The Wave Function of a Relativistic System

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Three methods of handling problems in relativistic quantum mechanics are discussed. One is the Tamm-Dancoff method, one is the Bethe-Salpeter method, and the third is a new method intermediate between the other two. The connections between the three methods are investigated. It is shown that the new method provides a kind of bridge between the 3- and 4-dimensional points of view. The new method is illustrated with applications to simple examples.

I. INTRODUCTION

THE purpose of this paper is to clarify the physical meaning of a wave function in relativistic quantum mechanics. The subject is full of obscurities and unsolved problems, most of which will not be discussed here. Only some facts will be pointed out, which it is hoped will stimulate further thought on these questions.

A relativistic system can be described from two points of view, 3 dimensional and 4 dimensional. In the 3-dimensional point of view, the wave function describes completely the configuration of the system at a particular time. The typical example of this point of view is the formalism of Tamm[1] and Dancoff.[2] In the 4-dimensional point of view, the wave function describes the probability amplitude for finding particle 1 at position 1 and time 1, particle 2 at position 2 and time 2, and so on. This is the point of view of Bethe and Salpeter.[3] Each point of view fails to include important aspects of the situation. The 3-dimensional picture conceals the relativistic invariance, and thereby makes practical calculations much more complicated. The 4-dimensional picture is manifestly relativistic, but the wave function does not give a complete description of the system in the sense of elementary quantum mechanics. The physical meaning of the 4-dimensional wave function is quite unclear.[4] Worst of all, there still exists no exact method[5] of relating the two wave functions to one another so as to make use of the advantages of both in the same problem.

This paper is concerned mainly with establishing connections between the 3- and 4-dimensional wave functions. The connection is made via a new formulation of the Tamm-Dancoff method which was briefly explained in a recent letter.[6] Section V deals with the important unsolved problem of the correct normalization conditions to be imposed on the 4-dimensional wave function. The later sections discuss the wave equations of relativistic systems and the boundary conditions which should go with them.

II. DEFINITIONS

We consider for definiteness a system of two quantized spinor fields $\psi^P(x)$ and $\psi^N(x)$ representing protons and neutrons, interacting with a neutral pseudoscalar meson field $\phi(x)$. The operators $\psi(x)$, $\phi(x)$ are Heisenberg operators defined at a space-time point x and are relativistically covariant. The actual state of the system is denoted by Ψ, a constant state vector in the Heisenberg representation. The vacuum state of the interacting fields is Ψ_0. We suppose that Ψ is a state consisting of one proton and one neutron in interaction, but not necessarily bound together.

The 4-dimensional wave function of the 2-particle state Ψ is, according to Gell-Mann and Low,[7] defined as the matrix element

$$\psi_{\alpha\beta}(x, y) = (\Psi_0{}^* T(\psi^P{}_\alpha(x),\ \psi^N{}_\beta(y))\Psi), \qquad (1)$$

where T represents the chronological product as defined by Wick.[8] This wave function is a 16-component spinor and is a function of the two independent space-time points x and y. In particular, x and y may be taken to be two space points r and r' at the same time $t=0$. Then $\psi_{\alpha\beta}(x, y)$ specializes to

$$\psi_{\alpha\beta}(r, r') = (\Psi_0{}^* \psi^P{}_\alpha(r)\psi^N{}_\beta(r')\Psi). \qquad (2)$$

We may identify the Heisenberg operator $\psi^P{}_\alpha(r)$ at $t=0$ with the time-independent operator which describes the proton field in the Schrödinger representation. Likewise $\psi^N{}_\beta(r')$ is a neutron field operator in the Schrödinger representation. With this identification the state vector of the actual state in the Schrödinger representation is

$$\Psi \exp(-iEt), \qquad (3)$$

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[1] I. Tamm, J. Phys. (U.S.S.R.) **9**, 449 (1945).
[2] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
[3] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[4] J. Goldstein (to be published).
[5] A big step in the understanding of the relation between 3- and 4-dimensional wave functions was taken by E. E. Salpeter, Phys. Rev. **87**, 328 (1952) and M. M. Lévy, Phys. Rev. **88**, 72, 725 (1952). But these authors deal only with systems in which the interactions are almost instantaneous and the particles can be considered to be at rest as a first approximation.
[6] F. J. Dyson, Phys. Rev. **90**, 994 (1953).
[7] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).
[8] G. C. Wick, Phys. Rev. **80**, 268 (1950).

supposing the actual state to have a given total energy E. The Schrödinger state vector of the vacuum state is

$$\Psi_0 \exp(-iE_0t), \tag{4}$$

where E_0 is the vacuum energy. We are using natural units so that $\hbar=c=1$.

The Schrödinger operator $\psi^P{}_\alpha(r)$ may be represented as a superposition of plane waves obeying periodic boundary conditions in a finite box of volume V, thus

$$\psi^P{}_\alpha(r)=V^{-\frac{1}{2}}\sum_k\sum_u u_\alpha b_{ku}\exp(ik\cdot r), \tag{5}$$

where k is summed over the normal frequencies of the box and u is summed over the 4 spinors satisfying the Dirac equation for given k. For each k the u are normalized by

$$(u^*u')=\delta_{uu'}. \tag{6}$$

The b_{ku} are proton absorption operators and antiproton creation operators, and satisfy the anticommutation rules

$$[b_{ku}, b_{k'u'}{}^*]_+=\delta_{uu'}\delta_{kk'}. \tag{7}$$

The neutron field is similarly

$$\psi^N{}_\alpha(r)=V^{-\frac{1}{2}}\sum_k\sum_u u_\alpha d_{ku}\exp(ik\cdot r), \tag{8}$$

and the b_{ku} anticommute with the d_{ku}.

The meson field in the Schrödinger representation is

$$\phi(r)=\sum_k (2V\omega_k)^{-\frac{1}{2}}(a_k+a_{-k}{}^*)\exp(ik\cdot r),$$

with

$$[a_k, a_{k'}{}^*]=\delta_{kk'},\quad \omega_k=(\mu^2+k^2)^{\frac{1}{2}}. \tag{9}$$

Finally the interaction between the fields in the Schrödinger representation is

$$H'=G\int(\psi^{P*}(r)\gamma\psi^P(r)+\psi^{N*}(r)\gamma\psi^N(r))\phi(r)d_3r \tag{10}$$

$$=G\sum_k\sum_{k'}(2V\omega_k)^{-\frac{1}{2}}(a_k+a_{-k}{}^*)\sum_u\sum_v(u^*\gamma v)\times(b_{k'u}{}^*b_{k'-k,v}+d_{k'u}{}^*d_{k'-k,v}), \tag{11}$$

where γ is written for the Dirac matrix $i\beta\gamma_5$.

Let N be a set of occupation numbers for the noninteracting fields. Thus, N is a set of integers $(N_1, N_2, \cdots)$, one corresponding to each state of a free proton, antiproton, neutron, antineutron or meson. Let $A(N)$ be the product of the absorption operators b_{ku}, $b_{ku}{}^*$, d_{ku}, $d_{ku}{}^*$, and a_k, which annihilate the particles represented by N. Let $C(N)$ be the product of the corresponding creation operators. Then the normalized state vector of the state in which the particle occupation numbers have the values N is

$$\Phi(N)=[\Pi(N)]^{-\frac{1}{2}}C(N)\Phi_0, \tag{12}$$

where Φ_0 is the vacuum state of the noninteracting fields, and

$$\Pi(N)=(N_1!)(N_2!)(N_3!)\cdots. \tag{13}$$

The $\Phi(N)$ form a complete orthonormal set of states for the system of fields. The 3-dimensional wave function of the system in the state Ψ is given according to Tamm[1] and Dancoff[2] by the set of amplitudes

$$\alpha(N)=((\Phi(N))^*\Psi)=[\Pi(N)]^{-\frac{1}{2}}(\Phi_0{}^*A(N)\Psi). \tag{14}$$

The $\alpha(N)$ define the state Ψ completely in the sense of elementary quantum mechanics. We denote by $\beta(N)$ the Tamm-Dancoff amplitudes for the interacting vacuum state Ψ_0,

$$\beta(N)=[\Pi(N)]^{-\frac{1}{2}}(\Phi_0{}^*A(N)\Psi_0). \tag{15}$$

Thus

$$\Psi=\sum_N\alpha(N)\Phi(N),\quad \Psi_0=\sum_N\beta(N)\Phi(N). \tag{16}$$

There is no clear relation between the 4- and 3-dimensional wave functions (1) and (14). Substituting from (5) and (8) into (2),

$$\psi_{\alpha\beta}(r, r')=V^{-1}\sum_k\sum_{k'}\sum_u\sum_{u'}u_\alpha u_\beta'\times\exp(ik\cdot r+ik'\cdot r')(\Psi_0{}^*b_{ku}d_{k'u'}\Psi). \tag{17}$$

Thus $\psi(r, r')$ is a linear combination of the coefficients $(\Psi_0{}^*b_{ku}d_{k'u'}\Psi)$, which are a special case of the coefficients

$$a(N, N')=[\Pi(N)\Pi(N')]^{-\frac{1}{2}}(\Psi_0{}^*C(N)A(N')\Psi). \tag{18}$$

We take the $a(N, N')$ as an alternative 3-dimensional wave function[9] of the state Ψ. We will call the theory using $\alpha(N)$ the "old Tamm-Dancoff formalism," and the theory using $a(N, N')$ the "new Tamm-Dancoff formalism."

III. PHYSICAL MEANING OF THE WAVE FUNCTIONS

The meaning of the old Tamm-Dancoff wave function $\alpha(N)$ is clear. It is just the probability amplitude for finding the free particles specified by N, if the system is first put into the state Ψ, and the field interaction is then instantaneously switched off.

The physical meaning of the wave function $a(N, N')$ cannot be expressed so directly. Intuitively we can say, by looking at Eq. (18), that $a(N, N')$ is related to the probability amplitude for ending in the true vacuum state Ψ_0, if we start in the state Ψ and instantaneously annihilate the particles N' and create the particles N. Thus $a(N, N')$ describes the probability in the state Ψ of finding N' particles more, and N particles less, than are to be found in the vacuum of the interacting fields. A less exact but briefer way of expressing it, is to say that $a(N, N')$ is the probability of finding N' plus particles and N minus particles in the state Ψ. A "minus particle" is defined as one which is absent in Ψ but present in the comparison state Ψ_0.

A great advantage of this introduction of minus particles is that it enables us to handle the positive and negative energy solutions of the Dirac equation together. Using the old wave function $\alpha(N)$, one is forced at every stage to separate the proton states from the antiproton states and write a separate equation for each. This makes the old Tamm-Dancoff method ex-

[9] The notation here differs from that of reference 6 by a numerical factor.

tremely cumbersome when applied to processes involving pair creation and annihilation. On the other hand, the one-proton component of the new wave function $a(N, N')$ is

$$a_1(ku)=(\Psi_0{}^*b_{ku}\Psi), \tag{19}$$

where b_{ku} is an annihilation operator for a proton when u is a positive-energy spinor. The same expression (19) also represents, when u is a negative-energy spinor, the minus-one-antiproton component of the wave function since b_{ku} is then an antiproton creation operator. We may always take the one-proton and the minus-one-antiproton parts of the wave function together in the equations. That is to say, we may proceed as if the Dirac hole theory had not been invented, as if the proton were actually allowed to occupy states of both positive and negative energy. The hole theory has to be taken into account only at one place in the calculations; when Eq. (18) is used to define $a(N, N')$ we must make sure that the creation operators $C(N)$ are written to the left of the annihilation operators $A(N')$.

Similarly the component,

$$a_1'(ku)=(\Psi_0{}^*b_{ku}{}^*\Psi), \tag{20}$$

of $a(N, N')$ represents both the one-antiproton and the minus-one-proton part of the wave function. In this way a physical interpretation is found for the negative-energy components of the wave function $\psi_{\alpha\beta}(r, r')$ in Eq. (17). This two-particle wave function represents the amplitude for finding in Ψ either one proton and one neutron, one proton minus one antineutron, one neutron minus one antiproton, or minus one antiproton minus one antineutron. The general 4-dimensional wave function $\psi(x, y)$ has no such simple interpretation, when x and y are separated by a time-like interval. But at least we have now an understanding of the meaning of $\psi(x, y)$ for equal times, including positive and negative energy components on an equal footing.

IV. CONNECTION BETWEEN THE WAVE FUNCTIONS

We have now established a direct connection between the 4-dimensional wave function and the 3-dimensional amplitudes $a(N, N')$. It remains to work out the connections between the $a(N, N')$ and the $\alpha(N)$ and $\beta(N)$. Substituting from (12) and (16) into (18),

$$a(N, N')=\sum_{N_1}\sum_{N_2}\beta^*(N_1)\alpha(N_2)\times[\Pi(N)\Pi(N')\Pi(N_1)\Pi(N_2)]^{-\frac{1}{2}}\times(\Phi_0{}^*A(N_1)C(N)A(N')C(N_2)\Phi_0). \tag{21}$$

The expectation value in (21) is zero unless there exists a set of nonnegative occupation numbers

$$M=N_2-N'=N_1-N, \tag{22}$$

and is then equal to

$$\Pi(N_2)\Pi(N_1)[\Pi(M)]^{-1}. \tag{23}$$

We define a symbolic binomial coefficient

$$\binom{N_1}{N}=\Pi(N_1)[\Pi(N)\Pi(N_1-N)]^{-1}, \tag{24}$$

and then (21) becomes

$$a(N, N')=\sum_M\beta^*(N+M)\alpha(N'+M)\times\binom{N+M}{M}^{\frac{1}{2}}\binom{N'+M}{M}^{\frac{1}{2}}. \tag{25}$$

This gives the formal definition of the $a(N, N')$ in terms of old Tamm-Dancoff wave functions.

The inverse of the relation (25) is

$$\beta^*(N_1)\alpha(N_2)=\sum_M(-1)^M a(N_1+M, N_2+M)\times\binom{N_1+M}{M}^{\frac{1}{2}}\binom{N_2+M}{M}^{\frac{1}{2}}, \tag{26}$$

where $(-1)^M$ means $(-1)^{\Sigma M}$ and $\sum M$ is the sum of the occupation numbers M. To verify (26), substitute back (25) into (26). The result is

$$\beta^*(N_1)\alpha(N_2)=\sum_M\sum_{M'}(-1)^M\beta^*(N_1+M+M')\alpha(N_2+M+M')\binom{N_1+M+M'}{N_1}^{\frac{1}{2}}\binom{N_2+M+M'}{N_2}^{\frac{1}{2}}\binom{M+M'}{M}. \tag{27}$$

But it is easily proved that, keeping $M+M'$ fixed and summing over M,

$$\sum_M(-1)^M\binom{M+M'}{M}=0, \tag{28}$$

except when $M+M'=0$. Therefore the right side of Eq. (27) reduces to the term $M=M'=0$ and is equal to the left side. This verifies Eq. (26). In consequence of Eq. (26) the amplitudes $a(N, N')$ include a complete, and indeed redundant, determination of the $\alpha(N)$ and $\beta(N)$. Therefore, the $a(N, N')$ in principle provide a complete description of the states Ψ and Ψ_0 and determine all measurable properties of these states.

V. NORMALIZATION AND CONVERGENCE CONDITIONS

The old Tamm-Dancoff wave function, being defined directly as a probability amplitude, must be normalized by the conditions

$$\sum_N|\alpha(N)|^2=\sum_N|\beta(N)|^2=1. \tag{29}$$

In a particular theory the sums (29), which are really multiple integrals, may or may not converge. If they diverge then the $\alpha(N)$ and $\beta(N)$ cannot rigorously be defined, and the whole Tamm-Dancoff picture makes no exact sense. It is likely that this is the case for the

field theories now in general use. Nevertheless the method of renormalization can be incorporated into the Tamm-Dancoff formalism,[10] and probably some kind of "renormalized" $\alpha(N)$ can be defined for which the integrals (29) become convergent. We may say that if the Tamm-Dancoff point of view finally makes sense in this context, Eq. (29) will remain as the correct normalization condition for the $\alpha(N)$.

We now discuss the appropriate normalization condition for the $a(N, N')$. Let z be any positive number, and write z^M for $z^{\Sigma M}$ as in Eq. (26). From Eq. (25) we derive

$$\begin{aligned}&\sum_N \sum_{N'} z^{N+N'} |a(N, N')|^2 \\ &= \sum_N \sum_{N'} \sum_M \sum_{M'} \beta^*(N+M)\alpha(N'+M) \\ &\quad \times \beta(N+M')\alpha^*(N'+M')z^{N+N'} \binom{N+M}{M}^{\frac{1}{2}} \\ &\quad \binom{N'+M}{M}^{\frac{1}{2}} \binom{N+M'}{M'}^{\frac{1}{2}} \binom{N'+M'}{M'}^{\frac{1}{2}}. \quad (30)\end{aligned}$$

Using Cauchy's inequality, the right side of (30) does not exceed $X^{\frac{1}{2}}Y^{\frac{1}{2}}$, where

$$\begin{aligned}X = Y &= \sum_N \sum_{N'} \sum_M \sum_{M'} |\alpha(N'+M)|^2 \\ &\quad \times |\beta(N+M')|^2 z^{N+N'} \binom{N'+M}{M} \binom{N+M'}{M'} \\ &= \sum_{N_1} \sum_{N_2} |\alpha(N_1)|^2 |\beta(N_2)|^2 (z+1)^{N_1+N_2}, \quad (31)\end{aligned}$$

by the binomial theorem. Thus for every z,

$$\begin{aligned}\sum_N \sum_{N'} z^{N+N'} |a(N, N')|^2 &\leqslant (\textstyle\sum_N (z+1)^N |\alpha(N)|^2) \\ &\quad \times (\textstyle\sum_N (z+1)^N |\beta(N)|^2). \quad (32)\end{aligned}$$

The same argument applied to (26) gives

$$\begin{aligned}&\sum_N \sum_{N'} (z+1)^{N+N'} |a(N, N')|^2 \\ &\qquad \geqslant (\textstyle\sum_N z^N |\alpha(N)|^2)(\sum_N z^N |\beta(N)|^2). \quad (33)\end{aligned}$$

The inequalities (32) and (33) show that there is a close connection between the normalization integrals for the $\alpha(N)$ and the $a(N, N')$. In particular, suppose a theory is such that *the old Tamm-Dancoff amplitude for finding more than 85 particles simultaneously present in the state Ψ or in Ψ_0 is strictly zero.* (Here 85 is chosen only as an example of a large but finite number.) Then the right side of Eq. (32) converges for all z if it converges for $z=0$. In this case the normalization condition,

$$\sum_N \sum_{N'} |a(N, N')|^2 = \text{finite}, \quad (34)$$

may be imposed, and the sum (34) will automatically converge provided that (29) is convergent.

[10] M. Cini, Nuovo cimento **10**, 526 and 614 (1953) has shown how to renormalize the Tamm-Dancoff theory. See remarks in Sec. VII below.

In practice the Tamm-Dancoff method necessarily requires that the amplitudes $\alpha(N)$ be calculated with a cut-off theory, in which all the amplitudes involving more than say Q particles are set equal to zero by definition. It is taken as a fundamental hypothesis that, if Q is sufficiently large, the results of the calculation will be insensitive to the value of Q and will tend to a limit as Q tends to infinity. If this hypothesis fails then again the Tamm-Dancoff point of view makes no sense. Therefore, we must always suppose in practice that we are working with a cut-off theory with some finite value of Q. This being so, the previous argument applies, and we may take (34) as a practical normalization condition for the $a(N, N')$ which will always be valid in circumstances where the Tamm-Dancoff method itself is valid.

If $\alpha(N)$ and $\beta(N)$ are set equal to zero for $\sum N > Q$, the $a(N, N')$ defined by (25) will also be zero for $\sum N > Q$ or $\sum N' > Q$, and Eqs. (25) and (26) will still be consistent. Thus, the old Tamm-Dancoff formalism with cutoff is equivalent to a theory in which a cutoff is applied directly to the amplitudes $a(N, N')$ involving more than Q particles.

Supposing that we are not content with the above arguments and wish to obtain a rigorous normalization condition for the $a(N, N')$ independent of cutoff, then Eq. (32) is just not strong enough to deduce anything nontrivial from (29) alone. In order that (34) should hold, we must assume

$$\sum_N 2^N |\alpha(N)|^2 < \infty, \quad \sum_N 2^N |\beta(N)|^2 < \infty. \quad (35)$$

Thus the normalization condition (34) will apply, provided that the expectation values of 2^N in the states Ψ and Ψ_0 are finite. More generally, a practical normalization condition for the $a(N, N')$ will result from Eq. (32), provided that the expectation values of z^N in the states Ψ and Ψ_0 are finite for some $z > 1$. In practice it is very likely that these conditions will be fulfilled in cases where Eq. (29) holds. In any case, the order of magnitude of the Tamm-Dancoff amplitudes for large N is so hard to estimate accurately that there is practically no likelihood of being able to distinguish between (35) and (29). Therefore again we may conclude that (34) is a correct normalization condition for the $a(N, N')$ in all cases to which the Tamm-Dancoff point of view is applicable.

From (34) we may derive in turn a normalization condition for the wave function $\psi_{\alpha\beta}(r, r')$ defined by (2). Using Eqs. (17), (6), and (34), we find

$$\int\int d_3r d_3r' \sum_\alpha \sum_\beta |\psi_{\alpha\beta}(r, r')|^2 = \text{finite}. \quad (36)$$

It is convenient to take (36) equal to 1, although $\psi(r, r')$ is not strictly a probability amplitude. The space integrations in (36) extend over the whole space, since the volume V of the normalization box can be made to tend to infinity without introducing any dif-

ficulties. Equation (36) gives important information about the normalization of the 4-dimensional wave function $\psi(x, y)$, although it still does not throw any light on the behavior of the function for x and y separated by a time-like interval. At least Eq. (36) is a necessary condition for the 4-dimensional wave function to have a physical meaning.

VI. AN ELEMENTARY EXAMPLE

To illustrate the preceding sections in a simple way, consider a neutral scalar meson field $\phi(x)$ interacting with a classical source. Let the interaction be

$$H'=\sum_k \rho(k)(a_k+a_k^*), \tag{37}$$

where $\rho(k)$ is a given function of k. As is well known,[11] the stationary states of this system can be found explicitly. Let S be the anti-Hermitian operator

$$S=\sum_k r(k)(a_k^*-a_k), \quad r(k)=\omega_k^{-1}\rho(k). \tag{38}$$

A complete set of stationary states of the system is

$$\Psi(N)=\exp(-S)\Phi(N), \tag{39}$$

with $\Phi(N)$ given by Eq. (12). Let Ψ be the state in which just one real meson is present with momentum p. The old Tamm-Dancoff amplitudes of Ψ and Ψ_0 are

$$\alpha(N)=[\Pi(N)]^{-\frac{1}{2}}(\Phi_0^* A(N)\exp(-S)a_p^*\Phi_0), \tag{40}$$

$$\beta(N)=[\Pi(N)]^{-\frac{1}{2}}(\Phi_0^* A(N)\exp(-S)\Phi_0). \tag{41}$$

An elementary calculation gives[11]

$$\beta(N)=[\Pi(N)]^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\sum_k[r(k)]^2\}\times\Pi_k[-r(k)]^{N(k)}, \tag{42}$$

$$\alpha(N)=\beta(N)[r(p)-(N(p)/r(p))]. \tag{43}$$

The necessary and sufficient condition that the old Tamm-Dancoff wave functions be normalizable is that the sum

$$\sum_k[r(k)]^2=\sum_k \omega_k^{-2}[\rho(k)]^2 \tag{44}$$

be convergent.

For the same state Ψ, the wave function $a(N, N')$ is

$$\begin{aligned} a(N,N')&=(\Pi(N)\Pi(N'))^{-\frac{1}{2}}(\Phi_0\exp(S)C(N)A(N')\\ &\quad\times\exp(-S)a_p^*\Phi_0)\\ &=(\Pi(N)\Pi(N'))^{-\frac{1}{2}}[-N'(p)/r(p)]\\ &\quad\times\Pi_k[-r(k)]^{N(k)+N'(k)}. \end{aligned} \tag{45}$$

This gives immediately

$$\begin{aligned} &\sum_N\sum_{N'}|a(N,N')|^2\\ &\quad=\{1+[r(p)]^2\}\exp\{2\sum_k[r(k)]^2\}. \end{aligned} \tag{46}$$

Hence Eq. (34) also is satisfied if, and only if, the series (44) converges. Thus, in this example, the normalization condition (34) for the $a(N, N')$ is precisely equivalent to the condition (29) for the old Tamm-Dancoff wave function to be definable.

[11] See R. J. Glauber, Phys. Rev. **84**, 395 (1951).

VII. EQUATIONS OF MOTION

The Schrödinger equation for the old Tamm-Dancoff wave function is

$$(E-E_M)\alpha(M)=\sum_N(\Phi^*(M)H'\Phi(N))\alpha(N), \tag{47}$$

where E_M is the sum of the energies of the free particles specified by M. The corresponding equation for the $a(N, N')$ is

$$\begin{aligned} &(\epsilon+E_N-E_{N'})a(N,N')\\ &\quad=[\Pi(N)\Pi(N')]^{-\frac{1}{2}}(\Psi_0^*[C(N)A(N'),H']\Psi), \end{aligned} \tag{48}$$

where $\epsilon=E-E_0$. The $a(N, N')$ were originally introduced in order to avoid difficulties with the vacuum energy, which arise with Eq. (47) but not with Eq. (48). This aspect of the $a(N, N')$ has been already discussed,[6] and no more will be said about it here.

It is important to distinguish between two possible methods of solving Eqs. (47) and (48). The first method is the Tamm-Dancoff method proper, as discussed in Sec. V. All amplitudes $\alpha(N)$ or $a(N, N')$ involving more than Q particles are set equal to zero, and the result is a finite set of coupled linear integral equations for the $\alpha(N)$ or $a(N, N')$ involving not more than Q particles. The finite set of integral equations is then to be solved, numerically or otherwise, without any further approximations. The only hypothesis made in this method is that the $\alpha(N)$ or $a(N, N')$ for sufficiently large numbers of particles are unimportant. The second method is the method of Lévy[5] and Klein.[12] In this method we single out the components of the wave function involving the smallest possible number of particles for the given system. Thus, for the state Ψ of one proton and one neutron, we single out the $\alpha(M)$ in which M denotes one proton and one neutron and no other particles, and we call this lowest-order part of the wave function α_{11}. Similarly, we single out the components a_{11} of the $a(N, N')$ representing one proton (or minus one antiproton) and one neutron (or minus one antineutron), which are just the components appearing in $\psi_{\alpha\beta}(r, r')$ according to Eq. (17). Equations (47) and (48) are then converted into integral equations for the α_{11} or a_{11} alone, successively eliminating all the rest of the $\alpha(N)$ and $a(N, N')$ by substituting from one equation into another. The result of the elimination is a linear integral equation for the α_{11} or a_{11}, in which the kernel appears as a power-series expansion in the coupling constant G. The integral equation is in principle exact, if the substitution process could be carried to completion. In practice the series for the kernel has to be broken off at some finite power of G. So the Lévy-Klein method is based on the assumed convergence of the power-series expansion of the kernel, a much stronger hypothesis than that used by the Tamm-Dancoff method.

Klein has shown[12] by actual calculation that the power series for the kernel diverges badly, even if one considers only those meson exchange processes in

[12] A. Klein, Phys. Rev. **90**, 1101 (1953).

which ultraviolet divergences of the self-energy type do not occur. For this reason the Lévy-Klein method must be considered unworkable, and we must in future rest our hopes on the Tamm-Dancoff method. Unfortunately the solution of the Tamm-Dancoff equations becomes a formidable task as soon as the Lévy-Klein expansion is abandoned. There is no evidence showing whether or not the solutions converge as the number of particles considered increases, because nobody has ever solved the equations taking into account more than a very few components of the wave function.

The 4-dimensional wave function $\psi(x, y)$ satisfies the Bethe-Salpeter equation,[3] which is again a linear integral equation with a kernel which is expressed as a power-series expansion in G. This equation is open to the same objections that were raised against the Lévy-Klein treatment of Eqs. (47) and (48). Only in this case the situation is worse, because there is no known way of expressing the equation in any form other than the series expansion. There exist no 4-dimensional equations analogous to (47) and (48) from which the Bethe-Salpeter equation might be derived. Up to now the 4-dimensional theory has been inextricably tied up with the power-series expansion.

Whichever form of the equations of motion is chosen, in order to solve the equations it is necessary to use the method of renormalization to eliminate self-energy and other divergences. The renormalization is at present only possible in terms of a power-series expansion in G. Thus, Lévy[5] renormalized the power-series expansion of the Bethe-Salpeter kernel, and he found an approximate connection between that kernel and the kernel of the integral equation for the old Tamm-Dancoff wave function α_{11}. In this way he could obtain a divergence-free integral equation for α_{11}, using an expansion in powers of G and various other approximations. More recently Cini[10] has succeeded in writing the old Tamm-Dancoff Eqs. (47) in a covariant form, and so he is able to renormalize directly the kernel of the integral equation for α_{11}. But Cini's method is still based on the power-series expansion; he can renormalize the Lévy-Klein treatment of the Eqs. (47), but he cannot yet renormalize the Tamm-Dancoff treatment.

The method of Cini can be used with Eq. (48) just as well as with Eq. (47). Also, since the $a(N, N')$ is much more closely related than the $\alpha(N)$ to the 4-dimensional wave function, it seems likely that it will be easier to renormalize the Tamm-Dancoff treatment for Eq. (48) than for Eq. (47). It remains an outstanding task for the future to avoid the series expansions and to renormalize Eq. (48) directly.

VIII. THE ONE-PARTICLE SYSTEM

In this section we illustrate the use of Eq. (48) by setting up the equations for a state Ψ consisting of a single proton. In this case the lowest-order component of $a(N, N')$ is the one-proton wave function $(\Psi_0 b_{pu}\Psi)$. It is convenient to use instead of this wave function the related quantity,

$$\psi_\alpha(p)=\sum_u(\Psi_0{}^*b_{pu}\Psi)u_\alpha, \tag{49}$$

which is a Dirac spinor, the sum being taken over the 4 solutions u of the Dirac equation for a free particle of momentum p. We write

$$E_p=(M^2+p^2)^{\frac{1}{2}}, \tag{50}$$

and let

$$\eta(p)=E_p{}^{-1}(\alpha\cdot p+\beta M) \tag{51}$$

be the Dirac operator whose eigenvalues are $+1$ for the positive energy states and -1 for the negative energy states of momentum p. The equation (48) for $\psi_\alpha(p)$ is then

$$[\epsilon-\eta(p)E_p]\psi_\alpha(p)=\sum_u(\Psi_0{}^*[b_{pu}, H']\Psi)u_\alpha. \tag{52}$$

Using (11) and (7), the right side of (52) becomes

$$G\sum_k(2V\omega_k)^{-\frac{1}{2}}\sum_v(\gamma v)_\alpha(\Psi_0{}^*b_{p-k,v}(a_k+a_{-k}{}^*)\Psi). \tag{53}$$

Equation (52) couples the one-proton wave function $\psi_\alpha(p)$ only to the one-proton-one-meson wave function

$$\psi_\alpha{}^+(p, k)=\sum_v(\Psi_0{}^*b_{pv}a_k\Psi)v_\alpha, \tag{54}$$

and to the one-proton-minus-one-meson wave function

$$\psi_\alpha{}^-(p, k)=\sum_v(\Psi_0{}^*a_{-k}{}^*b_{pv}\Psi)v_\alpha. \tag{55}$$

With these notations, Eq. (52) becomes

$$[\epsilon-\eta(p)E_p]\psi(p)=G\sum_k(2V\omega_k)^{-\frac{1}{2}}\times[\gamma\psi^+(p-k, k)+\gamma\psi^-(p-k, k)]. \tag{56}$$

The equation (48) for $\psi^+(p-k, k)$ is

$$[\epsilon-\eta(p-k)E_{p-k}-\omega_k]\psi^+(p-k, k)=\sum_v(\Psi_0{}^*[a_kb_{p-k,v}, H']\Psi)v. \tag{57}$$

Equation (57) couples the one-proton-one-meson wave function to the one-proton wave function and to various components $a(N, N')$ involving 3 particles. To obtain the first approximation of the Tamm-Dancoff method,[13] we set all the 3-particle amplitudes equal to zero. To do this we must arrange the commutator in (57) as a sum of terms written in normal form[8] with creation operators standing to the left of absorption operators, and then keep only the term which involves a single b_{pu} operator. This gives for the right side of (57)

$$G(2V\omega_k)^{-\frac{1}{2}}\Lambda^+(p-k)\gamma\psi(p), \tag{58}$$

with

$$\Lambda^+(p)=\tfrac{1}{2}[1+\eta(p)],\quad \Lambda^-(p)=\tfrac{1}{2}[1-\eta(p)]. \tag{59}$$

Because of the Λ^+ in (58), Eq. (57) becomes

$$(\epsilon-E_{p-k}-\omega_k)\psi^+(p-k, k)=G(2V\omega_k)^{-\frac{1}{2}}\Lambda^+(p-k)\gamma\psi(p). \tag{60}$$

[13] The Tamm-Dancoff treatment in this first approximation gives the same results as the Lévy-Klein treatment with the interaction kernel calculated to order G^2. The two methods become different only in the next approximation.

Similarly, the equation for $\psi^-(p-k, k)$ is

$$(\epsilon+E_{p-k}+\omega_k)\psi^-(p-k, k) = G(2V\omega_k)^{-\frac{1}{2}}\Lambda^-(p-k)\gamma\psi(p). \quad (61)$$

Substituting Eqs. (60) and (61) into Eq. (56), and letting $V\to\infty$, we derive an integral equation for $\psi(p)$ alone,

$$(\epsilon-\alpha\cdot p-\beta M)\psi(p)=(G^2/16\pi^3)\left\{\int d_3k(\omega_k)^{-1} \times\gamma[\epsilon-\eta(p-k)(E_{p-k}+\omega_k)]^{-1}\gamma\right\}\psi(p). \quad (62)$$

Equation (62) gives the second-order self-energy of a proton interacting with the meson field. The Dirac operator multiplying $\psi(p)$ on the right side is the self-energy operator. This operator in general has matrix elements between positive and negative energy components of $\psi(p)$. Thus, Eq. (62) is a set of 4 coupled equations and will have four independent solutions $\psi(p)$, two with a positive eigenvalue ϵ and two with a negative ϵ. The positive ϵ is the energy of a proton of momentum p, including self-energy, and the negative ϵ is minus the energy of an antiproton of momentum p.

It is noteworthy that Eq. (62) differs from the corresponding equation obtained using the old Tamm-Dancoff wave function $\alpha(N)$. The form of the energy denominators is here symmetrical between positive and negative energy intermediate states, whereas in the equation for $\alpha(N)$ the negative intermediate states give denominators of a quite different and unsymmetrical form. The denominators in Eq. (62) are much closer to those which appear in the covariant 4-dimensional perturbation theory. This removes one of the major difficulties found by Lévy[5] and Klein[12] in connecting the Bethe-Salpeter equation with the 3-dimensional treatment.

IX. THE TWO-PARTICLE SYSTEM

We now use the method of Sec. VIII to construct from Eq. (48) the wave equation for the neutron-proton system, setting equal to zero all components $a(N, N')$ involving 4 or more particles. We define the 2-particle wave function

$$\psi_{\alpha\beta}(p, q)=\sum_u\sum_v(\Psi_0{}^*b_{pu}d_{qv}\Psi)u_\alpha v_\beta. \quad (63)$$

This is just the momentum transform of the configuration-space wave-function (17). Similarly we write

$$\psi^+{}_{\alpha\beta}(p, q, k)=\sum_u\sum_v(\Psi_0{}^*b_{pu}d_{qv}a_k\Psi)u_\alpha v_\beta, \quad (64)$$

$$\psi^-{}_{\alpha\beta}(p, q, k)=\sum_u\sum_v(\Psi_0{}^*a_{-k}{}^*b_{pu}d_{qv}\Psi)u_\alpha v_\beta. \quad (65)$$

The equation (48) for $\psi(p; q)$ is

$$[\epsilon-\eta^1(p)E_p-\eta^2(q)E_q]\psi(p, q) = G\sum_k(2V\omega_k)^{-\frac{1}{2}}[\gamma^1\psi^+(p-k, q, k) +\gamma^1\psi^-(p-k, q, k)+\gamma^2\psi^+(p, q-k, k) +\gamma^2\psi^-(p, q-k, k)], \quad (66)$$

where the indices 1, 2 denote Dirac matrices operating on the first and second spinor suffix of ψ, respectively. The equation for $\psi^+(p-k, q, k)$, dropping the 4-particle components, is

$$[\epsilon-\eta^1(p-k)E_{p-k}-\eta^2(q)E_q-\omega_k]\psi^+(p-k, q, k) = G(2V\omega_k)^{-\frac{1}{2}}[\Lambda^{+1}(p-k)\gamma^1\psi(p, q) +\Lambda^{+2}(q)\gamma^2\psi(p-k, q+k)]. \quad (67)$$

Similarly,

$$[\epsilon-\eta^1(p-k)E_{p-k}-\eta^2(q)E_q+\omega_k]\psi^-(p-k, q, k) = G(2V\omega_k)^{-\frac{1}{2}}[\Lambda^{-1}(p-k)\gamma^1\psi(p, q) +\Lambda^{-2}(q)\gamma^2\psi(p-k, q+k)]. \quad (68)$$

Substituting from (67) and (68) into (66), we find the two-particle wave equation

$$[\epsilon-(\alpha^1\cdot p+\beta^1M)-(\alpha^2\cdot q+\beta^2M)]\psi(p, q) = (G^2/16\pi^3)\int d_3k(\omega_k)^{-1}[K\psi(p, q) +L\psi(p-k, q+k)]. \quad (69)$$

The kernel K gives the effect of the self-energies of the two particles,

$$K=\gamma^1[\epsilon-\eta^1(p-k)(E_{p-k}+\omega_k)-\eta^2(q)E_q]^{-1}\gamma^1 +\gamma^2[\epsilon-\eta^2(q+k)(E_{q+k}+\omega_k)-\eta^1(p)E_p]^{-1}\gamma^2. \quad (70)$$

The kernel L gives the interaction between the particles resulting from exchange of a single meson,

$$L=\gamma^1[\epsilon-\eta^1(p-k)E_{p-k}-\eta^2(q)(E_q+\omega_k)]^{-1}\gamma^2 +\gamma^2[\epsilon-\eta^2(q+k)E_{q+k}-\eta^1(p)(E_p+\omega_k)]^{-1}\gamma^1. \quad (71)$$

Again we see that Eq. (69) differs from the two-body equation derived from the old Tamm-Dancoff method. The energy denominators in Eq. (71) are similar to those which appear in the Bethe-Salpeter equation and are symmetrical between positive and negative energies.

To make use of Eq. (69), it is necessary to use the Cini method of renormalization[10] to eliminate the divergent parts from the self-energy kernel K. The resulting equation is free of divergences and suitable for practical calculations. By separating the center-of-mass motion and considering a state with a definite angular momentum, we can reduce the equation to a one-dimensional integral equation for a 16-component function of one real variable. The number of components can be reduced by making further use of the symmetry properties of the equation. In any case, a one-dimensional integral equation of this kind can be solved numerically without a prohibitive amount of work. We are, therefore, in a position to explore quantitatively the behavior of the wave function $\psi_{\alpha\beta}(r, r')$ of a relativistic 2-body system, without assuming the velocities of the particles to be nonrelativistic and without assuming the negative-energy components of the wave function to be small.

The solutions of Eq. (69) will divide into four classes, characterized by the way in which they behave for small G. The first class tends as $G\to 0$ to a state of one free proton and one free neutron. The second class tends to one free proton minus one free antineutron, the third to one free neutron minus one antiproton, the fourth to minus one antiproton minus one antineutron. It is possible that for large G these classes of solutions will no longer be clearly distinguishable. However, in studying the two-body system we are only interested in the solutions of the first class. The other 3 classes represent systems in which real particles are present in the comparison state Ψ_0, so that Ψ_0 for these solutions is not the vacuum state. We have never used, in the whole analysis leading to Eq. (69), the information that Ψ_0 is the vacuum state. The fact that Ψ_0 is the vacuum state is an additional boundary condition which we impose on Eq. (69), restricting the allowed solutions to those of the first class.

A practical difficulty in the use of Eq. (69) is the fact that the energy denominators in Eq. (71) may vanish. When

$$\epsilon-E_{p-k}-E_q-\omega_k=0, \tag{72}$$

the energy of the system is sufficiently large to allow production of a real meson. In this case the wave function $\psi^+(p-k, q, k)$ has a singularity where (72) is satisfied, representing an out-going wave of created mesons. This singularity appears in the integral Eq. (69) as a Dirac δ_+ function, in consequence of the assumed boundary condition that the free mesons form an outgoing wave. The integration over the δ_+ function then has a well-defined meaning and does not introduce any ambiguity. The type of vanishing energy denominator typified by Eq. (72) is handled here in exactly the same way as in nonrelativistic scattering theory.

However, another type of singularity in Eq. (71) occurs when

$$\epsilon+E_{p-k}-E_q-\omega_k=0. \tag{73}$$

This corresponds to a singularity in $\psi^+(p-k, q, k)$ at the point where one neutron and one meson and minus-one antiproton are present. Such a singularity could exist only if a real antiproton were present in the comparison state Ψ_0. So again we use the fact that Ψ_0 is the vacuum state as an extra boundary condition, implying that there is no δ-function singularity in $\psi^+(p-k, q, k)$ where Eq. (73) is satisfied. This means that the integrations over vanishing energy denominators in Eq. (69) are always to be taken as Cauchy principal values, except in the case where they are of the form (72) and are associated with real particle creation.

The appearance of the "spurious" vanishing energy denominators in Eq. (69) is a defect of the new Tamm-Dancoff method, which may considerably complicate the practical use of the method. But it is clear that this defect is also inherent in the Bethe-Salpeter equation. The Bethe-Salpeter equation is also derived formally without using the fact that Ψ_0 in Eq. (1) is the vacuum state. Hence the fact that Ψ_0 is the vacuum state must be used as a boundary condition in order to obtain the physically meaningful solutions of the equation. Only because there is no known method of solving the Bethe-Salpeter equation except in a quasi-static approximation, this need for additional boundary conditions has hitherto been concealed.

Note added in proof:—W. M. Visscher (Cornell thesis, 1953, to be published) has calculated the finite part of the self-energy kernel (70) which remains after mass-renormalization. He used the Cini method, adapted to this problem as explained in an earlier letter [F. J. Dyson, Phys. Rev. **91**, 421 (1953)].