Two-Particle Potential from the Bethe-Salpeter Equation

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(Received April 24, 1953)

A NONRELATIVISTIC derivation of the two-particle potential between two fermions, due to interaction with a boson field, was outlined by Tamm[1] and Dancoff[2] and generalized by Lévy,[3] including higher-order terms in the interaction potential. Following Lévy's treatment, the interaction terms of higher order can be represented by nonrelativistic graphs, distinguishing in addition the sequence of transitions and intermediate states. In this treatment, it does not seem to be understandable why the strongly diverging vacuum graphs are not considered, although according to the derivation they should appear, because the influence of them on the energy denominators is such that they are not separable. Further it is impossible to recognize the other divergent parts of the graphs as terms of mass and charge renormalization. This is impossible because the relativistic invariance has been destroyed in the very beginning of the derivation of the one-time formalism. In these cases Lévy goes back to the Bethe-Salpeter formalism,[4] showing an approximate correspondence of the two formalisms in two special examples and presuming it for the whole.

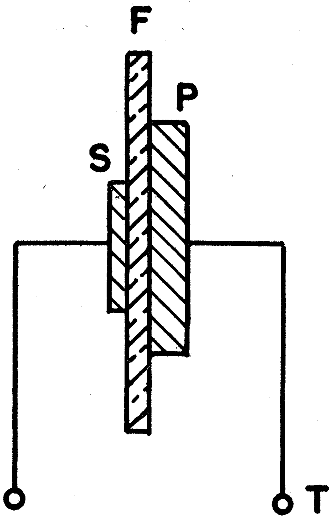

FIG. 1. Example of a Lévy graph with $n=2$ and $m=0$. The energy denominators are

$$\Delta E_1=E^{(1)}(\mathbf{p}+\mathbf{k}_1)+E^{(1)}(\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2)+\omega(\mathbf{k}_2),$$
$$\Delta E_2=E^{(1)}(\mathbf{p})+E^{(1)}(\mathbf{p}+\mathbf{k}_1)+E^{(1)}(\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2)+E^{(2)}(-\mathbf{p}-\mathbf{k}_2)-W,$$
$$\Delta E_3=E^{(1)}(\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2)+E^{(2)}(-\mathbf{p}-\mathbf{k}_2)+\omega(\mathbf{k}_2)-W.$$

To clear up this situation, a rigorous treatment of the two-particle potential was carried through, starting with the relativistic Bethe-Salpeter equation[4] and passing on from this to the one-time formalism. We give the results briefly here, whereas explicit calculations and results will be outlined in detail in ananother place.[5] The results correspond to those of Lévy in those special cases he took into consideration. For the difference coordinates in momentum space the wave equation

$$[W-(m_1^2+\mathbf{p}^2)^{\frac{1}{2}}-(m_2^2+\mathbf{p}^2)^{\frac{1}{2}}]a(\mathbf{p})=-Va(\mathbf{p}') \qquad (1)$$

holds, where W is the total energy of the system, $-V$ is the operator of the interaction potential, and $a(\mathbf{p})$ is a four-component wave function, containing the spins of both particles. V is represented by the totality of all Feynman graphs with two fermion lines running from the left to the right (see Fig. 1), but without vacuum graphs. The right ends should not contain self-energy parts but run into the wave function $a(\mathbf{p}')$ of (1). The different graphs should be distinguished by the number of meson lines (n), the number of closed loops (m), and the different topologies τ which are possible with given n and m. Further, we have to distinguish the order of the $2n$ points where the meson lines end. In this way, every Feynman graph ($nm\tau$) consists of $(2n)!$ ordered graphs, built up by all permutations (π) of the points without alteration of the topology. Our interaction operator consists of the sum over all these graphs,

$$V=\Sigma V_{nm\tau\pi}, \qquad (2)$$

where for every term $V_{nm\tau\pi}$ there is an analytic representation of the general form

$$V_{nm\tau\pi}=(-1)^l\lambda^n\int\frac{d\mathbf{k}_1\cdots d\mathbf{k}_{n-m}d\mathbf{q}_1\cdots d\mathbf{q}_m}{2\omega_1\cdot 2\omega_2\cdots 2\omega_n}\cdot\frac{\Gamma_1\cdots\Gamma_{2n}}{\Delta E_1\cdots\Delta E_{2n-1}}. \qquad (3)$$

The meaning of the terms of this formula is

$$\lambda=g^2(2\pi)^{-3},\quad \omega_\nu=(\mu^2+\mathbf{k}_\nu^2)^{\frac{1}{2}},\quad \Delta E=E_{\text{ex}}-W, \qquad (4)$$

where g is the coupling constant of the meson field, and μ is the mesonic mass. Every point of the graph means a transition to another virtual state; E_{ex} is the energy of the virtual state and depends on the lines lying between the two points. If two fermion lines of such a virtual state are both in the initial state or in the final state, the energy of both is W (see Fig. 1). All other fermion lines give $E(\mathbf{p})=(m_{1,2}^2+\mathbf{p}^2)^{\frac{1}{2}}$, and the meson lines $\omega(k)=(\mu^2+\mathbf{k}^2)^{\frac{1}{2}}$. Graphs that contain a virtual state with only two fermion lines are to be cut out. The connection between the variables $\mathbf{p}_\nu$ and $\mathbf{k}_\nu$ is given by the graph in the usual way. The Γ's have the same meaning as in Lévy's paper.[3] The lines running backwards are related to negative energy states, the lines with normal direction are related to positive energy states. l in (3) is the number of lines running backwards.

If a graph contains no closed loop ($m=0$), the integrations go over all meson variables $d\mathbf{k}_\nu$. If it contains one closed loop ($m=1$), one variable $\mathbf{k}_\nu$ of the adjacent meson lines can be eliminated, because $\Sigma\mathbf{k}_\nu=0$ holds for those lines. But here the integration goes over the momenta $d\mathbf{q}$ of the particle in the closed loop. All other prescriptions remain unchanged. If the interaction takes place between different particles, every closed loop appears twice, once for m_1 and once for m_2.

By these results an exact description of the interaction operator V is given, being valid for all graphs and orders in the coupling constant. It gives the Tamm-Dancoff equation with the following exceptions: first, it does not contain the vacuum graphs, and second, the excitation energies E_{ex} differ in the case in which the two fermion lines are in the initial or final state. Further, this derivation gives us a direct possibility to carry through the renormalization of the remaining divergent parts; a paper on this matter is in preparation.

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Hyperfine Structure of the Metastable Triplet State Helium-3†

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(Received May 18, 1953)

AN experimental determination has been made of the hyperfine structure of the $^3S_1$ metastable state of helium-3. The atomic beam magnetic resonance method was used in a manner similar to that used in experiments on the magnetic moment of the same state in helium-4.[1]

A total of 3 cc NTP of He^3 was available. This was circulated continuously through the system by a mercury diffusion pump. Air leakage into the system was removed by the clean-up action of the aluminum electrodes in the discharge tube. Hydrogen generated within the system in small amounts was oxidized by CuO at a temperature around 500°C. Liquid air traps removed all condensibles.

The rf transition field was supplied by a rectangular cavity with two slits through which the beam entered and left. The cavity was driven in the TE_{101} mode by a Sperry 2K-44 klystron on a conventional regulated power supply, and further stabilized against frequency changes by an Automatic frequency control circuit. The cavity had a Q of about 1000. Major tuning was accomplished by moving a strip of Teflon inside the cavity to regions of weaker or stronger rf electric field. Since the width of the cavity response was considerably more than the width of observed lines, tuning the cavity in the course of a run was not necessary.

A copper loop located near one of the cavity slits served the dual purpose of a cavity monitor and a source of low-frequency magnetic fields when low-frequency resonances were observed.

Transitions $(F, m_F)=(\frac{3}{2}, -\frac{1}{2})\rightarrow(\frac{1}{2}, \frac{1}{2})$ and $(\frac{3}{2}, -\frac{3}{2})\rightarrow(\frac{1}{2}, -\frac{1}{2})$ were observed at fields of about 2/3 gauss. The ultimate accuracy of the measurement was limited by the inhomogeneity of the magnetic field. Line widths due to this were about 0.3 to 1 Mc/sec, depending on the field-dependence of the particular transition.

Our final result for the hyperfine splitting of the state is 6739.71 ±0.05 Mc/sec. This is in agreement with a previous optical measurement[2] of 6630±150 Mc/sec.

The theoretical value of the hyperfine structure has not yet been computed to an accuracy comparable to the accuracy of the present experimental value. The principal contribution to the hfs due to a point dipole nucleus has been computed using a variationally determined electronic wave function which yields a value for the ionization potential of $^3S_1$ helium accurate to within 5 percent. The hfs splitting thus calculated is 6660 Mc/sec with an estimated accuracy of 5 percent.[2] There will be a relativistic correction of order α^2 which is similar in origin to the Breit correction to the hfs of one-electron atoms.[3] The principal radiative correction will be attributable to the anomalous magnetic moment of the electrons; in addition there will be radiative corrections of order α^2 due to polarization and fluctuation energy effects.[4] The effect of nuclear structure on the hfs is estimated in the accompanying letter to be ~1.3 parts in 10^4.[5]

A more complete report is in preparation.

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† Research supported in part by the U. S. Office of Naval Research.

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Effect of Nuclear Structure on the Hyperfine Structure of He^3

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(Received May 18, 1953)

THE hyperfine structure of the He^3 atom in its metastable $^3S_1$ state has been measured by the atomic beam magnetic resonance method with an accuracy which is adequate to reveal effects due to the internal structure of the He^3 nucleus.[1] An estimate of the influence of the nuclear structure will be given under the same assumptions as were made by Bohr for the hfs of deuterium.[2] When the electron is close to the nucleus it moves rapidly compared to the nucleon velocities and will therefore be bound to the two apparently stationary protons rather than to the center-of-mass of the He^3 nucleus. This approximation will be valid for distances of the electron from the nucleus which are less than

$$\rho \sim (\hbar/mc)mc^2/\bar{W}_N,$$

in which $\bar{W}_N$ is some mean nuclear excitation energy.[3] The quantity ρ will be larger than the nuclear radius and much smaller that the atomic radius. The contribution to the hfs from inside ρ must be calculated using a wave function with the electrons centered on the two protons; the contribution from outside ρ will be relatively insensitive to where the electrons are centered. When an electron is in the nucleus, the neutron magnetic moment will appear as a moment density distributed around the protons and hence its effect on the hfs will be reduced compared to that of a point magnetic dipole. This reduction is the principal effect of nuclear structure on the hfs. To our approximation the protons will have their spins antiparallel and hence will not contribute to the hfs.

The neutron-electron hfs interaction operator is given by

$$H_n = -e\boldsymbol{\alpha}^{(1)}\cdot\mathbf{A}^{(1)} - e\boldsymbol{\alpha}^{(2)}\cdot\mathbf{A}^{(2)},$$

in which $\boldsymbol{\alpha}^{(1)}$ and $\boldsymbol{\alpha}^{(2)}$ are Dirac matrices operating on electrons 1 and 2,

$$\mathbf{A}^{(1,2)} = -\boldsymbol{\mu}_n \times \nabla r_{1,2}\frac{1}{|\mathbf{r}_{1,2}-\mathbf{R}|}.$$

μ_n is the magnetic moment of the neutron, $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the electrons relative to the mid-point of the line joining the two protons, $\mathbf{R}$ is the position vector of the neutron relative to this same point. The neutron-electron hfs energy will be

$$\langle H_n\rangle = \int \phi^*(r_1, r_2)\chi^*(R)H_n\chi(R)\phi(r_1, r_2)d\tau_1 d\tau_2 d\tau_R,$$

in which $\phi(r_1, r_2)$ is the $^3S_1$ electronic wave function and χ is the nuclear wave function.

The nuclear wave function[4] used was $\chi = N\exp[-\frac{1}{2}\lambda(r_{np_1}+r_{np_2}+r_{p_1p_2})]$, in which λ is an adjustable parameter, r_{np_1} is the distance between the neutron and proton 1, and so on. The normalization factor N is given by $N=(4/7)^{\frac{1}{2}}\lambda^3$. The electronic wave function is actually a 16-component spinor, but the spinor parts of the wave function will play the same role as in the usual hfs calculation; and for the purpose of computing the fractional change in the hfs due to the nuclear structure, it is sufficient to take only a Schrödinger wave function. The spatial electronic wave function used was an antisymmetrized product of $Z=2$ hydrogenic wave functions for the configuration $1s2s$. The result of this calculation can be generalized easily to apply to an admixture of s configurations; however, in view of the other approximations being made and of the low accuracy to which the usual point dipole hfs has been calculated thus far for helium,[5] this extension was not made. A possibly useful wave function which involves the configurations $1sns$ ($n=2, 3, 4, 5, 6$) and which gives a calculated binding energy for $^3S_1$ helium which is within 0.2 percent of the experimental value is being reported.[6]

The integral to be evaluated for the $1s$ contribution to the hfs is of the form[3]

$$\int d\tau_1 u_1^*(1s)u_1(1s)\frac{1}{r_1^2}\left[\int_{R<r_1}\chi^*(R)\chi(R)d\tau_R\right].$$

The integral over the nuclear coordinates was done using Hylleraas variables, $s=r_{np_1}+r_{np_2}$, $t=r_{np_1}-r_{np_2}$, and $r_{p_1p_2}$, in terms of which $R=\frac{1}{2}(s^2+t^2-r_{p_1p_2}{}^2)^{\frac{1}{2}}$. This integration was only approximate, with an estimated error of ±10 percent. The unwieldy results of the nuclear integration could then be integrated over the electron coordinates and the result expressed as the sum of two terms. One term gives the usual hfs due to a point dipole neutron, and the other involves the nuclear structure parameter