Many-Body Forces and Nuclear Saturation*†

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Nuclear saturation is studied with many-body interactions derived from pseudoscalar meson theory with pseudoscalar coupling. All parameters appearing in this calculation are fixed on the basis of the work of Lévy, who has shown that leading terms in a perturbation deduction of the two-body interaction are well fitted to reproduce the experimental data. The leading term in the n-body potential depends only on the interparticle distances and is repulsive (attractive) for n odd (even). The energy of the nucleus is calculated with potentials up through five-body interactions and with neglect of Coulomb and surface effects. Saturation properties derived from these considerations are in accord with experience. Antisymmetrization of the nuclear wave function reduces the many-body interaction energies considerably by inhibiting the close approach of many particles. Thus the Pauli exchange terms are found to reduce the five-body interaction energy by 76 percent and to give a rapid convergence for the expansion in n-body forces.

I. INTRODUCTION

ONE of the outstanding problems of nuclear structure is the explanation of the saturation of nuclear forces in complex nuclei. The main facts involved in this problem are the saturation of nuclear density and the saturation of binding energy. The first refers to the fact that the density of nuclear matter seems to be very roughly independent of mass number A, for all but the lightest nuclei. Observed nuclear radii equal, approximately,

$$R_s = 1.4A^{\frac{1}{3}}\times 10^{-13}\text{ cm} = A^{\frac{1}{3}}/\mu,$$

with μ^{-1} defined as the meson Compton wavelength. Saturation of nuclear binding energy refers to the fact that the average binding energy per nucleon is roughly independent of A and approximately equal to 8 Mev.

These saturation features limit the choice of acceptable theories of nuclear forces. For example, purely attractive forces between pairs of nucleons (Wigner forces) are excluded.[1]

In 1932, Heisenberg[2] borrowed the concept of exchange forces from molecular theory and applied it to the nucleon-nucleon interaction problem. With the introduction of such forces as these, which are attractive or repulsive depending on the symmetry of the interacting nucleon pairs, it has proved possible to account for nuclear stability in terms of static, central, two-body potentials of short range. Combinations of ordinary and exchange forces consistent both with the saturation requirements and with observations on deuterons, α particles, and nucleon-nucleon scattering have been given during the 1930's by Wigner, Breit, Feenberg, Kemmer, Volz, and others.[3]

Recently, however, the study of the n-p and p-p scattering cross sections in the energy region up to 300 Mev has cast some doubt upon the role of exchange forces as the main reason of saturation. The analysis by Christian and Hart[4] seems to reveal a mixture of ordinary and exchange forces (Serber force) inconsistent with the saturation requirements.[5] It must be said, however, that this analysis is based upon rather incomplete experimental data; and the result, therefore, cannot be taken to exclude definitely the explanation of saturation by two-body exchange forces.

It has often been recognized and remarked that the limitation of considerations within the framework of two-body, velocity-independent interactions was too restrictive. Primakoff and Holstein[6] and Wheeler[7] have discussed extensions of the basis of considerations to include many-body and velocity-dependent forces. Also the development of meson theories of nuclear forces has suggested that, especially, the many-body forces should play an important role in nuclear matter. The interaction between mesons and nucleons is strong enough so that meson exchanges will be frequent between more than two neighboring nucleons, and this phenomenon leads to many-body forces.

The present status of meson theory is too uncertain to permit us to draw any quantitative conclusions, let alone to rely upon them. In this paper we consider the pseudoscalar meson theory with pseudoscalar coupling

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[1] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (J. Wiley and Sons, New York, 1952), Chap. III. In the following, we refer to this book as B.W.

[2] W. Heisenberg, Z. Physik **77**, 132 (1932).

[3] G. Breit and E. P. Wigner, Phys. Rev. **53**, 998 (1938). For a comprehensive bibliography of the literature on this subject, see L. Rosenfeld, *Nuclear Forces* (North Holland Publishing Company, Amsterdam, 1948), Chap. XI.

[4] R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950).

[5] This question is discussed in reference 1, Chap. IV. More recently E. Lomon and H. Feshbach [Bull. Am. Phys. Soc. **28**, No. 3, 30 (1953)] have shown that the scattering data can also be analyzed with forces which give odd-state repulsions that are absent from a Serber mixture.

[6] H. Primakoff and T. Holstein, Phys. Rev. **55**, 218 (1939); L. Janossy, Proc. Cambridge Phil. Soc. **35**, 616 (1939); N. N. Svartholm, thesis, Lund, 1945 (unpublished).

[7] J. A. Wheeler, Phys. Rev. **52**, 1083 (1937).

as a guide with which to explore some possible forms which many-body forces may assume, and we study the implications of these forces for the saturation problem.

We are using a very primitive approach in the evaluation of the meson theory. Perturbation theory is employed and only the leading terms, in powers of the meson-nucleon coupling constant $G^2/4\pi$ multiplied by the meson-nucleon mass ratio μ/M are considered, although it has been shown that other terms contribute also and may change the results considerably.[8] The two-body interaction emerging from this calculation is identical with the one derived by Lévy,[9] and it seems to be well fitted to reproduce the two-body experimental data. We then choose for our work the same coupling constant as Lévy has used.

We note especially one feature of the Lévy two-body force: Outside of a repulsive core of radius $r_c=0.38/\mu$, it contains a strongly attractive short-ranged Wigner force and two longer-ranged forces, one proportional to $(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$, the other to $(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)S_{12}$, where S_{12} is the tensor operator. The combination of these terms reproduces a characteristic feature of the two-body interaction; namely, that the range of the tensor force is longer than the range of the central force. The strongly attractive short-ranged Wigner force comes from the exchange of meson pairs, and it is this exchange, when applied to more than two nucleons, which gives rise to the leading terms of the many-body forces. We are aware that this primitive evaluation of the pseudoscalar meson theory neglects terms which may be important and, indeed, not negligible. However, we plan to use it as a first exploration of the many-body forces and to make use of the fact that it gives us a set of many-body interactions without the introduction of any new adjustable constants. It is hoped that their orders of magnitude, shapes, and nonexchange properties are guessed correctly by this procedure.

One of the most striking properties of these many-body forces is their alternation in sign. The potentials between odd (even) numbers of nucleons are repulsive (attractive). The average distance between nucleons at which the forces act decreases with increasing number, and at the same time the Pauli exclusion principle makes it more difficult for the nucleons to approach close to one another. These features are the main reasons for the saturating effects of the many-body forces. In fact, the repulsive three-body forces contribute most decisively to the potential energy and prevent the nucleus from collapsing.

Another effect of possible importance derives from the many-body forces. The marked position correlation between nucleons resulting from strong, short-ranged two-body forces is difficult to harmonize with the shell theory model of a nucleon moving in an average potential hole. The introduction of many-body repulsive interactions will tend to weaken the correlation between nucleons and bring the two viewpoints into closer accord.

The program of this paper is to study the effect of many-body forces upon the energies of complex nuclei in their lowest states. We use the results of the pseudoscalar meson theory as a characteristic example of such forces, without implying their correctness.

The pioneering work of this nature is that of Wentzel[10] in 1942. He discussed these questions on the basis of a meson scalar pair theory and by treating the nucleus as an infinitely large source of uniform finite density. His methods and results will be discussed in Sec. IV.

We should mention here another recently developed approach to an understanding of nuclear saturation from the point of view of nonlinear meson theory. Schiff[11] has studied the possible forms of nonlinearities which may be introduced into the meson equations, in a classical field treatment, and which predict saturation. The effective mesic self-repulsion introduced by the nonlinearity is the primary agent preventing collapse of the nucleus in this approach.

II. OUTLINE AND SUMMARY

A brief outline and summary of our calculations are presented in this section. Lévy's difficult and important analysis of the low-energy properties of the two-nucleon system provides the point of departure for the present work. Lévy has applied the methods of Tamm[12] and Dancoff[13] to calculate an effective two-body interaction with the pseudoscalar meson theory through terms of order $(G^2/4\pi)(\mu/2M)^2$, $(G^2/4\pi)^2(\mu/2M)^2$, and $(G^2/4\pi)^2(\mu/2M)^3$, where $(G^2/4\pi)$ is the meson-nucleon coupling constant and μ and M are the meson and nucleon masses, respectively. Lévy argued that, for internucleon separations $r\gtrsim(M\mu)^{-\frac{1}{2}}$, terms of these orders in the coupling constant and mass ratio are of major importance in determining the potential shape and depth.[9] For separations $r\lesssim(M\mu)^{-\frac{1}{2}}$ a complete analysis was not possible, but Lévy developed arguments from the field theory indicating a strong repulsion in this range. With choice of $G^2/4\pi=10$ and of the radius of the repulsive core $r_c=0.38/\mu$, the Lévy potential has been impressively successful in matching the data for n-p and p-p systems with energies up through 40 Mev.

[8] We elaborate on this point in Sec. III, which is concerned with the deduction of the interaction potentials from the meson field theory.

[9] M. M. Lévy, Phys. Rev. **88**, 725 (1952). We consider just the two leading and important terms V_2 and $V_4^{(a)}$ in Lévy's work. Algebraic errors in his calculation of the correction terms, independently noticed by A. Klein, Phys. Rev. **89**, 1158 (1953), destroy the validity of his convergence arguments. This is discussed more fully in Sec. III.

[10] G. Wentzel, Helv. Phys. Acta **15**, 111 (1942); **25**, 569 (1952).

[11] L. I. Schiff, Phys. Rev. **83**, 1 (1951); B. J. Malenka, Phys. Rev. **86**, 68 (1952); D. Finkelstein, Ph.D. thesis, Massachusetts Institute of Technology, 1953 (unpublished).

[12] I. Tamm, J. Phys. (U.S.S.R.) **9**, 449 (1945).

[13] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).

Guided by Lévy's success and viewpoint, and adopting his parameters, we derive the interaction potential from pseudoscalar meson theory for n-body interactions. The leading term for an n-body interaction is proportional to λ^n with λ defined by $\lambda=(G^2/4\pi)(\mu/2M)$, is repulsive or attractive according as n is odd or even, and is a function of the magnitudes of the interparticle separations only. These potentials are identical with those of Wentzel's scalar pair theory,[10] as is readily shown with the Dyson[14] or Foldy[15] transformation. In our calculations of the energies, we consider those potential terms which are proportional to λ^2 and to $\lambda\mu/M$ for the two-body interaction, and to λ^n for n-body interactions ($n>2$). We then determine the nuclear potential energies by taking expectation values of these interaction potentials with a completely antisymmetrized wave function of A particles confined to a box of radius $R=\eta R_s$. In order to determine, firstly, whether the three-body repulsion derived from the pseudoscalar theory is qualitatively sufficient to prevent collapse of heavy nuclei and, secondly, under what conditions it is possible to make a convergent expansion of the interactions in a series of n-body forces, we approximate the wave function for each of the A nucleons by plane waves for *all* internucleon separations greater than r_c. Within the repulsive cores, the wave function is taken to vanish. The potential energy per nucleon for two- plus three-body interactions resulting from this calculation shows a minimum of depth 30 Mev at a nuclear radius of $R=1.0R_s$. If we include the contribution of four plus five-body interactions, we find only a slight shift in the minimum to a depth of 32 Mev at a radius $R=1.1R_s$. It is of interest to note just briefly here the primary reason for the smallness of the effect of many-body interactions for $n>3$. As n increases, the range of the many-body potentials decreases, and at the same time the Pauli exclusion principle makes it more difficult for the n bodies to approach close to one another.

For the kinetic energies of the nuclear particles, we refer to a calculation of Lenz[16] in 1929. Point nucleons obeying the Pauli exclusion principle have the usual Fermi kinetic energy due to the filling of free particle energy levels. The requirement that the wave function vanish whenever any two nucleons approach within a separation of r_c, the hard core radius, of one another increases the wave function curvature and, hence, the kinetic energy above the Fermi value. With Lenz's approximate result for this increase, we write for the kinetic energy,

$$T=T_F(1+0.82/\eta), \tag{1}$$

where T_F denotes the Fermi energy. If Eq. (1) is added to the potential energy, there emerges finally an expression for the total energy per nucleon with a minimum of depth 12-Mev at a radius $R=1.15R_s$. Since surface and Coulomb effects have been neglected in the calculations outlined above, the 12-Mev binding energy per nucleon that we obtain is to be compared with the volume binding energy term in the von Weizsacker semiempirical formula.[17] Feenberg's[18] analysis gives approximately 14-Mev per nucleon for this.

Once it is established that the three-body force of pseudoscalar theory used in this calculation can qualitatively explain nuclear saturation, and that an expansion in n-body forces has meaning, it becomes desirable to improve the quantitative value of our results with more realistic wave functions. If the step-function correlation form discussed above is applied literally in the calculation of the kinetic energy, it yields an infinitely large result because of its discontinuity at the edge of the hard core. One can construct trial functions that go to zero continuously at the core and can then perform a variational calculation for the minimum value of the kinetic plus potential energy. Jastrow[19] has suggested a function in the form of a Slater determinant for the A free-particle states multiplied by $\frac{1}{2}A(A-1)$ symmetric correlation terms, one for each pair. The correlation term for each pair is identical in form and is taken to fall smoothly to zero for interparticle separations $r\leqslant r_c$. The parameters that determine the shape of a correlation term then are varied and the minimum energy found. Calculations of this type have been carried through and yield reasonable saturation as above. The correlation term corresponding to the minimum energy is essentially constant for $r>r_c$, indicating no marked tendency for nucleons to cluster or to anticluster. This result suggests that many-body repulsive forces may be of importance in harmonizing the two different viewpoints of the meson theory, which predicts strong internucleon interactions, and of the shell theory, which is based on the independent particle model. Further discussion on the calculations and results is presented in the following sections.

III. POTENTIALS

The interaction potentials are derived from the pseudoscalar meson theory in this section. The problem of two interacting nucleons is analyzed, first in order to illustrate the method of our calculations, and to show clearly the approximations inherent in the work.

We write for the interaction term in the Hamiltonian for nucleons and charge-symmetric pseudoscalar mesons with pseudoscalar coupling,

$$H'=iG\sum_{\alpha=1}^{3}\int\bar{\psi}\gamma_5\tau_\alpha\varphi_\alpha\psi d\mathbf{r}, \tag{2}$$

where ψ and $\bar{\psi}=\psi^*\gamma_4$ are the quantized nucleon field amplitudes; φ_α, with $\alpha=1, 2, 3$, are the amplitudes of

[14] F. J. Dyson, Phys. Rev. **73**, 929 (1948).
[15] L. L. Foldy, Phys. Rev. **84**, 168 (1951).
[16] W. Lenz, Z. Physik **56**, 778 (1929).
[17] Reference 1, Chap. VI.
[18] E. Feenberg, Revs. Modern Phys. **19**, 239 (1947).
[19] R. Jastrow (private communication).

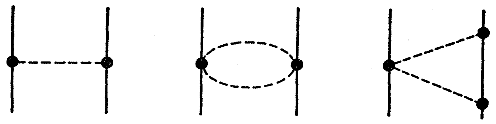


FIG. 1. Feynman diagrams for two-body interactions of order $(G^2/4\pi)(\mu/M)^2$, $(G^2/4\pi)^2(\mu/M)^2$, and $(G^2/4\pi)^2(\mu/M)^3$.

the meson field; and τ_α are the usual two-by-two isotopic spin (charge) matrices. We perform now the canonical transformation introduced by Dyson[14] to a new representation in which the velocity-independent aspects of the nucleon motion are more readily identified. This transformation is discussed by Lepore[20] and by Drell and Henley,[21] in whose work it is explicitly carried through in closed form. Keeping only the leading terms in G/M, the effects of which will be included in the calculations presented here, we write for the effective interaction Hamiltonian,

$$H_{\text{eff}}=(G^2/2M)\int\bar{\psi}\psi\varphi^2d\mathbf{r}+(G/2M)\int(\bar{\psi}\tau_\alpha\boldsymbol{\sigma}\psi)\cdot(\boldsymbol{\nabla}\varphi_\alpha)d\mathbf{r}. \quad (3)$$

The first term of Eq. (3) is recognized as the scalar pair term of Wentzel's theory, and the second one as the usual gradient coupling term. These two terms are the only ones that need be considered in order to derive n-body interaction potentials of order λ^n, $\lambda^{n-1}(\mu/M)$, and $\lambda^n(\mu/M)$, with $\lambda\equiv(G^2/4\pi)(\mu/2M)$.

With the interaction Hamiltonian cast in this representation, one can immediately apply adiabatic perturbation theory in the limit of fixed delta-function sources, for interparticle distances $r\gtrsim(M\mu)^{-\frac{1}{2}}$. The Feynman diagrams of the processes considered for two-body interactions appear in Fig. 1. The direct result is

$$\begin{aligned}V^{2\text{-body}}&=V_2+V_4{}^{(a)}+V_4{}^{(c)}\\&=-\left(\frac{G}{2M}\right)^2\frac{2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{2(2\pi)^3}\int d\mathbf{q}(\boldsymbol{\sigma}_1\cdot\mathbf{q})(\boldsymbol{\sigma}_2\cdot\mathbf{q})\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\omega_q{}^2}\\&\quad-\left(\frac{G^2}{2M}\right)^2\frac{2\cdot6}{4(2\pi)^6}\int\int d\mathbf{p}d\mathbf{q}\frac{e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}}}{\omega_p\omega_q(\omega_p+\omega_q)}\\&\quad-\left(\frac{G^2}{2M}\right)\left(\frac{G}{2M}\right)^2\frac{6}{4(2\pi)^6}\int d\mathbf{p}d\mathbf{q}(\mathbf{p}\cdot\mathbf{q})\frac{e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}}}{\omega_p\omega_q}\\&\quad\times\left[\frac{1}{\omega_p\omega_q}+\frac{1}{\omega_q(\omega_q+\omega_p)}+\frac{1}{\omega_p(\omega_q+\omega_p)}\right]\\&=\tfrac{1}{3}\lambda(\mu/2M)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+S_{12}[1+3/(\mu r)\\&\quad+3/(\mu r)^2]e^{-\mu r}/r\}-(3\lambda^2/\mu r^2)\{(2/\pi)K_1(2\mu r)\\&\quad-(\mu/M)[1+(1/\mu r)]^2e^{-2\mu r}\},\end{aligned}\quad(4)$$

where $S_{12}=(3/r^2)(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ is the tensor force operator, $\omega_q=(q^2+\mu^2)^{\frac{1}{2}}$, and $K_1(x)$ is the Hankel function of the first kind, with imaginary argument.[9] The last term $V_4{}^{(c)}$ in Eq. (4) differs from the corresponding term of order $\lambda^2\mu/M$ in Lévy's potential[9] (denoted as $V_4{}^{(b)}$). Whereas Lévy's $V_4{}^{(b)}$ was attractive and of minor importance and indicated the convergence of his procedure, the $V_4{}^{(c)}$ in Eq. (4) is repulsive, of comparable magnitude with $V_4{}^{(a)}$, which it largely cancels, and indicates the danger in neglecting terms of higher order[9,22] in μ/M. The terms in Eq. (4) appear in Fig. 2, from which it is clear that the above expression for $V^{2\text{-body}}$ cannot fit known properties of the two-nucleon system. It remains, however, as shown by Lévy that a potential of the form

$$V^{2\text{-body}}=\begin{cases}V_2+V_4{}^{(a)}, & r>r_c,\\ \infty, & r<r_c,\end{cases}\quad(5)$$

is spectacularly successful in describing the low-energy two-nucleon interaction. As a basis for further discussion, we adopt here the attitude that only the leading order terms in the pseudoscalar meson theory calculations are to be taken literally. We cannot give a rigorous argument to support the validity of this assumption, as we have no convergent procedure for calculating the many higher-order graphs that contribute. Thus, as shown by Lévy in his paper, there are radiative corrections to the lowest-order graphs in Fig. 1 that give large contributions.[22,23] Special classes

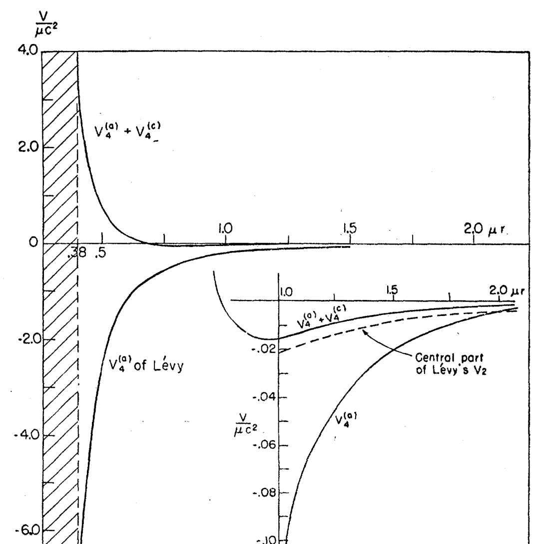


FIG. 2. The two-body interaction potential as expressed by Eq. (4). The dotted curve in the inset represents the contribution of the central part of V_2 for the deuteron ground state.

[20] J. V. Lepore, Phys. Rev. 88, 750 (1952).
[21] S. D. Drell and E. M. Henley, Phys. Rev. 88, 1053 (1952).
[22] A. Klein, Bull. Am. Phys. Soc. 28, No. 3, 36 (1953).
[23] M. Ruderman, Phys. Rev. 90, 183 (1953).

of these have been summed,[24] but the general problem remains unsolved. However, this approach gives a two-body potential, Eq. (5), that "works," and it permits us to derive many-body potentials consistent with it.

The leading term responsible for the three-body interaction is of order λ^3, corresponding to the pair term of Eq. (3) operating at each vertex. The diagram for the process appears in Fig. 3. The interaction potential is obtained readily with the identical methods as applied in the two-body case. It is, for interparticle distances $\mathbf{r}_{12}$, $\mathbf{r}_{23}$, and $\mathbf{r}_{31}$,

$$V^{3\text{-body}}=\left(\frac{G^2}{2M}\right)^3\frac{2\cdot 24}{8(2\pi)^9}\int\int\int d\mathbf{p}d\mathbf{q}d\mathbf{k}$$
$$\times\frac{e^{i(\mathbf{p}\cdot\mathbf{r}_{12}+\mathbf{q}\cdot\mathbf{r}_{23}+\mathbf{k}\cdot\mathbf{r}_{31})}}{\omega_p\omega_q\omega_k}\left[\frac{1}{(\omega_p+\omega_q)(\omega_p+\omega_k)}+\frac{1}{(\omega_q+\omega_p)(\omega_q+\omega_k)}+\frac{1}{(\omega_k+\omega_p)(\omega_k+\omega_q)}\right]$$
$$=-\left(\frac{G^2}{2M}\right)^3\left(\frac{3}{2\pi^6}\right)(r_{12}r_{23}r_{31})^{-1}\int\int\int_0^\infty dpdqdk$$
$$\times(p\sin pr_{12})(q\sin qr_{23})(k\sin kr_{31})$$
$$\times\left[\frac{1}{\omega_p(\omega_q^2-\omega_p^2)(\omega_k^2-\omega_p^2)}+\frac{1}{\omega_q(\omega_p^2-\omega_q^2)(\omega_k^2-\omega_q^2)}+\frac{1}{\omega_k(\omega_p^2-\omega_k^2)(\omega_q^2-\omega_k^2)}\right].$$

The singularities introduced in the process of rationalizing the energy denominators are simply and unambiguously handled by assigning small, numerically ordered imaginary masses to the mesons: *viz.*, $\omega_p\rightarrow(p^2+\mu^2+i\epsilon_p)^{\frac{1}{2}}$. We find readily

$$V^{3\text{-body}}=\left(\frac{\lambda}{\mu}\right)^3\left(\frac{24}{\pi}\right)(r_{12}r_{23}r_{31})^{-1}\times\int_0^\infty(k/\omega)\sin[k(r_{12}+r_{23}+r_{31})]dk$$
$$=12\lambda^3\left(\frac{2}{\pi}\right)\frac{K_1[\mu(r_{12}+r_{23}+r_{13})]}{\mu^2r_{12}r_{23}r_{31}}.\qquad(6)$$

Thus we see that the leading contribution to the three-body interaction is repulsive, spin and charge independent, and a function of the three interparticle distances only. The three-body terms of order $\lambda^2\mu/M$

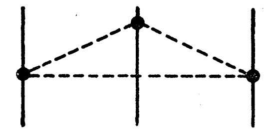


FIG. 3. Feynman diagram for the three-body interaction resulting from the pair term operating at each vertex.

and $\lambda^3\mu/M$, which arise when a pair of mesons is absorbed by a nucleon, one at a time, via the gradient coupling term in Eq. (3), are calculated in a similar manner. They are discussed in Appendix A and shown not to affect our saturation considerations because of their angular properties. It is, of course, not consistent to introduce them here since we have neglected the analogous term in the two-body analysis, $V_4^{(c)}$ in Eq. (4), in order to fit the data on the low-energy 2-nucleon system.

The general n-body interaction potential for $n\leqslant 5$ has been deduced directly as above and expressed as a sum over the $(n-1)!/2$ possible perimeters of a K_1 function of a perimeter divided by the product of the n legs forming the perimeter. An argument extending this result for all n is given in Appendix (B). The expression obtained is

$$V^{n\text{-body}}=-(3\mu/\pi)(-2\lambda)^n\sum_{i=1}^{(n-1)!/2}\frac{K_1(p_i)}{l_{i1}l_{i2}\cdots l_{in}},\qquad(7)$$

with p_i equal to μ times the ith perimeter, and $l_{i\alpha}$ equal to μ times the αth leg of the ith perimeter. Thus, for $n=4$, there are three possible perimeters, as illustrated in Fig. 4 together with the corresponding graphs. We notice that the even-body forces are attractive and the odd ones are repulsive.[25]

Wentzel[10] has given a very elegant procedure for obtaining the n-body interactions with the scalar pair theory, in which he takes the limit of fixed sources but makes no assumptions as to the magnitude of the coupling parameter. For interparticle distance greater than the radius of the Lévy-Jastrow[26] repulsive core, his results are of the same form as Eq. (7), but have

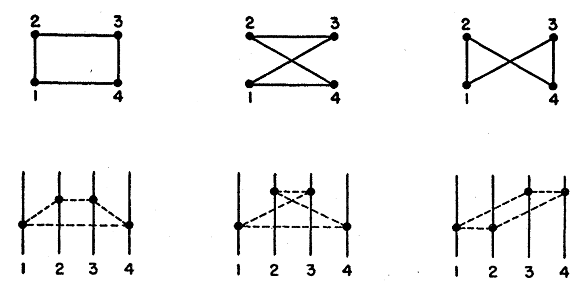


FIG. 4. The three distinct perimeters of a four-body configuration and their corresponding Feynman graphs for interactions via the pair term.

[24] Brueckner, Gell-Mann, and Goldberger, Phys. Rev. 90, 476 (1953).

[25] This sign alternation has its formal origin in our calculations in the fact that the n-body potential is obtained by nth-order perturbation theory, with each of the $(n-1)$ energy denominators negative in sign.

[26] R. Jastrow, Phys. Rev. 79, 389 (1950); 81, 165 (1951).

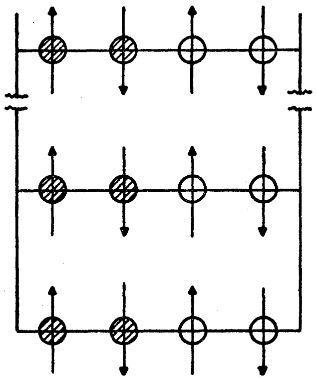


FIG. 5. Schematic picture of filled energy levels for a nucleus containing equal numbers of protons and neutrons, with one-half of each having spin up and spin down.

smaller effective coupling constants which reduce to the results of Eq. (7) in the perturbation limit of his calculations.

For want of a rigorous procedure, we accept the semiphenomenological, semifield theoretic potentials of Eqs. (4), (5), (6), and (7) as describing the interactions of nucleons.

IV. ENERGIES

In this section we apply the above model to a calculation of potential and kinetic energies as a function of the density of nuclear matter. Our initial goal is to establish that the three-body repulsion, Eq. (6), with Lévy's constants, is qualitatively sufficient to prevent collapse of nuclear matter to very high densities, and that an expansion in n-body forces, as given by Eq. (7), converges in the sense that, for normal nuclear densities, all n-body interactions with $n>5$ contribute negligibly.

To this end we calculate first the average value of the potential operator,

$$\langle V\rangle=\int\Psi^*V\Psi d\mathbf{r}_1\cdots d\mathbf{r}_A\bigg/\int\Psi^*\Psi d\mathbf{r}_1\cdots d\mathbf{r}_A, \quad (8)$$

where Ψ is the ground-state wave function for a nucleus of mass number A and is taken to be a Slater determinant of free-particle states satisfying periodic boundary conditions[27] in a box of radius $R=\eta R_s$ and volume $v=(4\pi/3)R^3$, multiplied by a symmetric function of the position coordinates of the A particles,

$$\Psi=\frac{1}{(A!)^{\frac{1}{2}}}\begin{vmatrix}\varphi_1(1)\cdots\varphi_1(A)\\ \cdot\quad\cdot\quad\cdot\quad\cdot\\ \varphi_A(1)\cdots\varphi_A(A)\end{vmatrix}\theta(\mathbf{r}_1\cdots\mathbf{r}_A), \quad (9)$$

with

$$\varphi_i(j)=v^{-\frac{1}{2}}e^{i\mathbf{k}_i\cdot\mathbf{r}_j}\chi_i(\sigma_j)\nu_i(\tau_j). \quad (10)$$

$\chi_i(\sigma_j)$ is a two-row spinor fixing the jth spin state of the ith particle. For spin up, it is $\begin{pmatrix}1\\0\end{pmatrix}$ and for spin down $\begin{pmatrix}0\\1\end{pmatrix}$. $\nu_i(\tau_j)$ is the analogous quantity fixing the nucleon charge in isotopic spin space. $\theta(\mathbf{r}_1\cdots\mathbf{r}_A)$ is an arbitrary symmetric function of the A nucleon position coordinates. We introduce position correlations between the A particles in order to satisfy the boundary condition that the wave function vanish whenever any two nucleons approach within $r_c=0.38/\mu$ of one another. This boundary condition expresses the repulsive core nature of the interaction potentials. We consider in these calculations a special form of $\theta(\mathbf{r}_1\cdots\mathbf{r}_A)$ that takes into account pairwise correlations of the nucleons only.[19] This form is

$$\theta(\mathbf{r}_1\cdots\mathbf{r}_A)=\sum_{i<j=1}^{A}f(r_{ij}), \quad (11)$$

with $r_{ij}\equiv|\mathbf{r}_i-\mathbf{r}_j|$. In order to attain the initial goal specified at the beginning of this section, we choose first for $f(r_{ij})$ a simple step function,

$$f(r_{ij})=\begin{cases}1, & r_{ij}>r_c\\ 0, & r_{ij}<r_c.\end{cases} \quad (12)$$

A variational calculation of the nuclear energy based on a two parameter trial form for $f(r_{ij})$ is presented in Sec. V.

Three assumptions underlie our calculation of the nuclear potential energy, Eq. (8), with the wave function expressed in Eqs. (9) to (12). Firstly, we neglect nuclear surface effects and calculate the "volume" potential energy of the nucleons. Formally, this implies treating the nucleus as a region of average particle density $A/v=(3/4\pi)\mu^3/\eta^3$ extending over all space. With this assumption the tensor interaction term in Eq. (4) and the angle dependent three-body potentials as deduced in Appendix A average to zero and do not influence our results. Secondly, we neglect Coulomb interactions between protons. The nucleus is then treated as composed of $A/2$ protons and $A/2$ neutrons, uniformly distributed, and equal numbers of each having spin up and spin down. For $A\gg1$, a negligible error is introduced in the cases of those nuclei with extra nucleons outside of the filled levels indicated in Fig. 5. Thirdly, we neglect the influence of the correlation function, Eqs. (11) and (12) on the orthogonality properties of the free-particle states. In the absence of a repulsive core ($r_c\rightarrow0$), the free-particle states, Eq. (10), are orthogonal in the nuclear "well." For repulsive cores of volume small in comparison with the average volume per nucleon in nuclear matter we expect the deviations of these states from orthogonality to be unimportant. Corrections resulting from such deviations are here neglected. An argument justifying their neglect is presented in Appendix C wherein are also discussed the analogous normalization corrections.

Two-Body Forces

For two-body interactions, Eq. (8) reduces in the standard way, with the above approximations, to[28]

[27] The effect of various boundary conditions on the kinetic energy has been studied by G. M. Volkoff, Phys. Rev. 62, 126 (1942).

[28] See Rosenfeld's book, reference 3; Chap. XI.

$$\langle V^{2\text{-body}}\rangle=\binom{A}{2}\frac{(A-2)!}{A!}\sum_{i,j=1}^{A}\int d\mathbf{r}_1 d\mathbf{r}_2 \varphi_i^*(1)\varphi_j^*(2)\times V(12)\begin{vmatrix}\varphi_i(1) & \varphi_i(2)\\ \varphi_j(1) & \varphi_j(2)\end{vmatrix} f^2(r_{12}). \quad (13)$$

The factor multiplying the integral expresses the number of interacting pairs among A nucleons times the number of possible state assignments (permutations) of the remaining $(A-2)$ particles, for each interacting pair, divided by the normalization constant appearing in Eq. (9).

In order to carry out products of spin and isotopic spin matrices in Eq. (13), we consider separately the two relevant terms of Eqs. (4) and (5). In the case of $V_4^{(a)}$, which is a spin and charge independent Wigner attraction, we obtain,[1,29] with neglect of $1\ll A$,

$$\begin{aligned}\sum_{i,j=1}\varphi_i^*(1)\varphi_j^*(2)[\varphi_i(1)\varphi_j(2)-\varphi_j(1)\varphi_i(2)]\\ &=v^{-2}\{A^2-\tfrac{1}{4}\sum_{i,j}e^{-i(\mathbf{k}_i-\mathbf{k}_j)\cdot(\mathbf{r}_1-\mathbf{r}_2)}\}\\ &=(A/v)^2\{1-\tfrac{1}{4}[3j_1(k_M r_{12})/(k_M r_{12})]^2\},\end{aligned} \quad (14)$$

where $j_1(x)$ is a spherical Bessel function, and $k_M=1.52\mu/\eta$ is the momentum of the highest filled energy level for the free-particle states in the nuclear well. The factor $\frac{1}{4}$ multiplying the exchange density term expresses the fact that, to leading order in $A\gg1$, $\frac{5}{8}$ of the pairs are antisymmetric in their space coordinates, and $\frac{3}{8}$ are symmetric. This factor results directly in the sum in Eq. (14), since states with different spin and/or isotopic spin are mutually orthogonal, and one-fourth of the nucleons are in each group of spin-up protons, spin-down protons, spin-up neutrons, and spin-down neutrons. For the central part of V_2, we have

$$\begin{aligned}\sum_{i,j=1}^{A}\varphi_i^*(1)\varphi_j^*(2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\\ \times[\varphi_i(1)\varphi_j(2)-\varphi_j(1)\varphi_i(2)]\\ &=v^{-2}\{0-(9/4)\sum_{i,j}e^{-i(\mathbf{k}_i-\mathbf{k}_j)\cdot(\mathbf{r}_1-\mathbf{r}_2)}\}\\ &=-(9/4)(A/v)^2[3j_1(k_M r_{12})/(k_M r_{12})]^2.\end{aligned} \quad (15)$$

The expression then for the potential energy of nuclear matter resulting from two-body interactions is

$$\langle V^{2\text{-body}}\rangle=(3/8\pi)A(\mu/\eta)^3\int_{r>r_c}d\mathbf{r}\times\left\{-(6/\pi)\lambda^2\frac{K_1(2\mu r)}{\mu r^2}[1-\tfrac{1}{4}D^2(k_M r)]-\tfrac{3}{4}\lambda(\mu/2M)(e^{-\mu r}/r)D^2(k_M r)\right\}, \quad (16)$$

where we introduce the notation

$$D(k_M r)\equiv 3j_1(k_M r)/k_M r,$$

and change coordinates in accord with the stated approximations,

$$\int d\mathbf{r}_1 d\mathbf{r}_2 w(r_{12})f^2(r_{12})=v\int_{r>r_c}d\mathbf{r}w(r). \quad (17)$$

Introducing dimensionless variables $x=\mu r$, $\alpha=k_M/\mu$, and $b=\mu r_c$, and integrating the classical density term, we obtain

$$\langle V^{2\text{-body}}\rangle=-\left(\frac{3A\mu}{2\eta^3}\right)\left\{\left(\frac{6}{\pi}\right)\lambda^2\left[\tfrac{1}{2}K_0(2b)-\tfrac{1}{4}\int_b^\infty dx K_1(2x)D^2(\alpha x)\right]+\tfrac{3}{4}\lambda\left(\frac{\mu}{2M}\right)\int_b^\infty dx\, x e^{-x}D^2(\alpha x)\right\}. \quad (18)$$

A graph of this expression for the two-body potential energy appears in Fig. 6, where we have used Lévy's parameters $G^2/4\pi=10$, and $r_c\mu=b=0.38$. At normal nuclear density $(\eta=1)$, the exchange density term in the Wigner interaction potential, $V_4^{(a)}$, reduces the contribution from the classical term by 19 percent.

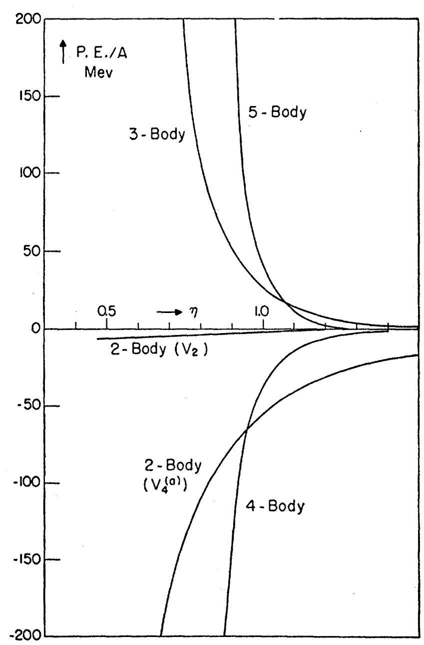


FIG. 6. Potential energy contributions, plotted as a function of η, as calculated from Eq. (18) for two-body interactions ($V_4^{(a)}$ and V_2), from Eq. (21) for three-body interactions, from Eq. (26) for four-body interactions, and from the corresponding expression for five-body interactions.

[29] E. P. Wigner and F. Seitz, Phys. Rev. **43**, 804 (1933); E. Feenberg, Phys. Rev. **60**, 204 (1941).

Three-Body Forces

For three-body interactions, Eq. (8) reduces to

$$\langle V^{3\text{-body}}\rangle=\binom{A}{3}\frac{(A-3)!}{A!}$$
$$\times\sum_{i,j,k=1}^{A}\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \varphi_i^*(1)\varphi_j^*(2)\varphi_k^*(3)V(123)$$
$$\cdot\begin{vmatrix}\varphi_i(1) & \varphi_i(2) & \varphi_i(3)\\ \varphi_j(1) & \varphi_j(2) & \varphi_j(3)\\ \varphi_k(1) & \varphi_k(2) & \varphi_k(3)\end{vmatrix} f^2(r_{12})f^2(r_{13})f^2(r_{23}), \quad (19)$$

in analogy with Eq. (13). For Wigner-type three-body interactions as in Eq. (6), the spin and isotopic spin matrix products may be worked out directly, giving

$$\sum_{i,j,k=1}^{A}\varphi_i^*(1)\varphi_j^*(2)\varphi_k^*(3)\begin{vmatrix}\varphi_i(1) & \varphi_i(2) & \varphi_i(3)\\ \varphi_j(1) & \varphi_j(2) & \varphi_j(3)\\ \varphi_k(1) & \varphi_k(2) & \varphi_k(3)\end{vmatrix}$$
$$=(A/v)^3\{1-\tfrac{1}{4}D^2(k_Mr_{12})-\tfrac{1}{4}D^2(k_Mr_{23})-\tfrac{1}{4}D^2(k_Mr_{31})$$
$$+(2/16)D(k_Mr_{12})D(k_Mr_{23})D(k_Mr_{31})\}. \quad (20)$$

The first four terms in Eq. (20) represent the classical density and the double exchange density for the interchange of any two of the interacting trio. The last term represents the two symmetric permutations of the three nucleons among themselves, the factor $\frac{1}{16}$ expressing the relative probability that all three of them have the same spin-isotopic spin quantum numbers, just as in the discussion for two-body interactions. The integrations in the above expressions are simplified by noting that

$$\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 w(r_{12},r_{23},r_{31})f^2(r_{12})f^2(r_{23})f^2(r_{31})$$
$$=v\int_{(r_{12},r_{23},r_{31})>r_c} d\mathbf{r}_{12}d\mathbf{r}_{13}w(r_{12},r_{23},r_{31})$$
$$=8\pi^2 v\int_{r_c}^{\infty}r_{12}dr_{12}\int_{r_c}^{\infty}r_{13}dr_{13}$$
$$\times\int_{>\{r_c,\,|r_{12}-r_{13}|\}}^{r_{12}+r_{13}} r_{23}dr_{23}w(r_{12},r_{23},r_{31}),$$

in accord with our stated approximations. Utilizing the symmetry in the three-particle separations r_{12}, r_{23}, and r_{31}, we obtain, in dimensionless form

$$\langle V^{3\text{-body}}\rangle=(3A\mu/4\eta^6)(24/\pi)\lambda^3$$
$$\times\int_b^{\infty}dx\int_b^{\infty}dy\int_{>\{b,\,|x-y|\}}^{x+y}dzK_1(x+y+z)$$
$$\cdot\{1-\tfrac{3}{4}D^2(\alpha x)+\tfrac{1}{8}D(\alpha x)D(\alpha y)D(\alpha z)\}\cdot \quad (21)$$

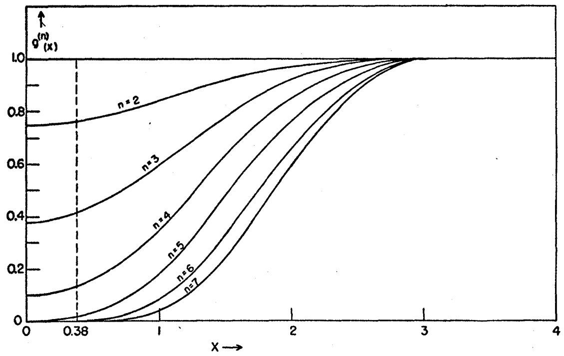


FIG. 7. The relative probability for n nucleons to be separated by a distance $x=\mu r$, for $n=2$ through 7. These curves are calculated for normal nuclear density $\eta=1$.

The following integral representation,[30] which decomposes a K_1 function of a sum of arguments into a product of factors of each argument, facilitates analysis of the integrals in Eq. (21):

$$K_1(a_1+a_2+\cdots+a_n)$$
$$=\int_1^{\infty}\frac{tdt}{(t^2-1)^{\frac{1}{2}}}e^{-a_1t}e^{-a_2t}\cdots e^{-a_nt}. \quad (22)$$

There results for the classical term

$$\int_b^{\infty}dx\int_b^{\infty}dy\int_{>\{b,\,|x-y|\}}^{x+y}dzK_1(x+y+z)$$
$$=\int_1^{\infty}\frac{tdt}{(t^2-1)^{\frac{1}{2}}}\frac{e^{-3bt}}{t^3}(1-\tfrac{3}{4}e^{-bt})$$
$$=0.121 \quad (\text{for } b=0.38). \quad (23)$$

The double exchange term reduces to

$$-\tfrac{3}{4}\int_b^{\infty}dx\int_b^{\infty}dy\int_{>\{b,\,|x-y|\}}^{x+y}dzK_1(x+y+z)D^2(\alpha x)$$
$$=-\frac{3\pi}{8}\left\{\int_b^{\infty}D^2(\alpha x)P(x)dx-\int_{2b}^{\infty}D^2(\alpha x)Q(x)dx\right\},$$

with

$$P(x)\equiv F(x+2b)-F(2x+2b),$$
$$Q(x)\equiv F(x+2b)-F(2x)+(2b-x)(2/\pi)K_0(2x),$$
$$F(x)\equiv(2/\pi)\int_x^{\infty}K_0(y)dy.$$

At normal nuclear density, it subtracts away 42 percent of the classical term. The triple exchange term can be evaluated only by approximate numerical procedures[31]

[30] W. Magnus and F. Obberhettinger, *Special Functions of Mathematical Physics* (Chelsea, New York, 1949).

[31] A more detailed discussion is presented in K. Huang, Ph.D. thesis, Massachusetts Institute of Technology, 1953 (unpublished).

and is found to equal 6 percent of the classical term for $\eta=1$. Antisymmetrization of the wave function thus reduces the three-body interaction energy by 36 percent at normal nuclear density; correspondingly, the two-body energy is reduced by 19 percent. The calculated results for Eq. (21) are plotted in Fig. 6 as a function of η.

Four- and Five-Body Forces

Increasing complexities enter into the calculations of higher-body forces. Additional exchange density terms appear and also the available integration volume for interacting hard spheres becomes more difficult to express and handle. The physical effect of the additional exchange terms resulting from the Pauli principle is to decrease the relative probability of finding many nucleons close to one another. We show this by calculating the probability distribution for n particles,

$$g^{(n)}(\mathbf{r}_1\cdots\mathbf{r}_n)=v^n\int d\mathbf{r}_{n+1}\cdots d\mathbf{r}_A\Psi^*\Psi\bigg/\int d\mathbf{r}_1\cdots d\mathbf{r}_A\Psi^*\Psi. \quad (24)$$

For two nucleons, $g^{(2)}(r_{12})$ is given in Eq. (14), multiplied there by $(A/v)^2$. The expression for $g^{(3)}(r_{12}, r_{23}, r_{31})$ appears in Eq. (20), multiplied there by $(A/v)^3$. The corresponding $g^{(n)}$ for higher n values are deduced from Eq. (24) in precisely the same manner. The result for $n=4$ is given below, with $a_{ij}\equiv k_M r_{ij}$,

$$\begin{aligned}g^{(4)}=1&-\tfrac{1}{4}\sum_{i<j=1}^{4}D^2(a_{ij})\\&+(2/16)\sum_{i<j<p=1}^{4}D(a_{ij})D(a_{jp})D(a_{pi})\\&-(2/64)\{D(a_{12})D(a_{23})D(a_{34})D(a_{41})\\&+D(a_{13})D(a_{34})D(a_{42})D(a_{21})\\&+D(a_{13})D(a_{32})D(a_{24})D(a_{41})\}\\&+(1/16)\{D^2(a_{12})D^2(a_{34})+D^2(a_{13})D^2(a_{24})\\&\qquad+D^2(a_{14})D^2(a_{23})\}. \quad (25)\end{aligned}$$

In Fig. 7 we graph the probability distributions $g^{(n)}(r)$, for $n=2$ through $n=7$, with all arguments $r_{ij}\equiv r\equiv x/\mu$. For $n=3$ this corresponds to arranging the three nucleons in an equilateral triangle of side r; for $n=4$, they form an equilateral tetrahedron. For $n>4$, there is no geometrical configuration with all $r_{ij}=r$. Figure 7 just indicates that the relative probability for finding nucleons close to one another decreases with increasing n. We see from Eq. (25) that $g^{(4)}$ reduces to $\frac{3}{32}$ in the limit of zero interparticle distances. All $g^{(n)}$ for $n>4$ vanish in this limit, in consequence of the exclusion principle and of the fact that there are but four spin and charge states for nucleons. We see from Fig. 7 that, for $n\geqslant 4$ the effective "Pauli repulsion" is of greater "range" and importance than the repulsive potential core.[32]

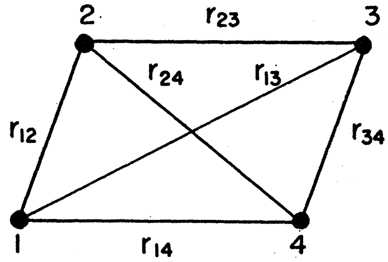


FIG. 8. Notation for four-body interaction integrals.

There is an additional factor weighting against close approach of n nucleons in the energy integrals. In going from the n to the $(n+1)$ body interactions, we see by Eq. (7) that the singularity in the potential for zero-particle separations increases by but one power of r, whereas the additional nucleon introduces one more volume element $d\mathbf{r}\propto r^2dr$ into the integral. Thus, for example, in the absence of a repulsive core, the two-body energy diverges, whereas the energies for all $n>2$ are finite.

On the basis of these observations we can make the calculation of four- and five-body interaction energies tractable by neglecting the repulsive potential cores and thereby vastly simplifying the integrals. We obtain then for $n=4$, by Eqs. (7)–(12), (25), and by symmetry of the potential in the coordinates,

$$\langle V^{4\text{-body}}\rangle=-\lambda^4\frac{48\mu}{\pi}\frac{3A^4}{4!v^4}\int d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3d\mathbf{r}_4\times\frac{K_1[\mu(r_{12}+r_{23}+r_{34}+r_{41})]}{\mu^4r_{12}r_{23}r_{34}r_{41}}g^{(4)}. \quad (26)$$

We can carry through the above integrals analytically for the first two terms in Eq. (25), corresponding to the classical density and the double exchange density describing the interchange of any pair. To do this, we use Eq. (22) and note that (see Fig. 8) for integrands that do not depend on r_{24},

$$\begin{aligned}\int d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3d\mathbf{r}_4&=v\int d\mathbf{r}_{12}d\mathbf{r}_{23}d\mathbf{r}_{14}\\&\rightarrow 2v(2\pi)^3\int_0^\infty r_{12}dr_{12}\int_0^\infty r_{23}dr_{23}\int_0^\infty r_{14}dr_{14}\\&\quad\times\int_{|r_{12}-r_{23}|}^{r_{12}+r_{23}}dr_{13}\int_{|r_{13}-r_{14}|}^{r_{13}+r_{14}}r_{34}dr_{34}. \quad (27)\end{aligned}$$

The resulting contribution to $\langle V^{4\text{-body}}\rangle$ of the first two

[32] We wish to thank Dr. E. P. Gross for an illuminating discussion on this point.

terms of $g^{(4)}$, labeled U^{4b}, is

$$U^{4b}=-\lambda^4\frac{81A\mu}{2\pi\eta^9}\int_1^\infty\frac{tdt}{(t^2-1)^{\frac{1}{2}}}\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz$$

$$\times\int_{|x-y|}^{x+y}dw\int_{|w-z|}^{w+z}du\,e^{-(x+y+z+u)t}\left[1-\tfrac{3}{2}D^2(\alpha x)\right]$$

$$=-\lambda^4\frac{27A\mu}{\pi\eta^9}\int_1^\infty\frac{tdt}{t^3(t^2-1)^{\frac{1}{2}}}\int_0^\infty dx$$

$$\times\left(\tfrac{1}{2}xt+x^2t^2\right)e^{-2xt}\left[1-\tfrac{3}{2}D^2(\alpha x)\right]$$

$$=-\lambda^4\frac{27A\mu}{4\pi\eta^9}\Bigg\{1+\frac{27(1-\alpha^2)}{4\alpha^4}$$

$$-\frac{27}{2\alpha^3}(1+\alpha^2)^{\frac{1}{2}}\tanh^{-1}\left(\frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\right)$$

$$+\frac{27}{4\alpha^6}(1+2\alpha^2)\left[\tanh^{-1}\left(\frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\right)\right]^2\Bigg\}.$$

For $\eta=1$, the double exchange term subtracts 83.3 percent of the classical term. The third term in $g^{(4)}$, describing symmetric permutations of three interacting nucleons, can be integrated after making exponential fits to the D functions,[31] and it yields 21 percent of the classical density term for $\eta=1$. The remaining, or "quartet," exchange terms present further difficulties, in that some of them involve both diagonals r_{13} and r_{24} of Fig. 8 simultaneously so that the reduction to Eq. (27) cannot be completely effected. It has proved possible to calculate these terms only approximately. However, their contribution is small,[31] being about 3.4 percent of the classical term for $\eta=1$. The net result of the exchange terms for four-body interactions is a reduction of the classical term by 59 percent at $\eta=1$.

The calculation of the five-body energy proceeds along a parallel if more arduous path. The exchange densities give a 76 percent reduction in this case for $\eta=1$.

The potential energies for up through $n=5$ body interactions are presented in Figs. 6 and 9 as functions of the nuclear radius. In analyzing these curves, we observe the following: For $\eta\gtrsim 1.5$, corresponding to nuclear densities of less than 30 percent of normal, the potential energy is due entirely to two-body interactions. At such densities it is relatively improbable to have three (or more) nucleons in a cluster of perimeter $\sim 1/\mu$ so that the many-body interactions contribute very little to the energy. In the region $1.3\lesssim\eta\lesssim 1.5$, three-body forces become important also. On the basis of two- and three-body forces alone, the potential energy per nucleon has a minimum of -30 Mev at $\eta=1.0$. When four- and five-body interactions are included, the minimum alters but slightly to -32 Mev at $\eta=1.1$.

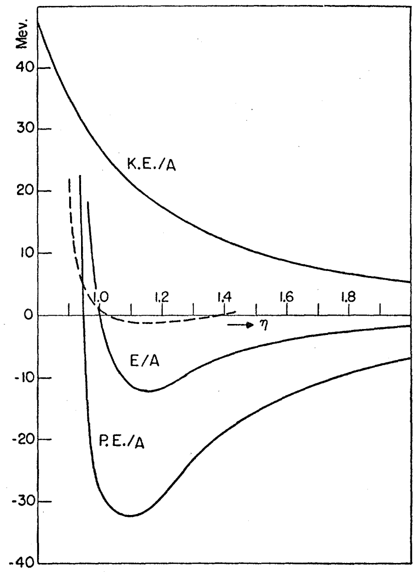


FIG. 9. Potential, kinetic, and total energy per nucleon as a function of η. The potential energy includes up through five-body interactions and is calculated for $G^2/4\pi=10$ and $r_c=0.38/\mu$. The kinetic energy is obtained from Eq. (29) expressing the Fermi plus Lenz contributions. The total energy has a minimum of depth 12 Mev per nucleon at $\eta=1.15$. The dashed curve is the total energy per nucleon calculated with $G^2/4\pi=15$ for the V_2 interaction, and $G^2/4\pi=7.5$ for all other interactions. It has a minimum of depth 2 Mev per nucleon at $\eta=1.15$.

Our aim in this calculation of the four- and five-body interaction energies has been only to demonstrate that their effect is quite small at normal nuclear density. We attach no significance to their numerical contributions as we have found it necessary to approximate in a relatively rough manner the "quartet" and "quintet" exchange terms and to neglect the repulsive potential cores. This neglect of the repulsive cores overestimates both of their contributions, and in particular, overestimates the four-body contribution more, as is evident from Fig. 7.

With regard to the higher-body forces, one can adopt either of two attitudes. In that the basis for the potentials used in these calculations is semiphenomenological, we can limit ourselves entirely to two- and three-body forces. In that the basis is semifield theoretic, however, we can accept and investigate all higher-body forces as appearing in Eq. (7). We have already seen that the four- and five-body interactions affect the energy minimum but slightly and indicate a rapid convergence for the series of n-body interactions for nuclear radii $\eta\gtrsim 1$. We can also see from Fig. 6 that this series oscillates wildly for $\eta\lesssim 1$. However, the contributions of two- to A-body forces can be summed approximately for high densities ($\eta\ll 1$). Wentzel[10] has deduced this directly with the meson scalar pair theory, which gives potentials of the form Eq. (7). In his calculation nucleons are treated as fixed sources uniformly distributed within an infinite nucleus. The repulsive potential core is simulated in Wentzel's calculation by a high-momentum cutoff for the meson field, given by the reciprocal Compton wavelength of the nucleon. In that the nucleons are treated as static sources, the Pauli principle does not operate. For a nucleus in the collapsed state ($\eta \ll 1$), however, it is well known that the exchange density contributions to the energy vanish. Thus we see by Eq. (15) that the interaction energy for a $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$ potential vanishes in the collapsed state. Whereas the exchange density reductions we calculated for the two- and three-body Wigner interactions were, respectively, 19 percent and 36 percent at $\eta = 1$, they are only 9.5 percent and 15 percent at $\eta = 0.5$.

In Appendix B, we outline a procedure for summing up the energy contributions with potentials of the form of Eq. (7) and with neglect of the Pauli exchange terms. The point to be made here is that the energy is positive and increasing with decreasing η, for $\eta \ll 1$. We can conclude that $n > 3$ body forces given by Eq. (7) are relatively unimportant at the potential energy minimum deduced from consideration of two- and three-body forces. For high nuclear densities, they contribute significantly and operate so as to oppose collapse. However, for moderately high density, i.e., $\eta \lesssim 1$, both the Pauli exchange terms and the higher-body forces are important. In this region we can only accept the indication of our calculations including the five-body forces that the potential energy curve is monotonically rising with decreasing η.

Kinetic Energy

Having calculated the potential energy of the neutrons and protons in the nucleus, we come next to a discussion of their kinetic energy. The sum of these two quantities gives us then the total energy.

The well-known Fermi kinetic energy for a gas of A noninteracting point nucleons ($A/2$ protons and $A/2$ neutrons) of mass M confined to a spherical box of radius R is[1]

$$T_F = 0.695 A^{5/3}/MR^2 = 14.7A/\eta^2 \text{ Mev.} \tag{28}$$

For the nuclear model under discussion here, the kinetic energy is greater than the Fermi energy because of the additional curvature introduced in the nucleon wave functions by the repulsive core boundary condition. This additional kinetic energy was calculated by Lenz[16] who considered the problem of noninteracting hard spheres. With the approximation of hard sphere radius small in comparison with interparticle separation Lenz obtained[33]

$$T = T_F\left(1 + \frac{2.16b}{\eta}\right) = 14.7A\left(\frac{1}{\eta^2} + \frac{0.82}{\eta^3}\right) \text{ Mev.} \tag{29}$$

Lenz's result, and in particular its density variation, can be qualitatively understood as follows. The kinetic energy for a point nucleon of wavelength $\lambda = R/n$ is $2\pi^2 n^2/MR^2$. For a nucleon of radius r_c, the wavelength shortens to $(R - r_c)/n$, the numerator here representing the available linear dimension after exclusion of the interior of the nucleon. For a nucleus of A nucleons, with $A^{\frac{1}{3}}$ along a diameter, the wavelength reduces to $(R - A^{\frac{1}{3}} r_c)/n = (R/n)(1 - b/\eta)$, and the kinetic energy increases to $T_F(1 - b/\eta)^{-2} \approx T_F(1 + 2b/\eta)$. We remark here that comparison of Eqs. (18) and (29) readily shows that the nucleus would tend to collapse under the influence of two-body forces alone. The coefficient of the leading η^{-3} term is negative with $G^2/4\pi = 10$ and $b = 0.38$. If we add the kinetic energy of Lenz, Eq. (29) to the potential energy, we obtain Fig. 9 for the total nuclear energy as a function of η. The curve shows a minimum at $\eta = 1.15$ corresponding to a binding energy of 12 Mev per nucleon. The dotted curve indicated in the same figure is calculated for an alternative choice of the coupling constants which Jastrow[34] has found to give an equally satisfactory account of low-energy two-body interactions. We obtain the dotted curve if we set $G^2/4\pi = 15\epsilon$, where $\epsilon = 1$ for the two-body $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$ interaction, and $\epsilon = 0.5$ for all other terms; $b = 0.38$ is used as a compromise between the two triplet and singlet repulsive core radii proposed by Jastrow. With these parameters the stable radius is given by $\eta = 1.15$, and the binding energy minimum corresponds to 2 Mev per nucleon.

In summary, the qualitative deductions of this section are twofold. A three-body repulsion of the form in Eq. (6), as suggested by pseudoscalar meson theory, prevents nuclear collapse and indicates nuclear saturation properties in accord with experience. The expansion in a series of many-body interactions rapidly converges for normal nuclear densities. This convergence is due chiefly to the Pauli exclusion principle, which serves to inhibit close approach of many nucleons.

Finally, we note just briefly that a system of nucleons interacting through the potentials discussed in this section is stable against lining up of the spins of the particles. This is because the kinetic energy increases by a factor of roughly four for the state in which all spins are parallel, whereas the change in the potential energy is considerably smaller since the dominating interactions are Wigner potentials.

V. VARIATIONAL CALCULATION

In this section we present a brief account of a variational calculation of the total energy based on more realistic wave functions. With saturation and convergence of the n-body expansion qualitatively established on the basis of our previous calculations, it is desirable to deduce more quantitatively significant numbers for the saturation radius and binding energy.

[33] An interpolation formula has been given by F. London, Proc. Roy. Soc. (London) **153**, 576 (1935), which is applicable for arbitrary interparticle separations. Applied to our case it yields a 12 percent increase above Lenz's result.

[34] R. Jastrow, *Proceedings of the Third Annual Rochester Conference* (Interscience Publishers, Inc., New York, 1953), p. 77.

Aside from this, however, there is an additional problem of considerable importance on which a variational calculation can shed some light. This is the degree of position correlation between the nucleons, as expressed by $f(r_{ij})$, which corresponds to the minimum value of the total energy. The great success of the shell theory based on the independent-particle model of the nucleus argues against strong correlations. The results of this calculation support this picture.

A two-parameter continuous trial form is introduced for the correlation function:

$$f_T(r)=\begin{cases}(1-r_c/r)[1+ae^{-\beta\mu(r-r_c)}], & r>r_c;\\ 0, & r<r_c.\end{cases} \tag{30}$$

The parameters a and β are varied and the energy minimum determined. The radial dependence of f_T^2 is shown in Fig. 10 for various values of (a,β). We see for $(a,\beta)=(2.5, 1)$ that there is a strong positive correlation between pairs, whereas for $a=0$, the probability of forming a pair is low. For parameters in the range $(a,\beta)=(1, 2)$ the correlation is weak; *viz*, the nucleon wave functions vanish at each other's core, but exhibit no strong tendence to cluster with or to avoid their neighbors, for $r>r_c$.

The potential energies for two- and three-body interactions have been calculated with Eq. (30). For two-body interactions this amounts to recalculating Eq. (16) with

$$\int_{r>r_c}^{\infty} d\mathbf{r}\rightarrow\int d\mathbf{r} f_T^2(r).$$

For three-body interactions, Eq. (21) is altered by

$$\int_b^\infty dx\int_b^\infty dy\int_{>\{b,|x-y|\}}^{x+y}dz$$
$$\rightarrow\int_b^\infty dx f_T^2(x)\int_b^\infty dy f_T^2(y)\int_{>\{b,\,|x-y|\}}^{x+y}dz f_T^2(z).$$

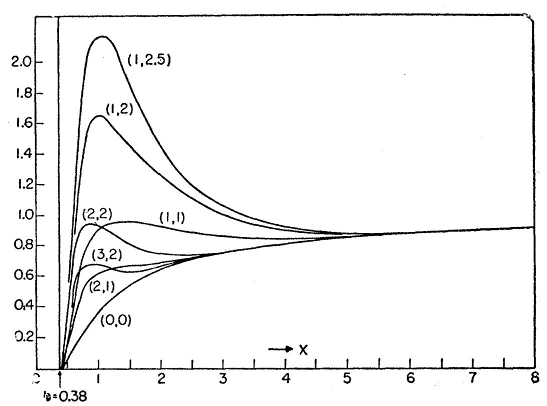


FIG. 10. The square of the correlation function of Eq. (30) plotted as a function of $x=\mu r$ for different sets of parameters a and β. The individual curves are labeled by the corresponding pair (β, a).

These integrations are then carried through numerically.

For the kinetic energy we calculate directly

$$T=-\frac{1}{2M}\frac{\sum_{i=1}^{A}\int\Psi^*\Delta_i\Psi d\mathbf{r}_1\cdots d\mathbf{r}_A}{\int\Psi^*\Psi d\mathbf{r}_1\cdots d\mathbf{r}_A}$$

$$=T_F-(A^2/2Mv)\int d\mathbf{r} f_T(r)\Delta f_T(r)[1-\tfrac{1}{4}D^2(k_Mr)], \tag{31}$$

consistent with the approximations explicitly stated in the preceding section. The second term of Eq. (31) represents the increase in kinetic energy due to the position correlation of the nucleons.

The calculations with the trial form of Eq. (30) indicate the existence of an energy minimum for approximate parameter values $a=1$, $\beta=2$. The method of calculation has been to determine the energy as a function of η for various sets of (a,β). For $a=1$ and $\beta=2$, the energy has a minimum at $\eta=0.95$ corresponding to a binding energy of roughly 4 Mev per nucleon. Though this result can be presented only in numerical form, its physical implications are clear. The two-body attractive forces favor strong position correlation of nucleons, corresponding to large values of a and small values of β as indicated in Fig. 10. However, the three-body repulsive forces operate in the opposite direction, since their positive contribution to the nuclear energy decreases with smaller a and larger β. Also the kinetic energy opposes strong positive correlation which increases the wave function curvature. The kinetic energy increase expressed by Eq. (31) varies roughly in proportion to a^2/β. The compromise effected by these antipodal tendencies falls at $a=1$, $\beta=2$. The graph of f_T^2 in Fig. 10 shows that for this set of parameter values no marked correlation of nucleon coordinates is evident for r greater than the repulsive core radius. We interpret this result as evidence in favor of the independent particle model.

VI. CONCLUSION

To summarize, we have deduced many-body potentials from the pseudoscalar meson theory with pseudoscalar coupling. Guided by Lévy's success in fitting low-energy properties of the two-body system and by his parameters, we have kept only the potentials of leading order in an expansion in powers of $(G^2/4\pi)(\mu/2M)$. It is quite possible that further developments in field theory will show that the terms we have studied are of lesser importance than others neglected in our analysis or that there are different effective values of $G^2/4\pi$ for the many-body potentials we have studied. Indeed, Wentzel's[10] calculations with the scalar pair theory predict a considerable reduction in the strength of these terms.[24] At present, however, it is impossible to say to what extent a complete covariant calculation in the framework of the renormalization program will remove the damping found by Wentzel.

However, if we accept the potentials in Eqs. (4), (5), (6), and (7) as a semiphenomenological working basis for our calculations, we find that the many-body forces, and in particular the three-body repulsion, provide a satisfactory qualitative understanding of nuclear saturation. The effect of the Pauli exclusion principle in these calculations has been to reduce considerably the contributions of $n>3$ body Wigner forces for nuclei at normal density.

In that the binding energies are obtained as relatively small differences of larger potential and kinetic energy contributions, their numerical values are to be understood primarily as order of magnitude indications. However, the resulting saturation radii are more accurately determined in this work since the position of the energy minimum is fixed to a large extent by the minimum in the curve of the two- plus three-body potential energies. This is because the kinetic energy is a considerably smoother function of $\eta=R/R_s$ than is the potential energy. We can best illustrate this point by comparing the step function calculation of Sec. IV with the variational calculation of Sec. V. The continuous correlation form of Eq. (30) with $a=1$, $\beta=2$ reduces the potential energy contribution by a factor of roughly two in comparison with the step function result. This potential reduction serves to decrease the magnitude of the binding energy from 12 Mev to 4 Mev per nucleon while altering $\eta_{\min}$ from 1.15 to 0.95.

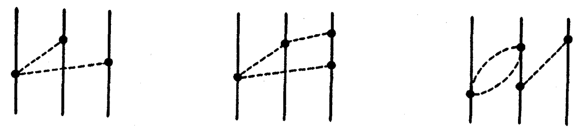
The energy minimum for the variational calculation is obtained with a trial form f_T showing no marked correlation of the nucleon coordinates. This result suggests that many-body repulsive forces may aid in harmonizing the shell theory and its underlying independent particle model with the meson theory and its prediction of strong internucleon interactions.

Further variational calculations are now in progress with more flexible trial forms. Energies deduced by such variational procedures must necessarily lie above the true eigenvalue so that one can in this way establish only an upper limit on the energy for a given choice of effective coupling constants for the n-body interactions. Better wave functions serve to lower the upper limit and to bring it closer to the true value.[35]

ACKNOWLEDGMENT

It is a pleasure to acknowledge our great indebtedness to Professor Victor F. Weisskopf for his initiating and continuing stimulus and contributions to this work.

[35] Additional information which may be culled from these calculations includes a minimal value for the ratio of the effective coupling constant for the three-body interaction to the two-body constant which is consistent with saturation.



(a) (b) (c)

FIG. 11. Feynman diagrams for three-body interactions.

APPENDIX A

We outline here the various contributions[36] to three-body interactions of orders $\lambda^2\mu/M$ and $\lambda^3\mu/M$. The Feynman graphs of the processes are drawn in Fig. 11.

The potentials are deduced from Eq. (3) in the adiabatic perturbation limit as discussed in Sec. III. All possible permutations and time orderings of process (a) contribute to the one pair interaction of order $\lambda^2\mu/M$. The result is

$$V_{3a}=\frac{G^4}{(2M)^3}\frac{2\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3}{(2\pi)^6}\int\int d\mathbf{p}d\mathbf{q}(\boldsymbol{\sigma}_2\cdot\nabla_2)(\boldsymbol{\sigma}_3\cdot\nabla_3)\frac{e^{i\mathbf{p}\cdot\mathbf{r}_{12}+i\mathbf{q}\cdot\mathbf{r}_{13}}}{\omega_p^2\omega_q^2}$$

$$+\text{symmetric terms in } (1,2) \text{ and } (1,3)$$

$$=\lambda^2(\mu^2/M)\bigg[(\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)\frac{(\boldsymbol{\sigma}_2\cdot\mathbf{r}_{12})(\boldsymbol{\sigma}_3\cdot\mathbf{r}_{13})}{r_{12}r_{13}}\left(1+\frac{1}{\mu r_{12}}\right)$$

$$\times\left(1+\frac{1}{\mu r_{13}}\right)\frac{e^{-\mu r_{12}}}{\mu r_{12}}\frac{e^{-\mu r_{13}}}{\mu r_{13}}+\text{symmetric terms}\bigg].$$

Two pair terms of the form illustrated by process (b) contribute an interaction of order $\lambda^3\mu/M$ which has the form

$$V_{3b}=\frac{G^6}{(2M)^4}\frac{12}{(2\pi)^9}\int\int\int d\mathbf{p}d\mathbf{q}d\mathbf{k}$$

$$\times(\nabla_{12}\cdot\nabla_{23}+\nabla_{13}\cdot\nabla_{32}+\nabla_{21}\cdot\nabla_{13})\frac{e^{i\mathbf{p}\cdot\mathbf{r}_{12}+i\mathbf{q}\cdot\mathbf{r}_{23}+i\mathbf{k}\cdot\mathbf{r}_{31}}}{\omega_p^2\omega_q^2\omega_k^2}$$

$$=\lambda^3(6\mu^2/M)(1/\mu^2)(\nabla_{12}\cdot\nabla_{23}+\nabla_{13}\cdot\nabla_{32}+\nabla_{21}\cdot\nabla_{13})$$

$$\times(e^{-\mu r_{12}}/\mu r_{12})(e^{-\mu r_{13}}/\mu r_{13})(e^{-\mu r_{23}}/\mu r_{23}).$$

Irreducible graphs of type c, a typical one of which appears in the time ordered Fig. 11(c), contribute a

[36] Similar potentials together with Eq. (7) for $n=3$ and 4 have been deduced independently by A. Klein [Phys. Rev. **90**, 1101 (1953)] in the framework of the Tamm-Dancoff formalism.

potential, of order $\lambda^3\mu/M$, which we write in the form

$$V_{3c}=\frac{G^6}{(2M)^4}\frac{3}{(2\pi)^9}(\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}_{23})(\boldsymbol{\sigma}_3\cdot\boldsymbol{\nabla}_{23})$$
$$\times\left\{\int d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\mathbf{r}_{23}}}{\omega_k^2}\int d\mathbf{q}d\mathbf{p}\frac{e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}_{12}}}{\omega_p\omega_q(\omega_p+\omega_q)^2}+\int d\mathbf{k}\frac{e^{i\mathbf{k}\cdot\mathbf{r}_{23}}}{\omega_k^3}\int\int d\mathbf{q}d\mathbf{p}\frac{e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}_{12}}}{\omega_p\omega_q(\omega_p+\omega_q)}\right\}$$
$$+\text{symmetric terms}$$
$$=\lambda^3(6\mu^2/\pi^2M)(1/\mu^2)(\boldsymbol{\tau}_2\cdot\boldsymbol{\tau}_3)(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}_{23})(\boldsymbol{\sigma}_3\cdot\boldsymbol{\nabla}_{23})$$
$$\times\left\{(e^{-\mu r_{23}}/\mu r_{23})(1/\mu^2r_{12}^2)\int_{\mu r_{12}}^{\infty}xK_0^2(x)dx+K_0(\mu r_{23})K_1(2\mu r_{12})/\mu^2r_{12}^2\right\}$$
$$+\text{symmetric terms.}$$

Interactions V_{3a} and V_{3b} average to zero in our nuclear model. Interaction V_{3c} reduces to two terms, a tensor interaction which averages to zero plus a $(\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b)(\boldsymbol{\tau}_a\cdot\boldsymbol{\tau}_b)$ term which was evaluated with the methods discussed in Sec. III and shown to give a three-body repulsive energy amounting to less than 5 percent of Eq. (21).

APPENDIX B

In the limit of high nuclear density, all exchange effects are negligible. The n-body probability density function $g^{(n)}$ then reduces to unity, and the n-body potential energy is given simply by the "classical term,"

$$\langle V^{n\text{-body}}\rangle\underset{\eta\to0}{\longrightarrow}C^{(n)},$$

$$C^{(n)}=(A/v)^n(1/n!)\int d\mathbf{r}_1\cdots d\mathbf{r}_nV^{n\text{-body}}\prod_{i<j}f^2(r_{ij}),\quad\text{(B1)}$$

where we replace $\binom{A}{n}$ by $A^n/n!$ for $A\to\infty$. The sum of $C^{(n)}$ from $n=3$ to ∞ can be carried out, if we neglect the repulsive core. (The two-body term is omitted because it diverges in the absence of core.) With this approximation we can then compare our result with that indicated by Wentzel.[10] The identity of these results then demonstrates the validity of $V^{n\text{-body}}$ deduced in Eq. (7).

We have evaluated this sum directly by establishing a recursion relation by means of term-by-term integration of Eq. (B1). We indicate here a more elegant procedure.[37]

Using Eq. (22), we can manipulate $C^{(n)}$ into the form

$$C^{(n)}=(-1)^{n+1}(2\mu/\pi)\frac{(3\lambda)^n}{n\eta^{3(n-1)}}\int_1^\infty\frac{tdt}{(t^2-1)^{\frac{1}{2}}}\times\int_{(n)0}\exp\{-(x_2+x_n+a_1+a_2+\cdots+a_n)t\},$$

where

$$x_k\equiv\mu|\mathbf{r}_k-\mathbf{r}_1|,\quad a_k\equiv\mu|\mathbf{r}_{k+2}-\mathbf{r}_{k+1}|.$$

The integration limits are given by

$$\int_{(n)0}=\int_0^\infty dx_2\int_0^\infty dx_3\cdots\times\int_0^\infty dx_n\int_{|x_2-x_3|}^{x_2+x_3}da_1\cdots\int_{|x_{n-1}-x_n|}^{x_{n-1}+x_n}da_{n-2}$$
$$=2^{n-2}\int_0^\infty dxe^{-xt}(-G_t)^{n-2}F(x,t),$$

where $F(x,t)\equiv e^{-xt}$ and G_t is an integral operator such that for any function $\beta(x,t)$,

$$G_t\beta(x,t)=\tfrac{1}{2}\int_0^\infty dy\beta(y,t)g_t(x,y),$$

with

$$g_t(x,y)\equiv-\int_{|x-y|}^{x+y}e^{-at}da=-\frac{1}{t}\begin{cases}e^{-xt}\sinh(yt), & x>y\\ e^{-yt}\sinh(xt), & y>x.\end{cases}$$

We see that $g_t(x,y)$ is a Green's function satisfying

$$\begin{aligned}\{d^2/dx^2-t^2\}g_t(x,y)&=\delta(x-y),\\ g_t(x,y)&=g_t(y,x),\\ g_t(0,y)&=g_t(\infty,y)=0.\end{aligned}\quad\text{(B2)}$$

By manipulation with Eq. (B2), we can show that, for the special form $F(x,t)=e^{-xt}$, we have

$$(G_t)^nF(x,t)=\frac{1}{n!}\left[\frac{d}{d(t^2)}\right]^ne^{-xt}.$$

Hence, if γ is a constant, and if we set $z\equiv t^2$, we obtain

$$\sum_{n=0}^\infty(\gamma G_t)^nF(x,t)=\sum_{n=0}^\infty\frac{\gamma^n}{n!}\left(\frac{d}{dz}\right)^n\exp(-xz^{\frac{1}{2}})$$
$$=\exp[-(t^2+\gamma)^{\frac{1}{2}}x].\quad\text{(B3)}$$

With the help of Eq. (B3), we obtain after a number of elementary integrations,

$$\left(\frac{1}{A}\right)\sum_{n=3}^\infty C^{(n)}=\left(\frac{\mu\eta^3}{8\pi}\right)\{\tfrac{1}{2}(1+\gamma)^2\ln(1+\gamma)-\tfrac{1}{2}\gamma-\tfrac{3}{4}\gamma^2\},$$

[37] We wish to thank Mr. W. Frank for helpful discussions on this method.

where $\gamma=6\lambda/\eta^3$. Exactly the same expression may be obtained by the method indicated by Wentzel.[10]

APPENDIX C

In this Appendix we discuss corrections to the orthogonality and normalization of the free-particle states introduced by the correlation function in Eqs. (9) to (12). Our aim here is to justify the reduction of Eq. (8) to Eqs. (13) and (19).

We develop here the argument for two-body interactions only, the generalization for n-body potentials being immediate.

We rewrite then Eq. (8) in the form for

$$V=\sum_{i<j} V(ij)\rightarrow\binom{A}{2}V(12),$$

$$\langle V\rangle=\binom{A}{2}\left\{\int d\mathbf{r}_1\cdots d\mathbf{r}_A\Psi^*\Psi\right\}^{-1}\times\int d\mathbf{r}_1\cdots d\mathbf{r}_A\sum_{(\alpha)}\sum_{(\beta)}s^{(\alpha)}s^{(\beta)}\times\{\varphi_{\alpha_1}^*(1)\varphi_{\alpha_2}^*(2)V(12)\varphi_{\beta_1}(1)\varphi_{\beta_2}(2)\}\times\{\varphi_{\alpha_3}^*(3)\varphi_{\beta_3}(3)\}\cdots\times\{\varphi_{\alpha_A}^*(A)\varphi_{\beta_A}(A)\}\prod_{i<j}f^2(r_{ij}), \quad (C1)$$

where we have expanded the Slater determinants with the following notations: (α) is a set of integers $(\alpha_1\cdots\alpha_A)$ representing a permutation of $(1, 2, \cdots, A)$, and $s^{(\alpha)}\equiv s^{\alpha_1\alpha_2\cdots\alpha_A}$ is the Levi-Civita tensor density of rank A, with the values ±1 for even (odd) permutations of the indices. We introduce the definition

$$X_{\alpha_1\alpha_2\beta_1\beta_2}(12)=\sum_{(\alpha)(\beta)}{}' s^{(\alpha)}s^{(\beta)}\int d\mathbf{r}_3\cdots d\mathbf{r}_A\times\{\varphi_{\alpha_3}^*(3)\varphi_{\beta_3}(3)\}\cdots\times\{\varphi_{\alpha_A}^*(A)\varphi_{\beta_A}(A)\}\prod{}'f^2(r_{ij}), \quad (C2)$$

where $\sum'$ represents sum over all permutations for fixed $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and $\prod'$ represents a product of all $f^2(r_{ij})$, excluding $f^2(r_{12})$. Equation (C1) is then written in the form

$$\langle V\rangle=\binom{A}{2}\frac{\int d\mathbf{r}_1d\mathbf{r}_2\sum_{\alpha_1\alpha_2\beta_1\beta_2}\{\varphi_{\alpha_1}^*(1)\varphi_{\alpha_2}^*(2)V(12)\varphi_{\beta_1}(1)\varphi_{\beta_2}(2)\}f^2(r_{12})X_{\alpha_1\alpha_2\beta_1\beta_2}(12)}{\int d\mathbf{r}_1d\mathbf{r}_2\sum_{\alpha_1\alpha_2\beta_1\beta_2}\{\varphi_{\alpha_1}^*(1)\varphi_{\alpha_2}^*(2)\varphi_{\beta_1}(1)\varphi_{\beta_2}(2)\}f^2(r_{12})X_{\alpha_1\alpha_2\beta_1\beta_2}(12)}. \quad (C3)$$

In terms of Eq. (C3) we formulate our problem as follows. We wish to demonstrate that, accurate to terms of order $\lesssim(1/10)(k_Mr_c)^2\approx0.03/\eta^2$, $X_{\alpha_1\alpha_2\beta_1\beta_2}(12)$ is independent of the particle coordinates and equal to $+c$ for $\alpha_1=\beta_1$, $\alpha_2=\beta_2$; $-c$ for $\alpha_1=\beta_2$, $\alpha_2=\beta_1$; 0 otherwise, where c is some constant. Once this result is established, Eq. (C3) reduces to Eq. (13) and our goal is achieved. This result is evidently rigorously valid in the absence of a repulsive core, since $f(r_{ij})\rightarrow1$ and

$$(1/v)\int d\mathbf{r}_je^{-i(\mathbf{k}_a-\mathbf{k}_b)\cdot\mathbf{r}_j}=\delta_{ab}.$$

With $r_c\neq0$, and $v_c\equiv(4\pi/3)r_c^3$, however, we write

$$f^2(r_{ij})\equiv1-U(r_{ij}),\quad U(r_{ij})\equiv\begin{cases}1, & r_{ij}<r_c,\\ 0, & r_{ij}>r_c,\end{cases} \quad (C4)$$

and note that

$$(1/v^2)\int d\mathbf{r}_id\mathbf{r}_je^{-i(\mathbf{k}_a-\mathbf{k}_b)\cdot(\mathbf{r}_i-\mathbf{r}_j)}f^2(r_{ij})=\delta_{ab}-(v_c/v)j_1(k_{ab}r_c)/k_{ab}r_c\approx\delta_{ab}-(v_c/v)(1-(1/10)k_{ab}^2r_c^2+\cdots), \quad (C5)$$

since in general the argument of the spherical Bessel function is small, its maximum value being $k_Mr_c=0.57/\eta$, so that, therefore, $(1/10)(k_{ab}r_c)^2<0.03/\eta^2$. Similarly, we observe, for $k_a\neq k_b\neq k_c\neq0$,

$$v^{-3}\int d\mathbf{r}_id\mathbf{r}_jd\mathbf{r}_pe^{i\mathbf{k}_a\cdot\mathbf{r}_{ij}+i\mathbf{k}_b\cdot\mathbf{r}_{jp}+i\mathbf{k}_c\cdot\mathbf{r}_{pi}}f^2(r_{ij})f^2(r_{jp})f^2(r_{pi})=v^{-3}\int d\mathbf{r}_id\mathbf{r}_jd\mathbf{r}_pe^{i\mathbf{k}_a\cdot\mathbf{r}_{ij}+i\mathbf{k}_b\cdot\mathbf{r}_{jp}+i\mathbf{k}_c\cdot\mathbf{r}_{pi}}\times\{U(r_{ij})U(r_{jp})+U(r_{jp})U(r_{pi})+U(r_{pi})U(r_{ij})-U(r_{ij})U(r_{jp})U(r_{pi})\}\approx3(v_c/v)^2(1-5/32). \quad (C6)$$

From Eqs. (C5) and (C6) we can immediately generalize for all permutations of $(\alpha_3\cdots\alpha_A)$ and $(\beta_3\cdots\beta_A)$ leaving $(\alpha_1, \alpha_2)=(\beta_1, \beta_2)$ or (β_2, β_1) that

$$X_{\alpha_1\alpha_2\alpha_1\alpha_2}(12)=-X_{\alpha_1\alpha_2\alpha_2\alpha_1}(12)\approx\text{constant}\approx(A-2)!\prod_{m=1}^{A-1}[1-m(v_c/v)]\times\left[1+\binom{A-2}{2}\frac{1}{4}\frac{(v_c/v)}{1-(v_c/v)}+\cdots\right]. \quad (C7)$$

The form of Eq. (C7) is understood as follows. The first term in the brackets is contributed by the "classical" term with $\alpha_i=\beta_i$, for all $i\geqslant 3$. The factor $(A-2)!$ represents the number of possible permutations of the $A-2$ particles in this set. The products of factors $\prod[1-m(v_c/v)]$ just represents the excluded volume effect calculated to first order in v_c/v for each factor; *viz.*,

$$\int d\mathbf{r}_1\cdots d\mathbf{r}_A \prod_{i<j=1}^{A} f^2(r_{ij})$$
$$\approx\{1-(A-1)v_c/v\}\int d\mathbf{r}_2\cdots d\mathbf{r}_A \prod_{i<j=2}^{A} f^2(r_{ij})$$
$$\approx\prod_{m=1}^{A-1}[1-m(v_c/v)].$$

The second term in the brackets in Eq. (C7) results from an interchange among any of the $\binom{A-2}{2}$ pairs that can be constructed such that $\alpha_i=\beta_j$, $\alpha_j=\beta_i$. Only $\frac{1}{4}$th of the possible pairs are not orthogonal in spin or isotopic spin space and these contribute according to Eq. (C5), with a plus sign because of the relation

$$s^{\cdots\alpha_i\cdots\alpha_j\cdots}s^{\cdots\alpha_j\cdots\alpha_i\cdots}=-1.$$

In this same manner, one can explicitly calculate further terms in Eq. (C7). As above, we find that to leading order they are constants independent of the argument r_{12} and the indices (α,β). The magnitude of the constants is unimportant since $\langle V\rangle$ is expressed in Eq. (C3) as a ratio.

It remains to be shown now that, for the class of permutations with

$$\alpha_1=\beta_1,\quad \alpha_2\neq\beta_2,\quad \text{or}\quad \alpha_2=\beta_2,\quad \alpha_1\neq\beta_1,\qquad \text{(C8)}$$

there results a negligibly small contribution to Eq. (C3). This conclusion is readily established formally. We just indicate the argument as follows. The second term in Eq. (C7) expresses contributions of the $\binom{A-2}{A}\approx A^2/2$ pairs with $\alpha_i=\beta_j$, $\alpha_j=\beta_i$, for $3\leqslant i<j$. However for the class of permutations in Eq. (C8), there are only $2(A-2)$ such pairs that can be formed; i.e., there are $2(A-2)$ ways in which to form pairs with $\alpha_1=\beta_1$, $\alpha_2=\beta_j$, $\alpha_j=\beta_2$, or with $\alpha_2=\beta_2$, $\alpha_1=\beta_j$, $\alpha_j=\beta_1$, for $j\geqslant 3$, and with all other $\alpha_p=\beta_p$. Thus the contribution to X of the pairs in the class of Eq. (C8) is small of order $1/A$ and can be neglected in Eq. (C3). The same conclusion follows directly for triple exchanges, in the class of Eq. (C8), of form $\alpha_1=\beta_1$, $\alpha_2=\beta_j$, $\alpha_j=\beta_k$, $\alpha_k=\beta_2$, for $j\neq k\geqslant 3$ in comparison with the $\binom{A-2}{3}\approx A^3/6$ triple exchanges of form $(\alpha_1,\alpha_2)=(\beta_1,\beta_2)$, $(\alpha_j,\alpha_k,\alpha_p)=(\beta_k,\beta_p\beta_j)$, which contribute to the next term in Eq. (C7).

The same reasoning extends simply to the cases corresponding to permutations described by

$$(\alpha_1,\alpha_2,\alpha_n,\alpha_m)=(\beta_n,\beta_m,\beta_1,\beta_2).$$