-maximum for Ag, and 3.2 ev for the L width of Ag. These widths actually apply to certain lines of the discrete spectrum, but should also give the right order of magnitude for a transition to the continuum since the width is due primarily to the probability of filling the hole in the K or L shell. The full width for the K transitions of Br was estimated as 4.7 ev from the values given by Compton and Allison[14] for the neighboring elements Ge and Sr. The L width of Br and the M widths of both elements were taken equal to the L width of Ag (3.2 ev). This value is probably of the correct order of magnitude.[15] The N widths are more uncertain and were taken as 1 ev. It should be noted that the Čerenkov loss is quite insensitive to the η_i. A calculation showed that if the width of ν_{14} ($N_{\text{IV–V}}$ of Ag) were 10^{-2} ev instead of 1 ev the Čerenkov loss would be increased by only $\sim$10 percent from the value 2.2×10^{-3} Mev/g cm^{-2} found with the 1 ev width. Therefore the precise choice of the η_i is not critical. The widths are listed in Table I. The η_i are obtained from these values by dividing by $2h\nu_p=7.10$ ry.

For AgBr, $c/\nu_p=2.56\times10^{-6}$ cm. Since the mean grain radius is $\sim$0.1–0.2μ, we took $b_p=5$ corresponding to $b=0.13\mu$. Equation (1) was integrated numerically for a few values of β. For $\beta=1$, Fig. 1 shows the real part of the integrand (called J) as function of ν. It is seen that J is large only for $\nu<0.13$ below the first absorption limit, which leads to the result that a large fraction of the Čerenkov loss comes from the visible and the near ultraviolet. The smallness of J for $\nu>\nu_{14}$ arises as follows. For $\beta=1$, the condition for Čerenkov radiation in the absence of absorption is $n>1$ or $\alpha>0$.

[9] A. Sommerfeld, *Atomic Structure and Spectral Lines* (Methuen and Company, London, 1934), third edition, p. 237.
[10] H. E. White, *Introduction to Atomic Spectra* (McGraw-Hill Book Company, Inc., New York and London, 1934).
[11] C. J. Bakker and E. Segrè, Phys. Rev. **81**, 489 (1951).
[12] A. H. Compton and S. K. Allison, *X-Rays in Theory and Experiment* (D. Van Nostrand Company, Inc., New York, 1935), second edition, p. 293.
[13] *International Critical Tables* (McGraw-Hill Book Company, Inc., New York and London, 1930), first edition, Vol. VII, p. 13.
[14] A. H. Compton and S. K. Allison, reference 12, p. 746.
[15] F. K. Richtmyer [Revs. Modern Phys. **9**, 391 (1937)] has shown that for Au, the L, M, and N widths have the same order of magnitude (10 ev). It seems likely that a similar result holds for the L and M widths of Ag and Br.

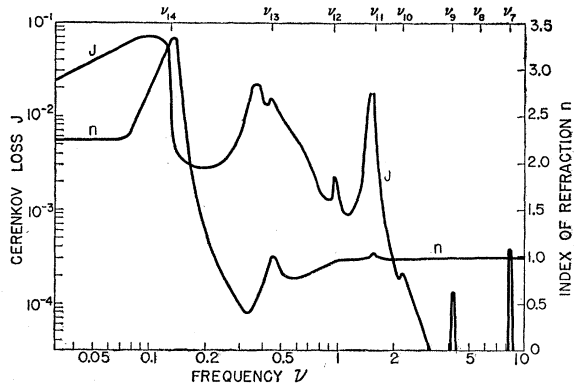

FIG. 1. Integrand J of Eq. (1) for the Čerenkov loss and refractive index n as a function of frequency ν for AgBr. These curves depend on the dispersion model used and are intended only to give a semiquantitative description of J and of the relation between J and n. The ν_i are the absorption limits (in units $h\nu_p=3.55$ ry).

For $\nu<\nu_{14}$ the contribution of all absorption limits to α is positive as can be seen from the contribution to α of a single transition which is

$$\alpha_i(\nu)\approx f_i/(\nu_i^2-\nu^2). \tag{7}$$

For $\nu>\nu_{14}$, ν lies in general between two absorption limits, $\nu_{i+1}<\nu<\nu_i$. Except for ν very close to ν_i the negative contribution of ν_{i+1} (and limits of lower energy) outweighs the positive contribution of ν_i and the higher limits. Figure 1 shows the curve of n as obtained from Eq. (6); n is generally <1 for $\nu>0.2$. Because α is complex (absorption) and b_p is not very large compared to 1 there is some energy escape ($J\neq 0$) even when $n<1$.

The fact that W_b is small for $n<1$ can be deduced directly from Eqs. (1) and (3). For $\beta=1$ consider the two cases Rl $\alpha<0$ and Rl $\alpha>0$, and assume Im α/Rl $\alpha\ll 1$, where Im denotes the imaginary part. For Rl $\alpha<0$ Rl $k_p^2>0$ and k_p is essentially real. The order of magnitude of k_p^2 is 1, as follows from Eq. (7) which gives $f_i\nu^2/(\nu^2-\nu_i^2)$ near the ith limit. For $\nu>\nu_i$, $\nu^2 f_i$ and $\nu^2-\nu_i^2$ are both $<\nu^2$ and of the same order of magnitude. Hence $\exp[-(k_p+k_p^*)b_p]$ is of order e^{-10} and thus neglegible. For Rl $\alpha>0$, Rl $k_p^2<0$ and k_p is almost pure imaginary so that $(k_p+k_p^*)b_p$ is small and the exponential is of order 1.

Using $A=0.0670$ Mev/g cm^{-2} the integration over J gives the result $W_b=2.10\times 10^{-3}$ Mev/g cm^{-2} for $\beta=1$. As pointed out in I this value is small compared to the relativistic rise of the total ionization loss $(1/\rho)(dE/dx)$ which is 0.12 Mev/g cm^{-2}. Equation (1) was also integrated for 2 other values of β giving $W_b(2)=1.26\times 10^{-3}$ Mev/g cm^{-2} and $W_b(10)=1.93\times 10^{-3}$ Mev/g cm^{-2}; here $W_b(a)$ denotes W_b for $p/\mu c=a$ and p, μ are the momentum and mass of the particle. Thus W_b reaches its asymptotic value for $p/\mu c\sim 20$. This result can be understood by considering Eqs. (1) and (3). The important quantities are $[1-\beta^2-(\alpha/1+\alpha)]$ and k_p. Since $J(\nu)$ is small for $\nu>10$ saturation of W_b sets in when these quantities have reached their final values in the range $\nu<10$. $\alpha(\nu)$ is of order ν^{-2} so that $k_p(\nu)$ reaches its final value when $\beta^{-2}-1\ll\nu^{-2}$ or $p/\mu c\gg\nu$. Thus the condition with respect to $k_p(\nu)$ is $p/\mu c\gg 10$. For $\nu\sim 10$, $\alpha/(1+\alpha)$ can be replaced by α so that the condition for the asymptotic behavior of $[1-\beta^2-(\alpha/1+\alpha)]$ is the same as for k_p. The early saturation of W_b is directly connected with the smallness of $J(\nu)$ for $\nu>10$.

The ionization loss in AgBr was calculated in order to compare with the experiments on the grain count. As shown in Eq. (50) below, the loss is given by

$$(1/\rho)(dE/dx)=(A/\beta^2)[B+\ln 2+2\ln(p/\mu c)+\ln T_{0,\,\mathrm{Mev}}-\beta^2-\delta], \tag{8}$$

where $B=\ln[mc^2(10^6\text{ ev})/I^2]=15.07$, $T_{0,\,\mathrm{Mev}}$ is the maximum energy transfer in Mev which was taken as 0.01, and δ is the correction for the density effect. Using the expression for δ given in II [Eqs. (10) and (10a)], one obtains

$$\delta=4.606X-5.14+0.160(3-X)^{3.18},\quad (0.10<X<3) \tag{9}$$

$$\delta=4.606X-5.14,\quad (X>3) \tag{10}$$

where $X=\log_{10}(p/\mu c)$. The resulting expression for $(1/\rho)(dE/dx)$ is

$$(1/\rho)(dE/dx)=(0.0670/\beta^2)[11.15+4.606X-\beta^2],\quad (X<0.10), \tag{11}$$

$$(1/\rho)(dE/dx)=(0.0670/\beta^2)\times[16.29-0.160(3-X)^{3.18}-\beta^2],\quad (0.10<X<3) \tag{12}$$

and the asymptotic value is 1.024 Mev/g cm^{-2} ($p/\mu c>10^3$). Figure 2 shows the theoretical curve of $(1/\rho)(dE/dx)$ as a function of $E/\mu c^2$, where E is the kinetic energy. Experimental values of the grain count obtained in 3 experiments are shown for comparison. The data of Pickup and Voyvodic[16] were obtained at sea level and at high altitude and include μ- and π-mesons, protons, and electrons. The data of Morrish[17] pertain to electrons in a plate exposed at high altitude. The points of Daniel *et al.*[18] correspond mostly to π-mesons resulting from collisions of energetic primaries with nuclei in the emulsion. In all 3 cases the momentum was obtained from the multiple scattering. The uncertainty of the momentum determination (or of $E/\mu c^2$) was indicated by Pickup and Voyvodic[16] and corresponds to the horizontal bars in Fig. 2. The experimental points were normalized to the asymptotic value 1.02 Mev/g cm^{-2}. It is seen that the theoretical curve is in good agreement with the data of Pickup and Voyvodic.[16] The other data confirm the relativistic rise

[16] E. Pickup and L. Voyvodic, Phys. Rev. **80**, 89 (1950).
[17] H. A. Morrish, Phil. Mag. **43**, 533 (1952).
[18] Daniel, Davies, Mulvey, and Perkins, Phil. Mag. **43**, 753 (1952).

predicted by theory and give a slight indication that the increase to the asymptotic value may occur somewhat faster than theory predicts. However, the lack of information on the momentum uncertainty of these data makes it impossible to draw any conclusions on this point. We note that Shapiro and Stiller[19] obtained a value 1.12 ± 0.04 for the ratio of the final ionization to minimum ionization for μ-mesons and protons, in good agreement with the present results.

Daniel *et al.*[18] have obtained curves of the grain density *vs* $p\beta$ for protons, π- and κ-mesons by interpolation of experimental points. More accurate values can be calculated from Eqs. (11) and (12). The theoretical curves agree well with those of Daniel *et al.*[18] except that the rise to the asymptotic value is less rapid.

In Eq. (1) only the first term of the expansions of $K_0(k_pb_p)$ and $K_1(k_p^*b_p)$ is taken into account. The higher terms of the asymptotic expansions[20] are given by

$$K_0(k_pb_p)=\left(\frac{\pi}{2k_pb_p}\right)^{\frac{1}{2}}\exp(-k_pb_p)\times\left(1-\frac{1}{8k_pb_p}+\frac{9}{128k_p^2b_p^2}-\cdots\right), \quad (13)$$

$$K_1(k_p^*b_p)=\left(\frac{\pi}{2k_p^*b_p}\right)^{\frac{1}{2}}\exp(-k_p^*b_p)\times\left(1+\frac{3}{8k_p^*b_p}-\frac{15}{128k_p^{*2}b_p^2}-\cdots\right). \quad (14)$$

A calculation was carried out for AgBr, with $\beta=1$, by introducing a factor $[1-(1/8k_pb_p)][1+(3/8k_p^*b_p)]$ in the integrand of Eq. (1). The resulting W_b is 2.19×10^{-3} Mev/g cm^{-2}, only 4 percent larger than the value obtained above neglecting the terms in b_p^{-1} in the integrand. This result shows that the higher terms in K_0 and K_1 can be safely neglected.

As pointed out in I, Eq. (1) reduces to the Frank and Tamm expression when there is no absorption (α=real),

$$W_b=\frac{2A}{\beta^2}\int_{\beta n>1}\nu\left(\frac{\alpha}{1+\alpha}-1+\beta^2\right)d\nu, \quad (\alpha=\text{real}), \quad (15)$$

where the integral extends over the ν for which $\beta n>1$.

It may be noted that the small value of W_b for AgBr grains is not due primarily to the use of the 14-absorption edge model of the dispersion described above. Preliminary calculations with a model consisting of 4 absorption edges gave very similar results. The 4 edges correspond to average positions of the K, L, M, and N edges of both elements. Thus the ν_i of the combined K edge was taken as the average of the ν_i's of the K edges of Ag and Br. The result for $b_p=5$ was $W_b(\infty)\approx0.02$ Mev/g cm^{-2}. Similarly to the more complete model, the 4-term calculation gave a low value of W_b because $n<1$ above the N absorption edge. The only essential difference between this crude model and the model described above is that the N edge is itself complex (two types of atoms, N_{I}, $N_{\text{II–III}}$, $\cdots$ splittings) and n exceeds 1 only below the first absorption edge ($N_{\text{IV–V}}$ of Ag). This reduces W_b still further, mainly because the region of integration of Eq. (1) for which J is large is decreased.

For macroscopic samples of AgBr, the Čerenkov loss is still smaller than for emulsion. The loss for $\nu>\nu_{14}$ is negligible since b_p is of order 10^5 so that $\exp[-(k_p+k_p^*)b_p]\sim e^{-10^5}$ in this region. The loss in the region $\nu<\nu_{14}$ is reduced because of absorption. A calculation of $W_b(\infty)$ was carried out assuming $b_p=10^5$ ($b=0.26$ cm). The N width was taken as 10^{-3} ev. With this width the absorption is small for $\lambda>3200$A. W_b was obtained as $\sim0.3\times10^{-3}$ Mev/g cm^{-2}. This is still smaller by a factor 3 than the part of the loss below ν_{14} for a grain ($\sim0.9\times10^{-3}$ Mev/g cm^{-2}). The difference is due to the absorption between $\lambda=3200$A and the assumed absorption edge at $\lambda=1940$A ($\nu_{14}=0.132$). The result obtained here for AgBr is typical of macroscopic crystals. For a different dispersion model one may obtain somewhat higher values, but it is unlikely that the Čerenkov loss for a macroscopic sample exceeds the order of 10^{-3} Mev/g cm^{-2}. Hence it can be neglected in comparing the theory with experiments on the ionization loss in crystals.[21]

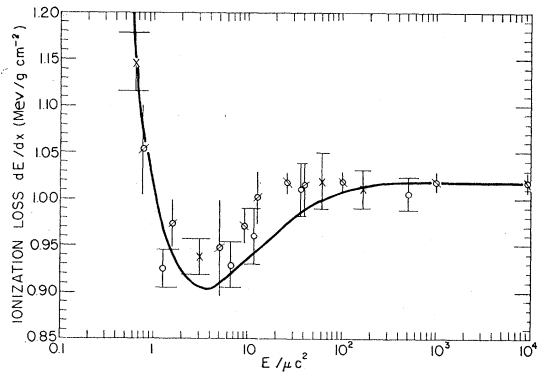

FIG. 2. The ionization loss in AgBr as a function of $E/\mu c^2$ (E=kinetic energy). The experimental points are the values of Morrish (marked ⊘), Daniel *et al.* (marked ∅), and Pickup and Voyvodic (the points marked ○ pertain to μ-meson decay electrons and μ-mesons in a sea level plate; the points marked × refer to high energy electrons, protons, and shower π-mesons in a high altitude plate).

[19] M. Shapiro and B. Stiller, Phys. Rev. **87**, 682 (1952).
[20] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944), second edition, p. 202.
[21] W. L. Whittemore and J. C. Street, Phys. Rev. **76**, 1786 (1949); F. Bowen and F. X. Roser, Phys. Rev. **85**, 992 (1952); A. Hudson and R. Hofstadter, Phys. Rev. **88**, 589 (1952).

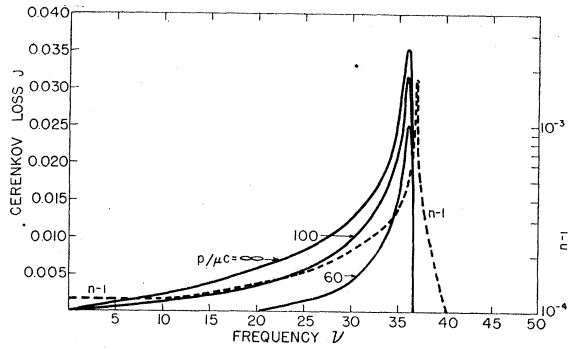

FIG. 3. Integrand J of Eq. (1) for the Čerenkov loss and $n-1$ as a function of ν for O_2. The different curves of J pertain to different values of $p/\mu c$.

III. ČERENKOV LOSS FOR GASES

The Čerenkov loss has been evaluated for He and O_2 for comparison with the experiments of Carter and Whittemore[22] on the droplet count in a cloud chamber filled with He and with a similar experiment by Ghosh, Jones, and Wilson[23] on O_2. For He the expression used for α differs from Eq. (4) by the inclusion of terms representing the effect of discrete lines. Thus we used

$$\alpha=-\sum_i\left(\frac{f_i}{\nu_i^2}\right)\frac{\ln[(\nu_i^2-\nu^2-2i\eta_i\nu)/\nu_i^2]+[(\nu^2+2i\eta_i\nu)/\nu_i^2]}{[(\nu^2+2i\eta_i\nu)/\nu_i^2]^2}+\sum_{i\ (\text{lines})}\frac{f_i}{\nu_i^2-2i\eta_i\nu-\nu^2},\quad(16)$$

where the second sum gives the effect of the lines. It is desirable to introduce the effect of lines for He because there is only one absorption limit so that the behavior of α as function of ν would be distorted by lumping the discrete lines with the absorption continuum. For AgBr a separate treatment of lines was not called for, because several shells are occupied so that the behavior of α is determined primarily by the different absorption limits. The data used for He are as follows. The $1s$-$2p$, $1s$-$3p$, and $1s$-continuum transitions occur at[24] 171 122 cm^{-1}, 186 197 cm^{-1}, and 198 298 cm^{-1}, respectively. These frequencies were used to obtain the ν_i. For the pressure $P=140$ cm of Hg used in the experiment,

$$h\nu_p=0.020\times(140/76)^{\frac{1}{2}}=0.027\text{ ry},$$

where 0.020 ry is the value for normal pressure (see Table I of II). The resulting values of the ν_i are $\nu_1=66.1$, $\nu_2=62.4$, $\nu_3=57.3$. In order to obtain the f_i, a result of Bethe[25] was used according to which the relative frequency of exciting the $2p$ level, the $3p$ level, and all higher p states (including the continuum) is 0.555, 0.089, and 0.356, respectively. The mean ionization potential of He is[5] 1.98 ry. In order to obtain agreement with this value, the p states above $3p$ were lumped with the continuum and the following values of f_i were used: $f_1=0.35$ for the continuum (whose mean ionization energy is $1.80e^{\frac{1}{2}}$ ry), $f_2=0.10$ for $1s$-$3p$, and $f_3=0.55$ for $1s$-$2p$. The widths were taken as 10^{-3} ev. As shown below, this value is of the same order of magnitude as that obtained from the theory of Weisskopf.[26] The corresponding η_i is 1.35×10^{-3}.

Equation (1) was integrated for several values of β using the above constants and Eq. (16) for α in which the first sum has one term ($i=1$) and the second sum has two terms ($i=2$, 3). The assumption is that W_b with b of the order of half the width of the track gives the energy which escapes beyond b and therefore does not contribute to the droplet count. According to Whittemore[27] the width of the track was 0.15–0.2 cm. For the pressure considered $c/\nu_p=3.35\times10^{-4}$ cm and b_p was taken as 300 ($b=0.10$ cm). The effect of a change of b_p is discussed below [see Eq. (36)]. The values of $W_b(p/\mu c)$ are given in Table II. It is seen that $W_b(\infty)$ is rather large, although still appreciably smaller than the relativistic rise of $(1/\rho)(dE/dx)$ which is 0.50 Mev/g cm^{-2}. W_b sets in at $p/\mu c\sim20$. Since most of the results of Carter and Whittemore[22] were obtained for lower momenta a detailed comparison is not possible. With the normalization used in the experiment and with $T_0=300$ ev, the theoretical curve of the number of ions/mm as a function of $p/\mu c$ would be decreased from 20 to 18.7 ions/mm when the Čerenkov loss is taken into account. It may be noted that the relativistic rise of $(1/\rho)(dE/dx)$ for He with $T_0=300$ ev in Eq. (8) is from 1.20 Mev/g cm^{-2} at $p/\mu c=4$ to 1.70 Mev/g cm^{-2} for $p/\mu c>10^3$. When the Čerenkov loss is included the amount of energy deposited rises only to 1.56 Mev/g cm^{-2}. Thus the Čerenkov loss is not negligible for He and must be included in comparing theory with experiment for $p/\mu c\gtrsim50$.

For O_2 two calculations were carried out. The one which did not involve a separate treatment of lines will be described first (model I). The values of ν_i are based on the ionization potentials previously used in II, $E_1=42.3$ ry ($1s$), $E_2=4.0$ ry ($2s$), $E_3=2.9$ ry ($2p$). With

TABLE II. Values of the Čerenkov loss W_b in gases for various $p/\mu c$. W_b is in units of Mev/g cm^{-2}; b is the radius of the cylinder for which W_b is evaluated.

$p/\mu c$	H_2	He	O_2 (model I)	O_2 (model II)
20	~0	0.0030	~0	0.0068
60	0.172	0.075	0.0152	0.0251
100	0.210	0.115	0.0319	0.0272
∞	0.232	0.140	0.0425	0.0300
b (cm)	0.10	0.10	0.081	0.081

[22] R. S. Carter and W. L. Whittemore, Phys. Rev. **87**, 494 (1952).
[23] Ghosh, Jones, and Wilson, Proc. Phys. Soc. (London) **A65**, 68 (1952).
[24] R. F. Bacher and S. Goudsmit, *Atomic Energy States* (McGraw-Hill Book Company, Inc., New York and London, 1932), p. 220.
[25] H. A. Bethe, Ann. Physik **5**, 325 (1930).
[26] V. Weisskopf, Physik Z. **34**, 1 (1933).
[27] W. L. Whittemore, private communication.

$f_1=2/8$, $f_2=2/8$, $f_3=4/8$ the geometric mean of the E_i is 6.14 ry or a factor 1.18 smaller than the Bakker-Segrè value, $I=7.26$ ry. Since the average ionization energy has a factor $e^{\frac{1}{3}}$, the E_i must be multiplied by $1.18e^{-\frac{1}{3}}$ to obtain agreement with the measured value of I. The plasma energy $h\nu_p$ is 0.0565 ry for normal pressure. One thus finds $\nu_1=536$, $\nu_2=50.6$, $\nu_3=36.7$. The width was taken as 10^{-3} ev, giving $\eta_i=0.65\times10^{-3}$. With these values of the constants, Eq. (4) was used for α. In view of $c/\nu_p=1.61\times10^{-4}$ cm, b_p was taken as 500 ($b=0.081$ cm) Eq. (1) was integrated numerically giving the results shown in Table II. $W_p(\infty)$ is appreciably smaller than for He. This is due mainly to the fact that the f_i of the lowest transition, which determines the Čerenkov loss, is smaller than for He (0.25 as compared to 0.55). Figure 3 shows the real part J of the integrand of Eq. (1) as a function of ν for various $p/\mu c$. The curve of $n-1$ is shown for comparison. As is also the case for He, $J=0$ above the lowest absorption line ν_3. This result arises because Rl k_p is of order 0.1 and $\exp[-(k_p+k_p^*)b_p]\approx0$ in view of $b_p=500$. This case differs from that of emulsion where, on account of $b_p=5$, some Čerenkov loss could take place in the region $\nu>0.2$ in spite of Rl $k_p=O(1)$.

For comparison with the results of Ghosh, Jones, and Wilson[23] $(1/\rho)(dE/dx)$ was calculated for μ-mesons in O_2. Equation (8) was used with $A=0.0765$, $B=17.78$. $T_{0,\text{Mev}}$ was taken as 0.93×10^{-3} (930 ev) which corresponds to 30 ion pairs and is the value used by Ghosh *et al.*[23] For δ the following expression is obtained from the constants given in Table III:

$$\delta=4.606X-10.71+0.145(4-X)^{3.52}, \quad (1.88<X<4) \quad (17)$$

$$\delta=4.606X-10.71, \quad (X>4)$$

where $X=\log_{10}(p/\mu c)$. The curve of dE/dx is shown in Fig. 4, together with the energy deposited in the track dE_d/dx which is given by

$$(1/\rho)(dE_d/dx)=(1/\rho)(dE/dx)-W_b, \quad (18)$$

where W_b was obtained from model I discussed above. The experimental points of the droplet count were normalized to the theoretical value of dE/dx at $p=2\times10^9$ ev/c. The theory is in reasonable agreement with the experimental results. In particular, the droplet count starts to rise more slowly in the region ($p/\mu c\sim10^2$) where the density effect sets in, as was already pointed out by Ghosh *et al.*[23] The correction for the Čerenkov loss is less important than for He.

The second model used for O_2 (model II) gave results very similar to model I. In this dispersion model the frequencies of the lowest transitions are taken into account exactly. These frequencies are[28] 76 794 cm^{-1} for $2p$-$3s$, 96 224 cm^{-1} for $2p$-$4s$, and 109 837 cm^{-1} for $2p$-continuum. The energies of the $1s$ and of the $2s$ continuum were taken as the ionization potentials

[28] Reference 24, p. 333.

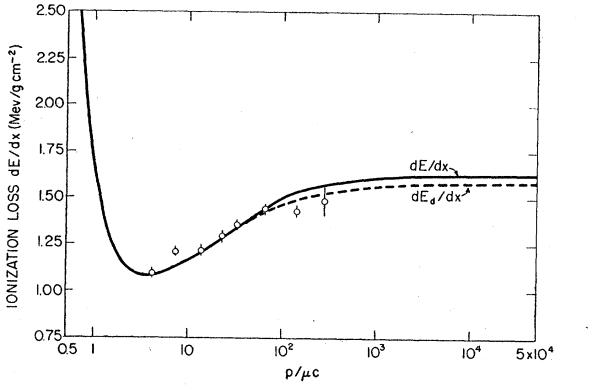

FIG. 4. The ionization loss of μ-mesons in O_2 as a function of $p/\mu c$. The broken curve (dE_d/dx) gives the energy deposited in the track after subtraction of the estimated Čerenkov loss. The experimental points are the data of Ghosh, Jones, and Wilson normalized to the theoretical curve at $p=2\times10^9$ ev/c.

given in Table I of II, $E_1=42.3$ ry and $E_2=4.0$ ry, respectively. In order to obtain agreement with the value $I=7.26$ ry of Bakker and Segrè[11] using these values of the ν_i, the f_i of the $2s$ and $2p$ transitions had to be taken somewhat different from the corresponding occupation numbers divided by Z. The values of ν_i and f_i are $\nu_1=749$, $f_1=0.25$; $\nu_2=70.8$, $f_2=0.49$; $\nu_3=17.6$, $f_3=0.091$; $\nu_4=15.6$, $f_4=0.026$; $\nu_5=12.4$, $f_5=0.143$. ν_4 and ν_5 correspond to the lines $2p$-$4s$ and $2p$-$3s$, respectively. With these constants and with $\eta_i=0.65\times10^{-3}$, $b_p=500$, Eq. (1) was integrated for $\beta=1$, using Eq. (16) for α. One finds $W_b(\infty)=0.0300$ Mev/g cm^{-2}. For the other values of β, an analytic expression given below [Eqs. (35), (36)] was used. The resulting values of W_b are given in Table II. It is seen that the results of the two models agree fairly well.

In order to derive an analytic expression for the Čerenkov loss in gases we consider the important practical case in which the dispersion in the optical and ultraviolet is determined by the absorption line ν_j corresponding to the first excited state. This condition was found to hold for the cases of He and O_2 discussed above and will be satisfied in general if f_j is not too small, say >0.1. Under this condition the exponential of Eq. (1) represents the effect of absorption. We have

$$k_p^2=\nu^2\left[\beta^{-2}-1-\frac{f_j}{\nu_j^2-2i\eta_j\nu-\nu^2}\right]. \quad (19)$$

Let

$$k_p^2=-|k_p|^2\exp(iy), \quad (20)$$

where $|k_p|$ is the absolute value of k_p and y is real. With the assumption $\eta_j\ll\nu_j$, Eqs. (19) and (20) give

$$|k_p|\approx\frac{\nu[f_j-(\beta^{-2}-1)(\nu_j^2-\nu^2)]^{\frac{1}{2}}}{(\nu_j^2-\nu^2)^{\frac{1}{2}}}, \quad (21)$$

$$y\approx\frac{2f_j\eta_j\nu}{(\nu_j^2-\nu^2)[f_j-(\beta^{-2}-1)(\nu_j^2-\nu^2)]}. \quad (22)$$

The exponential of Eq. (1) becomes

$$\exp[-|k_p|yb_p]$$
$$=\exp\left\{-\frac{2f_j\eta_j\nu^2 b_p}{(\nu_j^2-\nu^2)^{\frac{3}{2}}[f_j-(\beta^{-2}-1)(\nu_j^2-\nu^2)]^{\frac{1}{2}}}\right\}. \quad (23)$$

For a classical atomic system whose polarizability is $f_j/[4\pi(\nu_j^2-2i\eta_j\nu-\nu^2)]$ the absorption coefficient in units ν_p/c is[29]

$$\kappa_p=\frac{2f_j\eta_j\nu^2}{(\nu_j^2-\nu^2)^2+4\eta_j^2\nu^2}. \quad (24)$$

Assuming that the radiation originates along the particle path, the distance it traverses to reach the cylinder of radius b_p is $b_p/\sin\theta(\nu)$ where $\theta(\nu)$ is the angle of emission for frequency ν. $\theta(\nu)$ is given by

$$\cos\theta(\nu)=\frac{1}{\beta n(\nu)}=\frac{1}{\beta[1+f_j/(\nu_j^2-\nu^2)]^{\frac{1}{2}}}. \quad (25)$$

For $\nu_j\gg 1$, Eq. (25) gives

$$\sin\theta(\nu)\approx\left[\frac{f_j-(\beta^{-2}-1)(\nu_j^2-\nu^2)}{\nu_j^2-\nu^2}\right]^{\frac{1}{2}}. \quad (26)$$

Upon inserting Eqs. (24) and (26) into $\exp[-\kappa_p b_p/\sin\theta(\nu)]$ one obtains Eq. (23) which thus represents the absorption of the radiation.

We assume that the Čerenkov loss is that given by the Frank and Tamm expression [Eq. (15)] for frequencies ν for which $(k_p+k_p^*)b_p<1$ and is zero for $(k_p+k_p^*)b_p>1$. This takes into account absorption in an adequate manner for the present purposes. If ν_a denotes the frequency for which the exponent of (23) is 1, we write

$$\nu_a=\nu_j-\sigma. \quad (27)$$

Since $\sigma\ll\nu_j$, σ is determined by

$$\frac{2f_j\eta_j\nu_j^2 b_p}{(2\nu_j)^{\frac{3}{2}}\sigma^{\frac{3}{2}}[f_j-2(\beta^{-2}-1)\nu_j\sigma]^{\frac{1}{2}}}=1. \quad (28)$$

TABLE III. Values of the coefficients of Eqs. (40) and (40a) for for the density correction δ.

Material	A	B	$-C$	$10a$	m	X_1	X_0
Mg	0.07576	17.07	4.32	0.40	4.31	3	0.34
polystyrene	0.08257	18.76	3.06	4.2	2.84	2	0.10
CH_4	0.09573	19.56	9.41	9.9	2.24	3	1.55
$(CH)_2$	0.08255	18.76	9.86	12.2	2.13	3	1.58
CO_2	0.07676	17.92	10.26	1.2	3.61	4	1.71
O_2	0.07678	17.78	10.71	1.45	3.52	4	1.88

[29] See, for example, F. Seitz, *The Modern Theory of Solids* (McGraw-Hill Book Company, Inc., New York and London, 1940), first edition, p. 634.

This equation can be solved approximately by disregarding the second term in the square bracket. One finds

$$\sigma^{\frac{3}{2}}=(f_j\nu_j/2)^{\frac{1}{2}}\eta_j b_p. \quad (29)$$

The resulting values of σ are 1.38 for He and 0.45 for O_2 (model II) so that for $p/\mu c=60$, $2(\beta^{-2}-1)\nu_j\sigma=0.044$ for He and 0.0031 for O_2. These values are $\ll f_j$ (0.55 for He, 0.143 for O_2) which shows that the neglect of σ in the bracket of (28) is justified. From Eq. (15) the Čerenkov loss is given by

$$W_b=\frac{2A}{\beta^2}\int_{\nu_b}^{\nu_a}\left(\frac{\alpha}{1+\alpha}-1+\beta^2\right)\nu d\nu, \quad (30)$$

where

$$\alpha=f_j/(\nu_j^2-\nu^2), \quad (31)$$

and ν_b is the lower limit of integration as obtained from the condition $\beta n>1$,

$$\nu_b=[\nu_j^2-f_j(p/\mu c)^2]^{\frac{1}{2}}, \quad (p/\mu c<\nu_j f_j^{-\frac{1}{2}}) \quad (32)$$
$$\nu_b=0. \quad (p/\mu c>\nu_j f_j^{-\frac{1}{2}}) \quad (32a)$$

Upon inserting Eq. (31) into (30) and using $\nu_j^2\gg 1$ one finds

$$W_b=\frac{2Af_j}{\beta^2}\int_{\nu_b}^{\nu_a}\frac{\nu d\nu}{\nu_j^2-\nu^2}-\frac{2A}{(p/\mu c)^2}\int_{\nu_b}^{\nu_a}\nu d\nu. \quad (33)$$

The case $p/\mu c<\nu_j f_j^{-\frac{1}{2}}$ will be considered first. The first term of (33) is

$$W_{b,1}=\frac{Af_j}{\beta^2}\ln\left(\frac{\nu_j^2-\nu_b^2}{\nu_j^2-\nu_a^2}\right)=\frac{Af_j}{\beta^2}\ln\frac{f_j(p/\mu c)^2}{2\nu_j\sigma}. \quad (34)$$

The second term involves ν_a^2. Since $\sigma\ll\nu_a$, ν_a^2 can be replaced by ν_j^2. One obtains

$$W_b=Af_j\left[\frac{2}{3\beta^2}\ln\frac{f_j(p/\mu c)^3}{2\nu_j^2\eta_j b_p}-1\right]. \quad (p/\mu c<\nu_j f_j^{-\frac{1}{2}}) \quad (35)$$

For the case $p/\mu c>\nu_j f_j^{-\frac{1}{2}}$, $\nu_b=0$ and Eq. (33) gives

$$W_b=A\left[\frac{2f_j}{3\beta^2}\ln\frac{\nu_j}{2f_j^{\frac{1}{2}}\eta_j b_p}-\frac{\nu_j^2}{(p/\mu c)^2}\right]. \quad (p/\mu c>\nu_j f_j^{-\frac{1}{2}}) \quad (36)$$

Equations (35) and (36) give an approximate expression for W_b for gases. The preceding theory can be used to predict the approximate value of $p/\mu c$, say $p_0/\mu c$ where the Čerenkov loss sets in. As is seen from Fig. 3, this takes place when the lower limit ν_b equals $\nu_j-\sigma$. From Eq. (32) one finds

$$\nu_j^2-f_j(p_0/\mu c)^2\approx\nu_j^2-2\sigma\nu_j, \quad (37)$$

whence

$$p_0/\mu c=(2\nu_j^2\eta_j b_p/f_j)^{\frac{1}{3}}. \quad (38)$$

The value of $p_0/\mu c$ for He is 16.9. This result is in satisfactory agreement with the values of W_b in Table II which were obtained by numerical integration of Eq. (1). The Čerenkov loss reaches saturation in the region

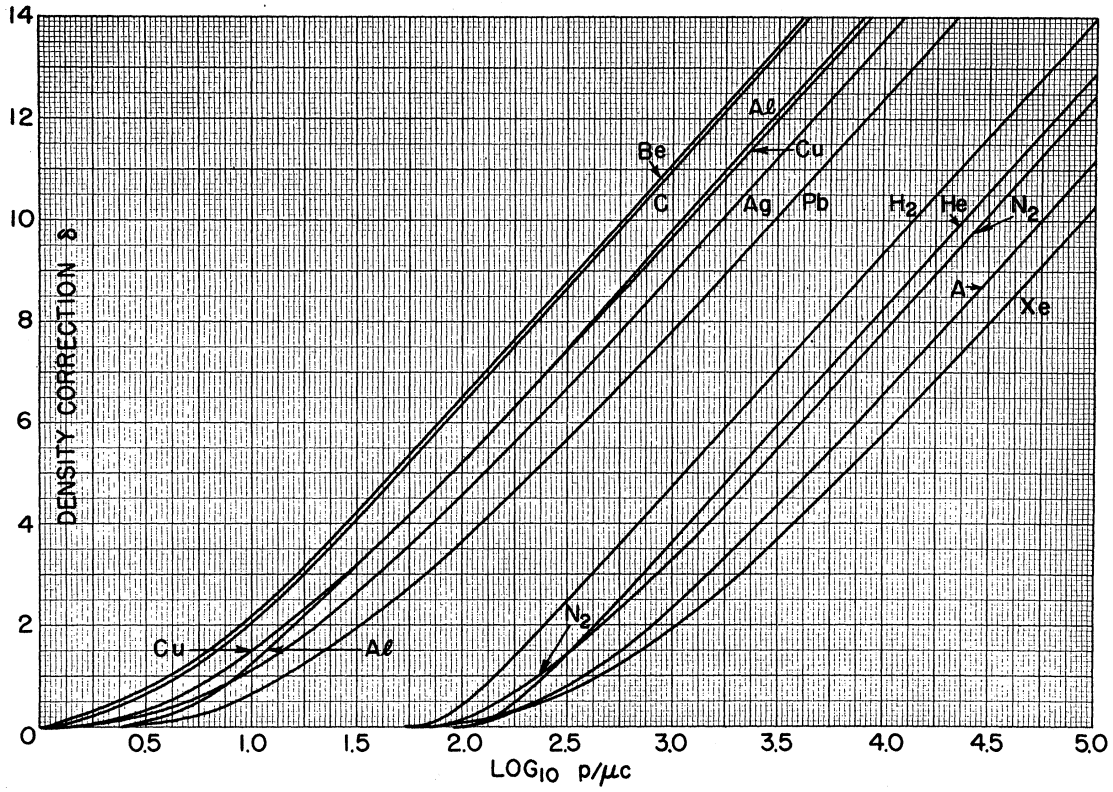

FIG. 5. Density effect correction δ as a function of the momentum/mass of the passing particle.

$p/\mu c > \nu_j f_j^{-\frac{1}{2}}$. For He, W_b attains $0.9W_b(\infty)$ at $p/\mu c = 140$. For O_2, the value $0.9W_b(\infty)$ is attained at $p/\mu c = 64$.

Equations (35) and (36) were used to obtain the Čerenkov loss in H_2. It was assumed that ν_j corresponds to the $1s$-$2p$ transition. The width was taken as 10^{-3} ev. The values of the parameters are: $\nu_j = 37.5$, $f_j = 0.55$, $\eta_j = 1.84\times10^{-3}$, $b_p = 219$ ($b = 0.1$ cm). The results are shown in Table II. $W_b(\infty) = 0.232$ Mev/g cm^{-2} may be compared with the relativistic increase of the total ionization loss $(1/\rho)(dE/dx)$ which is 0.81 Mev/g cm^{-2}. Thus the correction for W_b is important for H_2 for $p/\mu c \gtrsim 50$.

Equations (35) and (36) can be used to predict the effect of a change of the parameters on the Čerenkov loss. Thus if b_p or η_j is smaller by a factor of 5, $W_b(\infty)$ is increased by $(2/3)Af_j \ln 5$ which is 0.045 Mev/g cm^{-2} for He and 0.012 Mev/g cm^{-2} for O_2 (model II). Using the values of Table II one obtains the corrected values $W_b(\infty) = 0.185$ Mev/g cm^{-2} for He and $W_b(\infty) = 0.042$ Mev/g cm^{-2} for O_2.

According to Eq. (36) and with reasonable values of the parameters the Čerenkov loss decreases somewhat with increasing Z. Thus for Xe, taking ν_j to correspond to the $5p$-$6s$ transition which is at[30] 67 068 cm$^{-1} = 0.61$ ry, and with $f_j = 6/54$, a width of 10^{-3} ev and $b = 0.1$ cm ($b_p = 1143$), one finds $\nu_j = 5.88$, $\eta_j = 0.36\times10^{-3}$ so that $W_b(\infty) = 0.0144$ Mev/g cm^{-2}, as compared to 0.03 Mev/g cm^2 for O_2. The decrease with Z is due to the fact that the logarithm of (36) contains a factor $(\nu_{ev}/f_j^{\frac{1}{2}}\eta_{ev}b_{cm})\nu_p^{-1}$ where the subscripts of ν_{ev}, η_{ev}, and b_{cm} indicate that these quantities are in units ev and cm, respectively. On the assumption that ν_{ev}, η_{ev}, f_j, and b_{cm} have values comparable to those for O_2, the argument of the logarithm goes as ν_p^{-1} which decreases as Z is increased (see Table I of II).

The choice of the widths of 10^{-3} ev for He and O_2 is based on the study of Weisskopf[26] of the width of spectral lines in gases. According to Weisskopf, the half-width due to collisions is approximately given by

$$\bar{\eta}_{j,\text{coll}} = \frac{\pi e^2 n_0 f_j}{2m\omega_j}, \tag{39}$$

where the frequency of the line ω_j and $\bar{\eta}_{j,\text{coll}}$ are in radian units. The values of the full width $2\hbar\bar{\eta}_{j,\text{coll}}$ as found from (39) are 0.88×10^{-3} ev for He and 2.26×10^{-3} ev for O_2 according to model II (line at 0.70 ry, $\nu_5 = 12.4$). The value of 10^{-3} ev used in the calculations is of the same order of magnitude. Since η_j appears only in the logarithm of (36) the choice of η_j is not critical. The Doppler width is small compared to $2\hbar\eta_{j,\text{coll}}$ (e.g., 0.13×10^{-3} ev for He).

[30] Reference 24, p. 505.

IV. VALUES OF THE DENSITY EFFECT

The density effect has been evaluated for 6 additional substances not treated in II. The results are shown in Table III which lists the constants a, m, C, X_0, and X_1 which enter into the expression for δ. As shown in II [see Eqs. (10), (10a)], δ can be represented by

$$\delta=4.606X+C+a(X_1-X)^m, \quad (X_0<X<X_1) \tag{40}$$

$$\delta=4.606X+C, \quad (X>X_1) \tag{40a}$$

where $X=\log_{10}p/\mu c$; note that X, X_0, and X_1 correspond to x, x_0, and x_1 of II. Table III also gives the constants[31] A and B which enter into $(1/\rho)(dE/dx)$.

The excitation energies $h\nu_i'$ for the cases shown in Table III were taken from Table I of II, except for Mg where data were not obtained previously. The following data were used for this case:[9,10] $h\nu_1=95.8$ ry, $h\nu_2=6$ ry, $h\nu_3=1.4$ ry for the K, L, M shells, respectively, with $f_1=2/12$, $f_2=8/12$, $f_3=2/12$. The mean excitation energy obtained from the $h\nu_i$ is $h\nu_m=7.48$ ry, as compared to the experimental value $I=10.32$ ry obtained by interpolation from the table of Bakker and Segrè.[11] Using the same procedure as in II, the ν_i were multiplied by a factor $I/h\nu_m=1.38$ and divided by $h\nu_p=1.96$ ry to obtain the $\bar{\nu}_i$ used in Eq. (1) of II.

Figure 5 shows the values of δ for a few commonly used substances, as obtained in the earlier work.[2] This figure can be read to greater accuracy than the graphs given in II (Figs. 1 and 2). The values for gases pertain to normal pressure. The curve for Cu can also be used for Fe whose δ differs by $\lesssim 0.2$ from that of Cu. The curve for C applies to good approximation to polyethylene, for which $A=0.08774$, $B=19.07$. We note that Al has about the same δ as Cu because the smaller electronic density of Al is compensated by the larger polarizability per electron due to the lower Z. This result is similar to the case of NaI and U previously discussed.[2]

The expressions for dE/dx given in II made no distinction between electrons and particles heavier than electrons. Equation (11) of II applies approximately to electrons, but a small correction is necessary for heavy particles. Following Rossi[32] the average loss of heavy particles is given by

$$\frac{1}{\rho}\frac{dE}{dx}=\frac{2\pi n_0 e^4}{mv^2\rho}\left[\ln\frac{2mv^2T}{I^2(1-\beta^2)}-2\beta^2-\delta\right], \tag{41}$$

where T is the maximum energy transfer as given by Eq. (12) of II. The difference between $(1/\rho)(dE/dx)$ for heavy particles and electrons is due to the close collisions. The correct expression for electrons is[25]

$$\frac{1}{\rho}\frac{dE}{dx}=\frac{2\pi n_0 e^4}{mv^2\rho}\left[\ln\frac{mv^2T}{I^2(1-\beta^2)}+\frac{9}{8}-\beta^2-\delta\right], \tag{42}$$

which differs from Eq. (11) of II only through a term $\frac{1}{8}$ arising from close collisions; $T=E/2$, where E is the kinetic energy of the electron.

In order to clarify the difference between Eqs. (41) and (42) their derivation as given by Rossi[32] will be briefly outlined. The loss due to distant collisions is given by[33]

$$\frac{1}{\rho}\left(\frac{dE}{dx}\right)_{T_0}=\frac{2\pi n_0 e^4}{mv^2\rho}\left[\ln\frac{2mv^2T_0}{I^2(1-\beta^2)}-\beta^2-\delta\right], \tag{43}$$

where T_0 is the upper limit of the energy transfers considered and is assumed $\ll E$. T_0 is of order 10^4–10^5 ev. Equation (43) holds for all charged particles. To Eq. (43) one must add the energy loss in close collisions with transfers $>T_0$ in which the atomic electrons can be treated as free.

The case of a heavy particle will be considered first. Following Bhabha[34] the probability $\Phi_{\rm col}(E, E')$ that a particle of energy E (spin 0) transfers an energy between E' and $E'+dE'$ to an atomic electron (treated as free) is

$$\Phi_{\rm col}(E, E')dE'=\frac{A}{\beta^2}\left(1-\beta^2\frac{E'}{T}\right)\frac{dE'}{E'^2}. \tag{44}$$

The average loss from collisions with energy transfer $>T_0$ is

$$\int_{T_0}^{T}E'\Phi_{\rm col}(E, E')dE'=\frac{A}{\beta^2}\left(\ln\frac{T}{T_0}-\beta^2\right), \tag{45}$$

where we have used the fact that $T_0\ll T$. Upon adding Eq. (45) to (43) one obtains Eq. (41).

For the case of electrons, Møller[35] has shown that $\Phi_{\rm col}(E, E')$ is given by

$$\Phi_{\rm col}(E, E')dE'=AdE'\left[\frac{E}{E'(E-E')}-\frac{1}{E}\right]^2. \quad (\beta\approx 1) \tag{46}$$

The average loss due to close collisions is

$$\int_{T_0}^{E/2}E'\Phi_{\rm col}(E, E')dE'=A\left[\ln\frac{E}{T_0}+\frac{9}{8}-2\ln 2\right]. \quad (\beta\approx 1) \tag{47}$$

Upon adding Eq. (47) to (43) one obtains Eq. (42).

[31] The values of A in II were obtained from $0.153(Z/A_0)$ where A_0 is the atomic weight. According to the best values of the natural constants (J. W. Du Mond and E. R. Cohen, Phys. Rev. **82**, 555 (1951)), a more accurate value of A is $0.15355(Z/A_0)$.

[32] B. Rossi, *High Energy Particles* (Prentice-Hall, Inc., New York, 1952), p. 22.

[33] H. A. Bethe, reference 25 and Z. Physik **76**, 293 (1932).

[34] H. J. Bhabha, Proc. Roy. Soc. (London) **A164**, 257 (1938).

[35] C. Møller, Ann. Physik **14**, 531 (1932).

It can be verified that (42) is identical with the expression given by Rossi.[36]

Equation (43) should be used if one is interested in the loss due to collisions with a fixed maximum energy transfer T_0, as in the case of emulsion or of the droplet count in gases. Upon writing (41), (42), and (43) in terms of A and B, one obtains

$$\frac{1}{\rho}\frac{dE}{dx}=\frac{A}{\beta^2}\left[B+0.69+2\ln\frac{p}{\mu c}+\ln T_{\mathrm{Mev}}-2\beta^2-\delta\right], \quad \text{(heavy)} \quad (48)$$

$$\frac{1}{\rho}\frac{dE}{dx}=\frac{A}{\beta^2}\left[B+0.43+2\ln\frac{p}{mc}+\ln E_{\mathrm{Mev}}-\beta^2-\delta\right], \quad \text{(electrons)} \quad (49)$$

$$\frac{1}{\rho}\left(\frac{dE}{dx}\right)_{T_0}=\frac{A}{\beta^2}\left[B+0.69+2\ln\frac{p}{\mu c}+\ln T_{0,\mathrm{Mev}}-\beta^2-\delta\right], \quad (50)$$

where the subscript in T_{Mev}, E_{Mev}, and $T_{0,\mathrm{Mev}}$ indicates that T, E, and T_0 are to be expressed in Mev.

[36] Reference 32, p. 27, Eq. (11).

Equation (16) of II for the most probable loss ϵ_{prob} should be corrected. ϵ_{prob} is given by[37]

$$\epsilon_{\mathrm{prob}}=\frac{2\pi n_0 e^4 t}{mv^2\rho}\left[\ln\frac{2mv^2(2\pi n_0 e^4 t/mv^2\rho)}{I^2(1-\beta^2)}-\beta^2+0.37-\delta\right], \quad (51)$$

where t is the thickness in g cm^{-2}. Equation (51) can also be written

$$\epsilon_{\mathrm{prob}}=\frac{At}{\beta^2}\left[B+1.06+2\ln\frac{p}{\mu c}+\ln\frac{At}{\beta^2}-\beta^2-\delta\right]. \quad (52)$$

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Note added in proof:—In recent measurements of the grain count in emulsion, B. Stiller and M. M. Shapiro [Bull. Am. Phys. Soc. **28**, No. 3, 72 (1953)] found good agreement with the curve of $(1/\rho)(dE/dx)$ presented here (Fig. 2). Besides confirming the theoretical ratio of plateau to minimum ionization, these data lend support to the gradual rise of the ionization to the asymptotic value.

[37] L. Landau, J. Phys. (U.S.S.R.) **8**, 201 (1944); K. R. Symon, thesis, Harvard University, 1948 (unpublished).

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Absorption of Light by Trapped Electrons

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A new method of approach to the problem of absorption of light by trapped electrons in crystals is presented. The method is based upon the use of the Slater sum for an oscillator (density matrix). The absorption cross section is calculated under the assumption of an electron-lattice coupling which is linear in the normal coordinates of the lattice; this yields the shape of the absorption curve, but the maximum does not shift with temperature as required by experiment. This shift is then accounted for in a somewhat fundamental manner by considering the small change in the lattice frequencies which accompanies the photon induced electronic transition. The result is that the absorption maximum shifts with temperature, but the shape of the absorption curve is not effected by the change in the lattice frequencies.

HUANG and Rhys[1] published the first detailed quantum-mechanical calculation of the absorption of light in F centers. Their work was followed by two articles of Lax[2,3] in which a more general viewpoint is taken in the sense that the F center can be of more complicated structure (i.e., more than one electron) and the lattice is represented in a more general form (i.e., all modes optical and acoustical and a general frequency distribution). Lax obtains some of their results as a special case by setting all the frequencies $\omega_j=\omega$ (optical), where j indicates mode. He thereby avoids almost all of the analysis in their paper.[4] One finds in Lax's papers a complete formulation[2] of the problem and a complete discussion of the moments[3] of the absorption and emission spectral distributions.

The purpose of this paper is to present a third method of approach to the problem which is, in this author's opinion, simpler than either that of Huang-Rhys or Lax; the present method accomplishes the following things: (1) it avoids entirely the rather formal use of ordered operators in Lax by a straightforward application of the density matrix of a simple oscillator; (2) it yields the results of Huang-Rhys and the more general

[1] K. Huang, and A. Rhys, Proc. Roy. Soc. (London) **A204**, 406 (1950).
[2] M. Lax, Naval Research Laboratory Report 3973, 1952 (unpublished).
[3] M. Lax, J. Chem. Phys. **20**, 1752 (1952).
[4] Reference 3, p. 1760.