suggested for different reasons by Peaslee.[27] In any case, the (γ,α) reaction to the ground state should be due almost entirely to $H_0(E2)$ and seems to exhaust about half of the sum rule (10).

In the region of the giant $E1$ absorption resonance, Goward and Wilkins[28] have observed several peaks in the cross section for $O^{16}(\gamma,\alpha)C^{12*}\rightarrow 3\alpha$, of heights roughly equal to that of the ground state peak of Millar and Cameron;[24] as in C^{12}, the absence of a spectacular rise in the (γ,α) cross section is in agreement with our theory.

The threshold for the allowed $E1$-induced (γ,α)-process is $7+15+3=25$ Mev. Peaks in the cross section observed by Goward and Wilkins[28] above this energy (up to 30 Mev) are of roughly the same magnitude as in region "B," though the absorption cross section has presumably fallen considerably; again the theory appears to be borne out. It is, of course, extremely important to check whether an overwhelming proportion of the events in region "C" really proceeds via $T=1$ levels in C^{12}. It may be significant that Livesey and Smith[29] have reported a change in mechanism around 25 Mev.

At sufficiently high energies (>35 Mev), the most allowed transition should lead through $T=1$ states in O^{16}, C^{12}, and Be^8; the violation of isotopic spin conservation would be postponed as long as possible. It would be interesting to know whether such cascades do occur.

Experimental effects of the isotopic spin selection rules for γ-ray absorption should, on the whole, be less striking in odd-odd nuclei with $T_z=0$ than in the even-even ones. Perhaps the most noteworthy prediction that can be made is one concerning the competition between (γ,d)- and (γ,np)-reactions in the region of $E1$ absorption. Below the energy at which the residual even-even nucleus can be left in a $T=1$ state, the (γ,d) process is forbidden, while a corresponding reaction in which a neutron and a proton are emitted *together* in a $T=1$ state (particularly in the virtual $^1S_0$ state near zero energy) is fully allowed. Goward and Wilkins[21] appear to have observed an excess of (γ,np) over (γ,d) reactions. It would be useful to search for the allowed (γ,d) process as well; for example, above 25 Mev the reaction $B^{10}(\gamma,d)Be^{8*}$ should lead predominantly to the state at 17 Mev or higher $T=1$ states.‡

For a more detailed discussion of (γ,α)- and (γ,d)-reactions in light nuclei, the reader is referred to a forthcoming review article by one of us (V.L.T.) to appear in the *Reviews of Modern Physics.*

[27] D. C. Peaslee, Phys. Rev. **88**, 812 (1952).
[28] F. K. Goward and J. J. Wilkins, Proc. Phys. Soc. (London) **A65**, 671 (1952).
[29] D. L. Livesey and C. L. Smith, Proc. Phys. Soc. (London) **A65**, 758 (1952).

‡ *Note added in proof.*—Hsiao and Telegdi [Phys. Rev. **90**, 494 (1953)] have recently presented strong evidence that the reaction $O^{16}(\gamma,\alpha)C^{12*}\rightarrow 3\alpha$ does lead to $T=1$ states of C^{12} in energy region C. They have also reported cascades of the type predicted in the text. Our discussion of these cascades ought to be completed by the remark that their probability is greatly reduced through the fact that the emission of the second α suffers competition from nucleon emission.

The same authors, in unpublished work, have found that the reaction $N^{14}(\gamma,d)C^{12}\rightarrow 3\alpha$ proceeds predominantly through a 16-Mev level in C^{12}, preumably with $T=1$, as predicted.

PHYSICAL REVIEW VOLUME 91, NUMBER 1 JULY 1, 1953

A WKB-Type Approximation to the Schrödinger Equation

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(Received February 9, 1953)

A method of approximating solutions of the one-dimensional Schrödinger equation is presented in this paper. The method closely resembles the usual WKB approximation. Whereas in the ordinary WKB method the exponential function is used as the basis of the approximation, in this paper the solutions of an arbitrary Schrödinger equation are used. The general advantage is that by proper choice of the arbitrary equation an improved approximation can be obtained. The method is illustrated by treating the potential well and potential barrier problems when there are two turning points. The approximations to the wave functions are continuous even across the turning points. The barrier transmission problem is treated uniformly for energies above and below the peak of the barrier.

I. INTRODUCTION

THE WKB method, as well as showing the correspondence between classical and quantum mechanics, provides useful approximations to the solutions of the one-dimensional Schrödinger equation. A limitation on its usefulness as an approximation is that it becomes infinite at the classical turning points of the motion. Langer[1] introduced an approximation based on Bessel functions which remains finite at any one turning point and, far from the turning point, becomes identical with the WKB approximation. However, at a second turning point his result is infinite and, to obtain approximate solutions which are everywhere finite, one must join it to a similar approximation finite at the second turning point.

The ordinary WKB method is based on the exponential function and Langer's approximation on Bessel

[1] R. E. Langer, Phys. Rev. **51**, 669 (1937).

functions. In this paper it is shown that a general WKB-type approximation can be constructed based on the solutions of an arbitrary Schrödinger equation. A good approximation is obtained if the arbitrary equation is chosen to resemble the Schrödinger equation to be solved. In this respect the approximation is similar to perturbation theory. As a particular example of the general approximation, if the arbitrary equation is that of a free particle, one obtains the ordinary WKB approximation. As another special case, Langer's approximation can be interpreted as arising from the Schrödinger equation in which the kinetic energy is a power of the coordinate. Also, by properly choosing the equation, one may find approximate solutions which are finite at several turning points. The only example considered below is that of two turning points; many problems fall into this class.

An advantage of the method used here is that a problem does not need to be broken up into regions with connection formulas between them but instead a single approximate solution can be obtained which is continuous over the whole region. Because of this continuity one can use the approximate wave functions to discuss matrix elements. Ordinarily this is not done because of the infinities at the turning points. In general one must use numerical integration to calculate the matrix elements from the approximate wave functions. However, the numerical integration can be avoided in some special cases discussed in Sec. III. Another advantage of the method is that the problem of transmission through a potential barrier can be discussed uniformly for particle energies above and below the peak of the barrier. The usual WKB approximation gives complete transmission when the energy is above the peak.

The general treatment, for an unspecified basic function, is given in Sec. II. The application to a potential well problem with two turning points is given in Sec. III, basing the approximation on the solutions of the Schrödinger equation for a harmonic oscillator. In Sec. IV the complementary problem of transmission through a potential barrier is discussed.

II. GENERAL TREATMENT

The general problem is to find approximate solutions of the one-dimensional Schrödinger equation,

$$d^2\psi(x)/dx^2+[p^2(x)/\hbar^2]\psi(x)=0, \tag{1}$$

where

$$p^2(x)=W-V(x). \tag{2}$$

Here W is $2m$ times the total energy of a particle of mass m moving in a potential $V(x)/2m$. The approximate solutions are to be based on preselected functions $\phi(S)$ which satisfy

$$d^2\phi(S)/dS^2+[P^2(S)/\hbar^2]\phi(S)=0 \tag{3}$$

for some function $P(S)$. Ordinarily one will be guided to an appropriate choice of $\phi(S)$ by choosing $P(S)$ qualitatively similar to $p(x)$. In the ordinary WKB method[2] one finds approximations to the solutions of Eq. (1) by first changing the dependent variable from $\psi(x)$ to $S(x)$ through the substitution

$$\psi(x)=\exp[i\hbar^{-1}S(x)]. \tag{4}$$

The problem then reduces to finding a solution of

$$-S'^2+i\hbar S''+p^2=0, \tag{5}$$

and one thinks of solving this by a series expansion on $\hbar$, which is assumed to be small:

$$S(x)=S_0(x)+S_1(x)\hbar+S_2(x)\hbar^2+\cdots. \tag{6}$$

Ordinarily only the first two terms are calculated, with the result that

$$\psi\cong p^{-\frac{1}{2}}\exp\left[\pm i\hbar^{-1}\int p(x)dx\right], \tag{7}$$

where the usual connection formulas between regions on opposite sides of the turning points (points at which $p=0$) are to be used.

In parallel with the ordinary WKB method, the first step in attacking the general problem is to change the dependent variable to $S(x)$ by the substitution

$$\psi(x)=T(x)\phi[S(x)], \tag{8}$$

where the function $T(x)$ will be specified below. It is needed in order to obtain an equation in S, independent of ϕ. On substituting Eq. (8) into Eq. (1), one finds immediately that

$$\psi''+(p^2/\hbar^2)\psi=(\hbar^2T''/T-S'^2P^2+p^2)\hbar^{-2}T\phi +(2T'/T+S''/S')TS'd\phi/dS, \tag{9}$$

where Eq. (3) has been used to eliminate ϕ''. Here one sees that if

$$T=S'^{-\frac{1}{2}}, \tag{10}$$

then the problem reduces to finding a solution of

$$\hbar^2T''/T-S'^2P^2+p^2=0. \tag{11}$$

The value of T from Eq. (10) makes this a differential equation in $S(x)$ alone. The next step is to find a series solution for small $\hbar$. Since only $\hbar^2$ appears in Eq. (11) one may write

$$S(x)=S_0(x)+S_2(x)\hbar^2+S_4(x)\hbar^4+\cdots. \tag{12}$$

If, as usual, the approximation is carried to two orders in $\hbar$, only S_0 will be retained in this series, and evidently from Eq. (11)

$$S_0'^2P^2=p^2. \tag{13}$$

If the positive square root is used, the approximate solutions are

$$\psi_{\text{app}}=S'^{-\frac{1}{2}}\phi(S), \tag{14}$$

[2] See, for example, Leonard I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), first edition, p. 178; Edwin C. Kemble, *The Fundamental Principles of Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1937), first edition, p. 90.

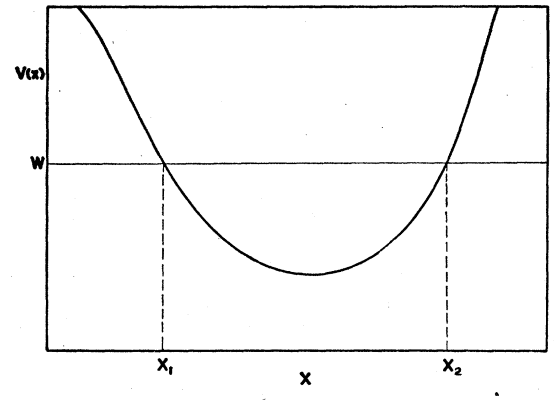

FIG. 1. Type of potential well considered.

where

$$\int_{s_0}^{S} P(\sigma)d\sigma=\int_{x_0}^{x} p(\xi)d\xi. \tag{15}$$

The two independent solutions $\phi(S)$ of Eq. (3) give the approximations to two independent solutions of Eq. (1). In many problems the integration constants s_0, x_0 may be chosen to make the approximate wave function continuous across the turning points, as will be illustrated in Secs. III and IV. With the choice $P^2(S)=1$, the approximation reduces immediately to the ordinary WKB approximation. By taking x_0 to be a turning point of $p^2(x)$, $P^2(S)=S^\nu$, and $s_0=0$, one obtains Langer's approximation for the case when $p^2(x)$ has a zero of the νth order at the turning point.

Whenever the function $S(x)$ is real, as in Secs. III and IV, the approximate wave function will exactly satisfy the conservation of probability equation,

$$\frac{\hbar}{2im}\frac{d}{dx}\left(\psi_{\rm app}^*\frac{d}{dx}\psi_{\rm app}-\psi_{\rm app}\frac{d}{dx}\psi_{\rm app}^*\right)$$
$$=\frac{\hbar}{2im}S'\frac{d}{dS}\left(\phi^*\frac{d}{dS}\phi-\phi\frac{d}{dS}\phi^*\right)=0. \tag{16}$$

This is a consequence of the choice of T in Eq. (10) and the fact that ϕ itself satisfies a Schrödinger equation. One may verify that

$$\psi_{\rm app}''+(p^2/\hbar^2)\psi_{\rm app}=\left[\tfrac{3}{4}(S''/S')^2-\tfrac{1}{2}S'''/S'\right]\psi_{\rm app}. \tag{17}$$

The accuracy of the approximation depends on the size of the term on the right.

III. APPLICATION TO POTENTIAL WELL PROBLEMS

In this section the approximation will be applied to the problem of a particle in a potential well with just two turning points x_1 and x_2. Only the simple case when the zeros of p^2 at x_1 and x_2 are of first order will be considered. The potential will then be qualitatively as shown in Fig. 1. The solutions of the harmonic oscillator problem,[3]

$$d^2\phi/dS^2+(E-S^2)\phi=0 \tag{18}$$

(here E is a parameter which will be chosen later), are convenient to use as the basis of the approximation in this problem because their properties are very well known and because $P^2(S)$ has the same type and number of zeros as $p^2(x)$.

Using the notation of Whittaker and Watson to describe the solutions of Eq. (18),[4] one finds the following approximate solutions of Eq. (1):

$$\psi\cong S'^{-\frac{1}{2}}D_{\frac{1}{2}(E-1)}(\sqrt{2}S), \tag{19}$$

$$\psi\cong S'^{-\frac{1}{2}}D_{\frac{1}{2}(E-1)}(-\sqrt{2}S), \tag{20}$$

where

$$\int_{-\sqrt{E}}^{S}(E-\sigma^2)^{\frac{1}{2}}d\sigma=\int_{x_1}^{x}p(\xi)d\xi. \tag{21}$$

These are independent if $(E-1)/2$ is not an integer. The integration constants in Eq. (21) have been chosen for the following reason. Since $S'=p(x)/P(S)$, if the function $S(x)$ is chosen so that the zero of $P(S)$ at $-\sqrt{E}$ corresponds to the zero of $p(x)$ at x_1, then $S(x)$ will be continuous at this point. This means that the approximate wave function itself will be continuous across the turning point x_1. Continuity across the second turning point x_2 is obtained similarly if one chooses the parameter E so that

$$\int_{x_1}^{x_2}p(\xi)d\xi=\int_{-\sqrt{E}}^{\sqrt{E}}(E-\sigma^2)^{\frac{1}{2}}d\sigma=\tfrac{1}{2}E\pi. \tag{22}$$

This makes S real when x is real and completely specifies two independent continuous approximate solutions of Eq. (1). In some problems it is of interest to have asymptotic expansions of the approximate wave functions for large x. From Eq. (21) it is seen that

$$\lim_{x\to\pm\infty}S(x)=\pm\infty$$

so that the asymptotic behaviors of the approximate wave functions depend on the asymptotic forms of the function $D_{(E-1)/2}$. Using these asymptotic forms as given in Eqs. (39) and (45) in the Appendix and expanding Eq. (21) for $S\to\pm\infty$, one finds that

$$\left[(2\pi)^{-\frac{1}{2}}(2e/E)^{\frac{1}{2}E}\Gamma(\tfrac{1}{2}E+\tfrac{1}{2})\right]$$
$$\times 2\cos(\tfrac{1}{2}E\pi)\,|p|^{-\frac{1}{2}}\exp\left(\int_x^{x_1}|p(\xi)|d\xi\right)$$
$$+\sin(\tfrac{1}{2}E\pi)\,|p|^{-\frac{1}{2}}\exp\left(-\int_x^{x_1}|p(\xi)|d\xi\right)$$
$$\underset{x\to-\infty}{\longleftarrow}2^{\frac{1}{2}}(2e/E)^{\frac{1}{2}E}(S')^{-\frac{1}{2}}D_{\frac{1}{2}(E-1)}(\sqrt{2}S)$$
$$\underset{x\to+\infty}{\longrightarrow}|p|^{-\frac{1}{2}}\exp\left(-\int_{x_2}^{x}|p(\xi)|d\xi\right), \tag{23}$$

[3] In this and subsequent equations units have been chosen so that $\hbar=1$.

[4] E. T. Whittaker and G. N. Watson, *Modern Analysis* (The Macmillan Company, New York, 1947), p. 347.

$$|p|^{-\frac{1}{2}}\exp\left(-\int_x^{x_1}|p(\xi)|d\xi\right)$$

$$\underset{x\to-\infty}{\longleftarrow}2^{\frac{1}{4}}(2e/E)^{\frac{1}{4}E}(S')^{-\frac{1}{2}}D_{\frac{1}{2}(E-1)}(-\sqrt{2}S)$$

$$\underset{x\to+\infty}{\longrightarrow}[(2\pi)^{-\frac{1}{2}}(2e/E)^{\frac{1}{4}E}\Gamma(\tfrac{1}{2}E+\tfrac{1}{2})]$$

$$\times 2\cos(\tfrac{1}{2}E\pi)|p|^{-\frac{1}{2}}\exp\left(\int_{x_2}^{x}|p(\xi)|d\xi\right)$$

$$+\sin(\tfrac{1}{2}E\pi)|p|^{-\frac{1}{2}}\exp\left(-\int_{x_2}^{x}|p(\xi)|d\xi\right). \quad (24)$$

When W is large, it is clear from Eq. (22) that the parameter E is large. In that case the factor in the square brackets in Eqs. (23) and (24) approaches one, as is seen by using the Stirling approximation for the Γ-function. The connection between the two exponential regions is then identical with the one obtained by the ordinary WKB method using the usual connection formulas across the turning points without regard to direction.

For the bound states one must use the solutions of Eq. (18) which are finite at infinite S. These solutions are

$$D_n(\sqrt{2}S)=2^{-\frac{1}{2}n}H_n(S)\exp(-\tfrac{1}{2}S^2), \quad (25)$$

where $n=(E-1)/2$ is a positive integer or zero and the H_n are the Hermite polynomials. The approximations to the bound state wave functions are then

$$\psi_n\cong A_nS'^{-\frac{1}{2}}\exp(-\tfrac{1}{2}S^2)H_n(S), \quad (26)$$

where S is still given by Eq. (21) and the A_n are normalizing constants. The eigenvalue condition, from Eq. (22), is

$$\int_{x_1}^{x_2}p(\xi)d\xi=(n+\tfrac{1}{2})\pi, \quad n=0, 1, 2, \cdots, \quad (27)$$

which is identical with the usual WKB eigencondition. If a different $P(S)$ had been used, a different condition might have been obtained. The potential[5]

$$V(x)=-1.922e^x(1+e^x)^{-1}-11.20e^x(1+e^x)^{-2} \quad (28)$$

has been chosen for an example. The reason for choosing this type of potential is that the exact solutions for the bound states can be expressed in terms of elementary functions. The constants have been chosen to allow only two energy levels and to make the wave functions unsymmetrical. The exact and approximate wave functions have been plotted in Fig. 2.

Since the approximate wave functions are continuous, they can be used to obtain approximations to matrix elements. In many problems diagonal matrix elements can be found by making a further approximation which avoids calculating the wave functions and making a numerical integration. Consider, for example,

$$\int_{-\infty}^{\infty}|\psi_n|^2x^m dx$$

$$\cong A_n^2\int_{-\infty}^{\infty}(S')^{-1}\exp(-S^2)H_n^2(S)x^m dx$$

$$=A_n^2\int_{-\infty}^{\infty}(S')^{-2}\exp(-S^2)H_n^2x^m dS, \quad (29)$$

where m is some integer. This quantity might be evaluated by expressing x and S'^{-1} as power series in S and performing the integration. As an approximation one may use, instead of power series, $(2K+1)$-degree poly-

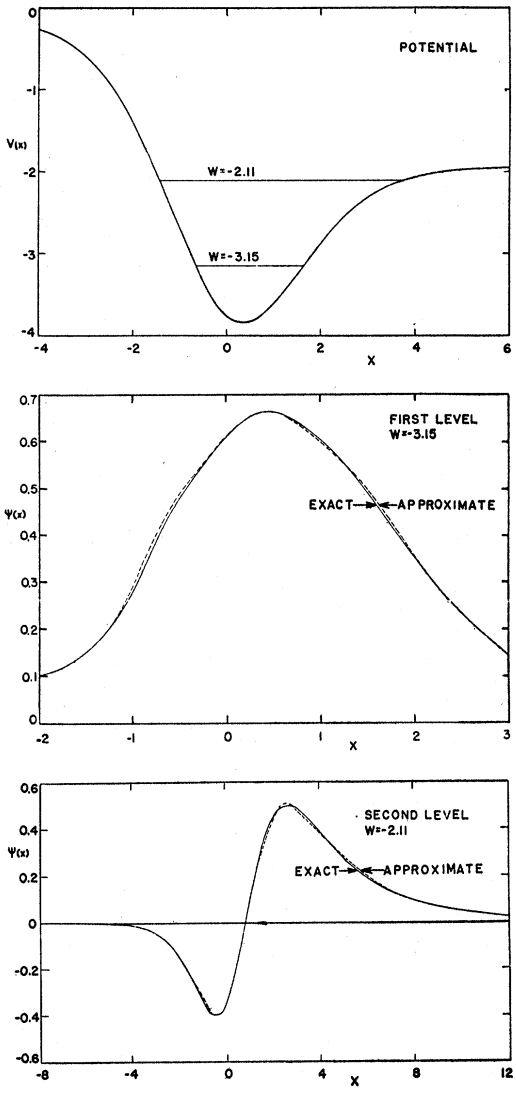

FIG. 2. The potential, $V(x)=-1.922e^x(1+e^x)^{-1}-11.20e^x\times(1+e^x)^{-2}$, and the wave functions of the two bound states. Each wave function separately has been normalized so that the integral-square is 1.

[5] This potential is of the type which was introduced by Carl Eckart, Phys. Rev. **35**, 1303 (1930).

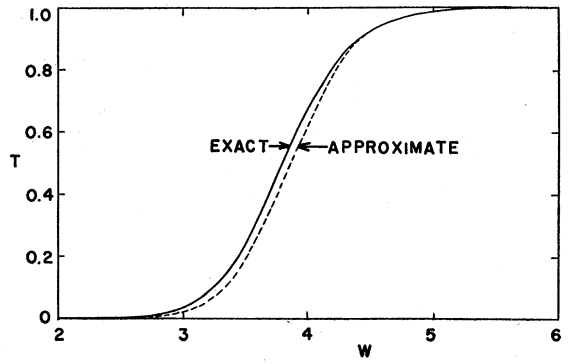

FIG. 3. Transmission coefficient for the potential, $V(x)=1.922e^x(1+e^x)^{-1}+11.20e^x(1+e^x)^{-2}$. (This is the negative of the potential shown in Fig. 2.)

nomials chosen to give the correct values of $x(S)$ and its derivatives up to the Kth order at both turning points. The values at other points could be used but the turning points are especially suitable since their locations are known immediately once the energy has been found and also since ordinarily the wave function will be large only in the region between these points. The needed derivatives at a first-order turning point can be found by using Eq. (15) to make a series expansion of the type

$$x(S)=x_0+[(dp^2/dx)_{x0}/(dP^2/dS)_{s0}]^{-\frac{1}{3}}(S-s_0)+\cdots. \quad (30)$$

One can easily express $(S')^{-1}$ in terms of S by differentiating the polynomial $x(S)$ with respect to S.

As a special case of Eq. (29) this procedure may be used to normalize the approximate wave functions of bound states so that their integral squares are one. For example, for the two levels of the Eckart potential used above, one finds for the normalizing constants A_0 and A_1:

	A_0 (first level)	A_1 (second level)
Numerical integration	1.627	1.686
First degree polynomial approximation ($K=0$)	1.600	1.483
Third degree polynomial approximation ($K=1$)	1.629	1.700

As a further example $\langle x^2\rangle_{Av}$, the expected value of x^2, has been calculated for the two bound states of the above Eckart potential. Third-degree polynomials were used, both for normalizing and for evaluating the integral. For comparison the same quantity has been calculated using the exact wave functions and also using the approximate wave functions with numerical integration throughout. The results are as follows:

	First level	Second level
$\langle x^2\rangle_{Av}$ (approximate wave function, polynomials)	1.237	9.52
$\langle x^2\rangle_{Av}$ (exact wave function, numerical integration)	1.223	10.19
$\langle x^2\rangle_{Av}$ (approximate wave function, numerical integration)	1.224	10.51

The polynomial method in general may be criticized because there is no way of estimating the error in the approximation of a matrix element and because it cannot be applied to off-diagonal matrix elements, since different functions $S(x)$ arise for different energies.

IV. APPLICATION TO POTENTIAL BARRIER PROBLEMS

The penetration of a particle through a potential barrier will be discussed in this section. The discussion will be restricted to those potentials for which, when the energy is below the peak of the barrier, $p^2(x)$ has two first order zeros x_1 and x_2, where $x_1<x_2$. As a further restriction on the type of potential discussed, it will be assumed that the two real turning points for energies below the peak of the barrier go unambiguously into two complex turning points for energies above the peak of the barrier. Since $p^2(x)$ is a real function, these two turning points will be complex conjugate; the one with the positive imaginary part will be called x_1 and the other x_2. As basic functions for the approximation the solutions of the equation

$$d^2\phi/dS^2+(E+S^2)\phi=0 \quad (31)$$

will be used. The zeros of $(E+S^2)$ will be labeled in parallel with those of $p(x)$ so that when E is negative $s_1=-\sqrt{(-E)}$, $s_2=+\sqrt{(-E)}$ and when E is positive $s_1=+i\sqrt{E}$, $s_2=-i\sqrt{E}$.

Following the general treatment, and in parallel with the potential well problem, one finds the following approximate solutions:

$$\psi\cong S'^{-\frac{1}{2}}D_{\frac{1}{2}(+iE-1)}(\sqrt{2}Se^{-i\pi/4}), \quad (32)$$

$$\psi\cong S'^{-\frac{1}{2}}D_{\frac{1}{2}(-iE-1)}(\sqrt{2}Se^{+i\pi/4}), \quad (33)$$

where

$$\int_{s1}^{S}(E+\sigma^2)^{\frac{1}{2}}d\sigma=\int_{x1}^{x}p(\xi)d\xi, \quad (34)$$

and the parameter E is chosen to satisfy

$$\int_{x1}^{x2}p(\xi)d\xi=\int_{s1}^{s2}(E+\sigma^2)^{\frac{1}{2}}d\sigma=-\tfrac{1}{2}iE\pi. \quad (35)$$

In performing the σ-integration here and in finding the results below, the branch of the square root used is the one for which, when $E>0$ and S is real, $\arg(E+S^2)^{\frac{1}{2}}$ is 0, and for which, when $E\leqq 0$ and S is real, $\arg(E+S^2)^{\frac{1}{2}}$ is 0 if $S<s_1$, $\pi/2$ if $s_1<S<s_2$, and 0 if $s_2<S$. The similar branch of $p(x)$ must be used in the ξ-integration. From Eq. (35) one sees that E is real and positive when W is above the peak of the barrier $V(x)$ and that E is real and negative when W is below the peak. Using Eqs. (34) and (35), one may verify that S is real when x is real. In order to find the transmission coefficient the asymptotic forms of the approximate solutions for

large x are needed. From Eq. (34) it is seen that

$$\lim_{x\to\pm\infty} S(x)=\pm\infty.$$

Performing the σ-integration in Eq. (34), expanding for $S\to\pm\infty$, and using the asymptotic forms of $D_n(z)$ as given in Eq. (40) and Eq. (41) in the Appendix, one finds that

$$\begin{aligned}&[(2/\pi)^{\frac{1}{2}}(|E|/2e)^{-\frac{1}{2}iE}\Gamma(\tfrac{1}{2}iE+\tfrac{1}{2})\\&\times\cosh(\tfrac{1}{2}E\pi)]e^{\frac{1}{4}E\pi}p^{-\frac{1}{2}}\exp\left(-i\,\mathrm{Re}\int_x^{x_1}p(\xi)d\xi\right)\\&+e^{-\frac{1}{4}i\pi}p^{-\frac{1}{2}}\exp\left(i\,\mathrm{Re}\int_x^{x_1}p(\xi)d\xi\right)\\&\underset{x\to-\infty}{\longleftarrow}(2e^{-\frac{1}{4}i\pi})^{\frac{1}{4}}(|E|e^{\frac{3}{4}i\pi}/2e)^{-\frac{1}{4}iE}S'^{-\frac{1}{2}}D_{\frac{1}{2}(iE-1)}(\sqrt{2}e^{-\frac{1}{4}i\pi}S)\\&\underset{x\to+\infty}{\longrightarrow}e^{\frac{1}{4}E\pi}p^{-\frac{1}{2}}\exp\left(i\,\mathrm{Re}\int_{x_2}^{x}p(\xi)d\xi\right),\end{aligned}\quad(36)$$

where Re indicates that the real part of the integral is to be used. The corresponding relation for

$$D_{\frac{1}{2}(-iE-1)}(\sqrt{2}e^{\frac{1}{4}i\pi}S)$$

is the complex conjugate of this one. The transmission coefficient T, defined as the ratio of the transmitted to the incident current, is then

$$T\cong(1+e^{-E\pi})^{-1}.\quad(37)$$

An example of the application of this result is given in Fig. 3. A connection similar to the one given in Eq. (36) was found by Guth and Mullin[6] using a method indicated by Kramers and Ittman.[7] Using the relation between the gamma and the trigonometric functions one finds that the absolute value of the quantity in the square brackets in Eq. (36) is $[2\cosh(\frac{1}{2}E\pi)]^{\frac{1}{2}}$. Also, using the Stirling approximation, one finds that for large E the quantity in the square brackets is $e^{\frac{1}{4}|E|\pi}$. For energies below the peak of the barrier one may discuss the same transmission problem with the ordinary WKB method, using the connection formulas without regard to direction. When the energy is so far below the peak of the barrier that $e^{E\pi}$ can be neglected compared to one, the resulting connection across the entire barrier is identical with Eq. (36) when the square bracket can be replaced by $e^{\frac{1}{4}|E|\pi}$.

APPENDIX

The asymptotic representations of $D_n(z)$ are needed in Sec. III when n is real and $\arg z=0$ or π and in Sec. IV when the real part of n is $-\frac{1}{2}$ and $\arg z=-\pi/4$ or $3\pi/4$. In general the asymptotic representation is of the form

$$\begin{aligned}D_n(z)\sim&\alpha\exp(-\tfrac{1}{4}z^2)z^n[1+O(z^{-2})]\\&+\beta\exp(\tfrac{1}{4}z^2)z^{-n-1}[1+O(z^{-2})],\end{aligned}\quad(38)$$

where α and β are independent of $|z|$ but vary with $\arg z$. The predominant terms in this representation are conveniently given by Whittaker and Watson.[8] From their results one may infer immediately that the complete representation is

$$D_n(z)\sim\exp(-\tfrac{1}{4}z^2)z^n,\quad\text{when}\quad\arg z=0,\quad(39)$$

because if β were not zero the term which β multiplies, containing $\exp(\frac{1}{4}z^2)$, would be predominant. Here and below the terms of order z^{-2} compared to one have been omitted. Similarly, when the real part of n is $-\frac{1}{2}$ and $\arg z=-\pi/4$ or $3\pi/4$, the terms multiplying α and β are each of order $|z|^{-\frac{1}{2}}$, so again the predominant terms give the complete representations:

$$D_n(z)\sim\exp(-\tfrac{1}{4}z^2)z^n,\quad\text{when}\quad\arg z=-\pi/4,\quad(40)$$

$$\begin{aligned}D_n(z)\sim&\exp(-\tfrac{1}{4}z^2)z^n-(2\pi)^{\frac{1}{2}}[\Gamma(-n)]^{-1}\\&\times\exp(n\pi i)\exp(\tfrac{1}{4}z^2)z^{-n-1},\quad\text{when}\quad\arg z=3\pi/4.\end{aligned}\quad(41)$$

These are the representations used in Sec. IV.

When $\arg z=\pi$, the β term is the predominant one and therefore $\beta=-(2\pi)^{\frac{1}{2}}e^{n\pi i}[\Gamma(-n)]^{-1}$. One may infer the value of α when n is real and $\arg z=\pi$ by an indirect argument using the fact that

$$\begin{aligned}D_n(z)=&(2\pi)^{-\frac{1}{2}}\Gamma(n+1)[e^{\frac{1}{2}n\pi i}D_{-n-1}(iz)\\&+e^{-\frac{1}{2}n\pi i}D_{-n-1}(-iz)],\end{aligned}\quad(42)$$

as is shown by Whittaker and Watson for all n and z. One may assume that, for z real and positive,

$$\begin{aligned}D_{-n-1}(e^{\frac{1}{2}i\pi}z)\sim&\exp(\tfrac{1}{4}z^2)(e^{\frac{1}{2}i\pi}z)^{-n-1}\\&+b\exp(-\tfrac{1}{4}z^2)(e^{im\pi}e^{\frac{1}{2}i\pi}z)^n,\end{aligned}\quad(43)$$

where b and m are real numbers to be determined. The first term on the right is simply Whittaker and Watson's predominant term and in the second term β has been written as $be^{imn\pi}$. It is seen that m must be an integer or zero since $D_{-n-1}(z)$ is a real function of z. Substituting Eq. (39), Eq. (43), and the complex conjugate of Eq. (43), into Eq. (42), one obtains

$$b=(2\pi)^{\frac{1}{2}}/2\Gamma(n+1)\cos((m+1)n\pi).\quad(44)$$

Here it is seen that m must be -1 if b is to remain finite for all n. Using these values for b and m and replacing z by $-z$ in Eq. (42), one may conclude that

$$\begin{aligned}D_n(z)\sim&\cos(n\pi)e^{-n\pi i}\exp(-\tfrac{1}{4}z^2)z^n\\&-(2\pi)^{\frac{1}{2}}[\Gamma(-n)]^{-1}e^{n\pi i}\exp(\tfrac{1}{4}z^2)z^{-n-1},\\&\text{when}\quad\arg z=\pi.\end{aligned}\quad(45)$$

This representation and the one given in Eq. (39) are used in Sec. III.

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[7] H. A. Kramers and G. P. Ittman, Z. Physik **58**, 225 (1929).

[8] Reference 4, pp. 347–348.