Properties of the Salpeter-Bethe Two-Nucleon Equation*

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The relativistic two-nucleon equation of Salpeter and Bethe is examined in the ladder approximation for large binding energy, with an invariant interaction function. The binding energy is considered an adjustable parameter and the coupling constant, $g^2/4\pi$, is taken as the eigenvalue of the problem. As a starting point for this study, the special case of two equal masses and binding energy equal to the total mass is considered. It is found that in this case the equation may be simplified to a remarkable degree. For zero quantum mass, a one-dimensional integral equation in momentum space is obtained, and solved, in closed form. The solutions can also be displayed in closed form in configuration space. Solutions exist corresponding to this binding energy for all positive g^2. Requiring normalizability on a space-like surface in configuration space eliminates solutions for sufficiently small g^2, but a normalizable continuum remains. Arguments are presented to show that this continuum is not due to the choice of binding energy, but is, in fact, characteristic of the invariant equation. It is shown that by introducing a high frequency cutoff into the particle propagators and then going to the limit of infinite cutoff, the remaining continuum is reduced to a single physically sensible solution.

I. INTRODUCTION

THE relativistic two-nucleon equation of Salpeter and Bethe,[1] which has been obtained field-theoretically by Gell-Mann and Low,[2] represents the first completely covariant treatment of bound states in quantum field theory. However, the only applications which have been possible have been in the semirelativistic or nonrelativistic regions; *e.g.*, the mass corrections to the hydrogen fine-structure;[3] the spectrum of positronium;[4] and the partial renormalization of the Dancoff equations.[5] All of these applications have involved the approximation of the SB equation by introducing an instantaneous interaction as the starting point of a perturbation or iteration procedure. It is therefore of considerable interest to study the properties of this equation, and to seek methods of solving it when the interaction is treated covariantly. In fact, it has not previously been shown that solutions to the SB equation exist if no nonrelativistic approximations are made; furthermore, it has not been shown that all of the relativistic solutions, if they exist, have nonzero nonrelativistic limits.

We shall, in this paper, consider the question of existence of solutions, but not the question of existence of the nonrelativistic limits. We shall treat the interaction in a Lorentz-invariant fashion, but we approximate the equation by restricting ourselves to the so-called "ladder" approximation; that is, we retain only the first term in the expansion of the interaction in powers of $g^2/4\pi$, the coupling constant. Salpeter and Bethe[1] have given qualitative arguments for the validity of this procedure when $g^2/4\pi$ is small; however, we will not consider here the validity of the ladder approximation, but only whether it admits solutions if no nonrelativistic approximations are made within it. We shall therefore not restrict ourselves to small values of $g^2/4\pi$, but shall imagine that the interaction is exactly represented by the first term in its expansion.

The existence of solutions to the SB equation, in the ladder approximation, is demonstrated by exhibiting exact solutions for a special case. Treating the binding energy as a fixed parameter and the coupling constant as the eigenvalue of the problem, we consider the case when the binding energy is equal to the total rest mass of the two particles.[6] The equation may then be simplified to a remarkable degree and solved exactly. It is found that solutions exist for all positive values of g^2; imposing the condition that the solutions be normalizable on a space-like surface eliminates solutions for values of g^2 below a certain critical value, but leaves a normalizable continuum of solutions belonging to values of g^2 larger than the critical value. The validity of such a requirement is discussed with reference to the lack of a direct physical interpretation of the solutions.

To show that this continuum does not arise from the choice $E=0$, but rather is due to the high degree of singularity of the kernel, the equation is solved also for $E=0$ and an instantaneous interaction (which reduces the degree of singularity). It is shown that a point spectrum results.

It is then shown that the continuum may be reduced to a point spectrum in an invariant fashion by introducing a high momentum cutoff for the relative momenta and allowing the cutoff to become arbitrarily large. Using this procedure, only one point in the spectrum appears; the solution corresponding to this value of the coupling constant satisfies the suggested normalizability condition.

* Part of this work was performed at Cornell University, and forms part of the author's doctoral thesis.

[1] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951). We shall refer to this equation as the SB equation.

[2] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

[3] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).

[4] R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952).

[5] M. M. Lévy, Phys. Rev. **88**, 72 (1952); A. Klein, Phys. Rev. **89**, 1158 (1953).

[6] It must be noted that this choice of binding energy implies a system at rest, with zero rest mass; this can only be considered physically if it is interpreted as the limiting case of a nonzero rest mass system.

In so far as is possible, the notation of reference 1 will be used throughout. The time component of the four-vector x will be denoted by x_0, and is real. Three-vectors will be represented in bold-face type; thus, $x=(x_0, \mathbf{x})$. Natural units ($\hbar=c=1$) will be used throughout.

II. PROPERTIES OF THE SALPETER-BETHE EQUATION

In center-of-mass coordinates, the SB equation for bound states may be written, in the ladder approximation, as

$$\Psi(x)=\frac{2ig^2}{(2\pi)^5}\int d^4x' S(x-x')V(x'^2)\Gamma\Psi(x'), \quad (1)$$

where $V(x^2)$ is the interaction function, assumed to depend only on the proper distance between the particles ($x^2=x_0^2-\mathbf{x}^2$). For a relativistic Yukawa interaction involving quanta of mass κ, this function is given by

$$V(x^2)=\frac{1}{(2\pi)^2}\int e^{-ip\cdot x}\frac{1}{p^2-\kappa^2}d^4p. \quad (2)$$

The quantity Γ in Eq. (1) is a direct product of two commuting Dirac matrices, describing the transformation properties of the interaction field (i.e., $\Gamma=1$, $\gamma_\mu{}^a\gamma_\mu{}^b$, $\gamma_5{}^a\gamma_5{}^b$, $\gamma_5{}^a\gamma_5{}^b\gamma_\mu{}^a\gamma_\mu{}^b$, for scalar, vector, pseudoscalar, and pseudovector neutral fields, respectively). The kernel $S(x-x')$ is compounded from two S_F functions;[2] it is given by

$$S(x)=\int d^4p e^{-ip\cdot x}\left[\frac{E+p_0+H_a(\mathbf{p})}{(E+p_0)^2-W_a{}^2}\right]\left[\frac{E-p_0+H_b(-\mathbf{p})}{(E-p_0)^2-W_b{}^2}\right]\times\beta_a\beta_b, \quad (3)$$

where $H_a(\mathbf{p})=\boldsymbol{\alpha}^a\cdot\mathbf{p}+\beta^a m_a$, $W_a{}^2=\mathbf{p}^2+m_a{}^2$, a and b refer to the two particles (assumed distinguishable), and $2E$ represents the total energy of the system. The Dirac matrices labeled a commute with those labeled b; Ψ is a 16-component function, the matrix properties of which will be discussed in detail below. The coordinate x is the relative separation of the particles, and the binding energy of the system is given by $\epsilon_B=2E-m_a-m_b$. We only consider the case when ϵ_B is negative.

The kernel $S(x-x')$ is strongly (quadratically) divergent on the relative light cone, $(x-x')^2=0$. As has been pointed out by Hayashi and Munukata,[7] in calculating the value of $\Psi(x)$ when $x^2=0$ from Eq. (1) the divergence in $S(x-x')$ will coincide with the δ_+-like divergence of $V(x'^2)$. These authors suggest that there may be no solutions to the equation as a result of this property; while this is not true, as we shall show, it seems likely that the character of the equation may be largely determined by this high degree of singularity. It should be noted that if an instantaneous interaction is used, the coincidence of divergences does not occur.

It is easily seen that the divergence in $S(x)$ is independent of the value of E, since it arises from large momenta; hence, this divergence is unaltered when $E=0$. For this reason, therefore, we shall consider E to be a parameter which may equal zero, and shall take $g^2/4\pi$ to be the eigenvalue of Eq. (1). Since one expects, ultimately, to obtain a relation between E and $g^2/4\pi$, it is clearly of no mathematical importance which of these two quantities is taken as the eigenvalue.

We shall show in the next section that it is possible to simplify Eq. (1) to a remarkable degree when $E=0$. As we have mentioned, the problem ceases to have any clear physical meaning at this point in the energy scale, but it seems reasonable to suppose that the mathematical nature of the problem is not seriously altered by this choice, due to the fact that the singular character of the kernel of the equation is independent of E. In the latter part of the paper, we shall attempt to determine to what extent the results are affected by the choice $E=0$, and to what extent they may be considered characteristic of the general problem.

III. SOLUTIONS FOR $E=0$

Equation (1) may be transformed into momentum space, yielding the equation

$$(E\beta_a+p_\mu\gamma_\mu{}^a-m_a)(E\beta_b-p_\mu\gamma_\mu{}^b-m_b)\Phi'(p)=i\lambda'\Gamma\pi^{-2}\int d^4k\frac{1}{(p-k)^2-\kappa^2}\Phi'(k), \quad (4)$$

where we have put $g^2/4\pi=\lambda'$. We consider first the matrix properties of Φ' and the mode of operation of the γ matrices.

The function Φ' is a 16-component function having the transformation properties of a product of two Dirac free-particle spinors. If the components of Φ' are arranged as the elements of a 4×4 matrix, the rows of which are numbered by the particle a index, and the columns by the particle b index, then the γ matrices operate as follows:[8]

$$\begin{aligned}\gamma_\mu{}^a\Phi'&\equiv\gamma_\mu\Phi',\\ \gamma_\mu{}^b\Phi'&\equiv\Phi'\gamma_\mu{}^+,\end{aligned} \quad (5)$$

where γ_μ is the usual 4×4 Dirac matrix, and $\gamma_\mu{}^+$ is the transpose of γ_μ. Ordinary matrix multiplication is implied. It should be noted that in forming the scalar quantity $\Phi'^*\Phi'$, it is necessary to take the trace of the matrix product.

[7] S. Hayashi and H. Munukata, Prog. Theoret. Phys. **7**, 481 (1952).

[8] L. de Broglie, *Théorie Générale des Particules à Spin (Methode de Fusion)* (Gauthier-Villars. Paris, 1943).

Using this representation of Φ', Eq. (4) becomes

$$(E\beta+p_\mu\gamma_\mu-m_a)\Phi'(p)(E\beta^+-p_\mu\gamma_\mu{}^+-m_b)$$
$$=i\lambda'\pi^{-2}\int\frac{d^4k}{(p-k)^2-\kappa^2}\rho\Phi'(k)\rho^+, \quad (6)$$

where we have made use of the fact that Γ can, in general, be represented as $\rho_a\rho_b$ with ρ defined according to the transformation properties of the interaction.

The charge conjugate matrix $A=\gamma_1\gamma_3$ has the following property:

$$A\gamma_\mu{}^+=\gamma_\mu A. \quad (7)$$

Hence, if we define $\Phi=\Phi'A$, it is seen that Φ satisfies

$$(E\beta+p_\mu\gamma_\mu-m_a)\Phi(p)(E\beta-p_\mu\gamma_\mu-m_b)$$
$$=i\lambda'\pi^{-2}\int\frac{d^4k}{(p-k)^2-\kappa^2}\rho\Phi(k)\rho, \quad (8)$$

and, with $E=0$,

$$(p_\mu\gamma_\mu-m_a)\Phi(p)(p_\mu\gamma_\mu+m_b)$$
$$=-i\lambda'\pi^{-2}\int\frac{d^4k}{(p-k)^2-\kappa^2}\rho\Phi(k)\rho. \quad (9)$$

We now put $m_a=m_b$, and assume Φ to be proportional to the unit matrix. We obtain[9]

$$(p^2-m^2)\Phi(p)=-i\lambda\pi^{-2}\int\frac{d^4k}{(p-k)^2-\kappa^2}\Phi(k), \quad (10)$$

where $\lambda=\lambda'\rho^2(\rho^2=1$ for scalar and pseudoscalar interaction, 4 for vector and pseudovector interaction). It is further seen that we may seek a solution to Eq. (10) which is a function of the *single* variable p^2. Such a solution, then, is a scalar under all Lorentz transformations including $\mathbf{p}\rightarrow-\mathbf{p}$, and must correspond to a ^1S state of the system.[10]

Equation (10) may be solved exactly when $\kappa=0$; hence we make this final step, and consider the interaction to take place by means of massless quanta. As we will show, no essential generality is lost by this step. For this case, the kernel of Eq. (10) is the Green's function for the d'Alembertian operator in momentum space; thus, Eq. (10) can be converted into a differential equation in momentum space, directly.

It is desirable, however, to obtain this differential equation somewhat differently; for the less direct method will serve also to replace Eq. (10) by a one-dimensional integral equation. We assume that the function $\Phi(p^2)$ may be written (the justification for this assumption will be provided below):

$$\Phi(p^2)=\int\int_T dxdyg(x,y)(p^2-x^{-1}+y+i\epsilon)^{-2}, \quad (11)$$

where ϵ is a positive infinitesimal constant, and the integration is over some region T in the xy plane. Substituting this expression into Eq. (10), with $\kappa=0$, and expressing all momenta and masses in units of m, we obtain, with $s=p^2$, and suppressing the $i\epsilon$,

$$s(s-1)\int\int_T dxdyg(x,y)(s-x^{-1}+y)^{-2}$$
$$=\lambda\int\int_T dxdyg(x,y)\ln\frac{(x^{-1}-y-s)}{x^{-1}-y}, \quad (12)$$

after the four-dimensional integration is carried out. When the logarithm in Eq. (12) is expressed as a double integral, we obtain

$$s(s-1)\int\int_T dxdyg(x,y)(s-x^{-1}+y)^{-2}$$
$$=\lambda\int_0^s dz\int_z^\infty dt\int\int_T g(x,y)dxdy(t-x^{-1}+y)^{-2}, \quad (13a)$$

or

$$s(s-1)\Phi(s)=\lambda\int_0^s dz\int_z^\infty dt\Phi(t). \quad (13b)$$

Rearranging the order of integration, we obtain

$$s(s-1)\Phi(s)=\lambda\int_0^s t\Phi(t)dt+\lambda s\int_s^\infty\Phi(t)dt. \quad (14)$$

The kernel of this integral equation is just the Green's function for d^2/ds^2; hence, the differential equation mentioned above is

$$\frac{d^2}{ds^2}[s(s-1)\Phi(s)]+\lambda\Phi(s)=0. \quad (15)$$

We may now replace $\Phi(t)$ under the integrals in Eq. (14) by a second derivative; integration by parts then removes the integrals and yields the boundary conditions auxiliary to Eq. (15). These conditions are

$$\lim_{s\rightarrow0}\{s[s(s-1)\Phi(s)]'+s\Phi(s)\}=0, \quad (16a)$$
$$\lim_{s\rightarrow\infty}[s(s-1)\Phi(s)]'=0. \quad (16b)$$

Equation (15) is recognizable as the hypergeometric equation; a solution satisfying conditions (16) is

[9] Equation (10) has been studied by Dr. S. F. Edwards [Phys. Rev. **90**, 284 (1953)] to whom the author is grateful for several useful discussions.

[10] Since no principal quantum member appears in the problem, it is not certain that $\Phi(p^2)$ describes the only ^1S state.

given by[11]

$$\Phi(s)=\int_0^1 t^\alpha(1-t)^{-\alpha}(1-[s+i\epsilon]t)^{\alpha-2}dt \tag{17a}$$

$$=B(\alpha,2-\alpha)F(1+\alpha,2-\alpha;2;s+i\epsilon), \quad |s+i\epsilon|\leq 1 \tag{17b}$$

$$=-\frac{1}{\alpha}\left\{\frac{\Gamma(1+\alpha)\Gamma(2\alpha-1)}{\Gamma(\alpha-1)}(-s-i\epsilon)^{-\alpha-1}\right.$$
$$\times F(1+\alpha,\alpha;2\alpha;[s+i\epsilon]^{-1})$$
$$\left.+\frac{\Gamma(2-\alpha)\Gamma(1-2\alpha)}{\Gamma(-\alpha)}(-s-i\epsilon)^{\alpha-2}\right.$$
$$\left.\times F(2-\alpha,1-\alpha;2-2\alpha;[s+i\epsilon]^{-1})\right\}, \quad |s+i\epsilon|\geq 1, \tag{17c}$$

where $B(m,n)$ is the beta function, $\Gamma(m)$ the gamma function, F the hypergeometric function, and α is related to λ by $\alpha(1-\alpha)=\lambda$. It is sufficient to restrict $\mathrm{Re}(\alpha)\leq\frac{1}{2}$ to obtain all the solutions.

In order to show that $\Phi(s)$, as defined by Eqs. (17), actually satisfies Eq. (10) with $\kappa=0$, it is only necessary to show that this function may be expressed in the form (11). Using the formula,

$$(x\pm i\epsilon)^{-n}=[B(m,n)]^{-1}\int_0^\infty y^{m-1}(x+y\pm i\epsilon)^{-m-n}dy, \quad (m,n>0), \tag{18}$$

valid for all real x, which may be obtained from the definition of the beta function,[12] and putting $m=\alpha$, $n=2-\alpha$, we obtain from Eq. (17a):

$$\Phi(s)=\frac{(-1)^{\alpha-2}}{B(\alpha,2-\alpha)}\int_0^\infty dy\int_0^1 dt y^{\alpha-1}t^{2\alpha-2}(1-t)^{-\alpha}\times(s+y-t^{-1}+i\epsilon)^{-2}, \tag{19}$$

which is of the required form.

Since all solutions of Eq. (10) must satisfy Eq. (15), any solutions not of the form (19) must also be hypergeometric functions. If we use well-known integral representations, the most general solution to Eq. (15) may be written down; with the help of formula (18), it is possible to show that no other solutions to Eq. (10) exist.

We have shown, therefore, that the SB equation has solutions belonging to $E=0$ for all positive values of the coupling constant; thus, for any coupling constant, the point $E=0$ is apparently contained in the ^1S energy spectrum, with no constants of the motion to prevent the system from falling into this state. This, of course, contradicts the physical fact that the lowest ^1S state of positronium lies far above $E=0$, and it contradicts the results of the nonrelativistic limit of the SB equation obtained by Salpeter.[3]

It therefore appears that additional regularity conditions must be imposed on the solutions in order to select the physically significant ones. Such conditions can only be obtained from the connection of the solutions to the laws of conservation of charge and energy, since it is only from this connection that the full physical meaning of the solutions can be understood. Unfortunately, this connection has not yet been obtained for the relativistic two-body problem. In the next section we consider imposing the requirement of space-like normalizability on the solutions, but since the interpretation of $\Phi(p^2)$ is incomplete, it must be recognized that there is no real reason to expect this requirement to be sufficient.

We shall find, in fact, that requiring Φ to be normalizable as a probability amplitude does exclude values of λ below a certain critical value, but leaves a normalizable continuum of solutions belonging to values of λ above this critical value.

IV. NORMALIZATION OF THE WAVE FUNCTION

The integrals which arise in carrying out the normalization of the wave function are more easily performed in configuration space. Furthermore, the functions themselves are perhaps more easily understood in this space, and for this reason, we shall first obtain the Fourier transform of the function defined by Eqs. (17).

It is very difficult to obtain these transforms directly, and an easier method exists. If we use the relation,[13]

$$\int e^{ip\cdot x}(p^2-m^2+i\epsilon)^{-2}d^4p=\pi^3H_0^{(2)}(mR), \tag{20}$$

where $H_0^{(2)}$ is the Hankel function of zero order, second kind, and where $R=(x^2-i\epsilon)^{\frac{1}{2}}$ (positive real if $x^2>0$, negative imaginary if $x^2<0$) we may establish the behavior of the Fourier transform of Eq. (19) for very large imaginary R; that is, Fourier transform of $\Phi(s)$ decreases exponentially in the space-like region. This, of course, is a condition we should have to impose in any event on a bound state solution, but it is important to note that the condition is actually satisfied by the solution $\Phi(s)$.

The Fourier transform, $\Psi(x)$, of $\Phi(p)$ satisfies the following differential equation, which is obtained by transforming Eq. (9). We exhibit the equation for $\kappa\neq 0$ to establish the role played by the quantum rest mass.

$$\left(-i\gamma_\mu\frac{\partial\rightarrow}{\partial x_\mu}-m\right)\Psi(x)\left(-i\gamma_\mu\frac{\partial\leftarrow}{\partial x_\mu}+m\right)=-\frac{2i\lambda\pi\kappa}{R}H_1^{(2)}(\kappa R)\rho\Psi(x)\rho, \tag{21}$$

[11] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1950).

[12] The author is grateful to Dr. N. Austern and Dr. S. Butler for bringing this useful device to his attention.

[13] J. Schwinger, Phys. Rev. **76**, 790 (1949).

where the arrows imply that the differential operators operate from the left. For $\kappa R \ll 1$ the interaction term reduces to $4\lambda R^{-2}$; hence it is clear that κ only affects the solutions at distances *large* compared to κ^{-1}. The behavior of the solutions near $R=0$ is therefore independent of κ.

Putting $\kappa=0$, and taking Ψ to be a scalar function of R, we obtain

$$\left(\frac{d^2}{dR^2}+\frac{3}{R}\frac{d}{dR}+1+\frac{4\lambda}{R^2}\right)\Psi(R)=0, \tag{22}$$

where we have again expressed all instances in units of m^{-1}. This equation is solved by [14]

$$\Psi(R)=R^{-1}Z_n(R), \tag{23}$$

where Z_n is any Bessel function, and where $n=(1-4\lambda)^{\frac{1}{2}}=1-2\alpha$. The condition that $\Psi(R)$ must be exponentially encreasing in the space-like region implies that

$$\Psi(R)=R^{-1}H_n^{(2)}(R) \tag{24}$$

be the unique solutions.

It should be noted that the behavior of $\Psi(R)$ for very large $|R|$ is virtually independent of n, apart from a phase factor. All of the solutions decay exponentially in the space-like region, and oscillate with slowly decreasing amplitude in the time-like region. However, near $R=0$, $\Psi(R)$ depends quite strongly on n; the behavior is like $R^{-(1+n)}$ for $n\neq 0$, and like $R^{-1}\ln R$ for $n=0$. For n imaginary ($\lambda>\frac{1}{4}$), $\Psi(R)$ oscillates with infinite amplitude and frequency at $R=0$.

The wave function Ψ is defined from field theory as the matrix element of two annihilation operators; as such, it is tentatively suggested that Ψ have the interpretation of probability amplitude whenever the two measurements which are involved commute. This is equivalent to the suggestion that the quantity

$$N=\int \Psi^*(0, \mathbf{r})\Psi(0, \mathbf{r})d^3r \tag{25}$$

should represent the normalization of the wave function on the surface $t=0$, since on this surface the two measurements never interfere.[15]

The integral (25) is performed in Appendix A. The result is finite for $n<\frac{1}{2}$, corresponding to $\lambda>\frac{3}{16}$; on the basis of this normalization condition, then, we are led to reject all solutions corresponding to λ smaller than this quantity as being nonphysical. This is equivalent to the statement that if the coupling is sufficiently weak, it is not possible to obtain a binding this strong without the collapse of the system.

In this connection, a characteristic radius of the state may also be calculated; we define this quantity by

$$\langle r\rangle=N^{-1}\int \Psi^*(0, \mathbf{r})\,|\mathbf{r}|\,\Psi(0, \mathbf{r})d^3r. \tag{26}$$

The integral is also carried out in Appendix A, the result being

$$\begin{aligned}\langle r\rangle&=4n\pi^{-1}\cot(n\pi), \quad n<\tfrac{1}{2};\\ &=0, \quad 1\geq n\geq\tfrac{1}{2}.\end{aligned} \tag{27}$$

Thus, the requirement of normalizability does not suffice to reduce the continuous spectrum completely. We shall first examine the source of this difficulty in the next section; subsequently, we shall consider a procedure which does have the effect of reducing the continuum.

V. SOURCE OF THE CONTINUUM

We have shown that for the special case $E=0$, the SB equation has solutions, in the ladder approximation, for all values of the coupling constant. Considerations of normalizability do not suffice to reduce this continuum to a point spectrum, and in any case, such considerations are on an unsure basis, due to the lack of physical interpretation for the solutions. It is therefore of some importance to attempt to isolate the mathematical properties of the SB equation which are responsible for this continuum; that is, we shall attempt to find out to what extent the continuum results from the choice $E=0$, and to what extent it results from the high singularity of the kernel of the equation.

It has been suggested that the nature of the spectrum may be largely determined by the fact that, in coordinate space, the singularity of the interaction on the relative light cone may coincide with the singularity in the function $S(x-x')$, defined by Eq. (3). If this is so, the nature of the spectrum should be altered if the interaction is taken to be instantaneous, for in that case, the two singularities can never coincide. We are therefore led to examine the equation [see Eq. (10)]:

$$(p^2-m^2)\Psi(\mathbf{p}^2, p_0)=-i\lambda\pi^{-2}\int\frac{d^4k}{(\mathbf{p}-\mathbf{k})^2}\Psi(\mathbf{k}^2, k_0), \tag{28}$$

the solution of which describes a ^1S state of a system for which $E=0$ and the interaction is of the instantaneous Coulomb type. If we define a new function,

$$\psi(\mathbf{p}^2)=\int_{-\infty}^{\infty}\Psi(\mathbf{p}^2, p_0)dp_0, \tag{29}$$

[14] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945). One may reasonably ask why, when Eq. (22) is so easily solved, the momentum-space treatment is at all necessary. The answer lies in the lack of a definitive set of boundary conditions [such as Eqs. (16)], which are extremely difficult to obtain in coordinate space.

[15] It is tacitly assumed in the field-theoretic derivation of the SB equation (reference 2) that the quantity N is finite. However, it has not been shown that this assumption is essential to the derivation. It is also to be noted that although N is defined by Eq. (25) in a noncovariant way, the result of normalizing over any arbitrary space-like surface can be shown to be finite if N is finite, and infinite otherwise.

we obtain at once the following three-dimensional equation

$$\psi(\mathbf{p}^2)=-i\lambda\pi^{-2}\int_{-\infty}^{\infty}\frac{dp_0}{p_0{}^2-\mathbf{p}^2-m^2}\int d^3k\frac{1}{(\mathbf{p}-\mathbf{k})^2}\psi(\mathbf{k}^2)$$
$$=\frac{\lambda\pi^{-1}}{(\mathbf{p}^2+m^2)^{\frac{1}{2}}}\int d^3k\frac{1}{(\mathbf{p}-\mathbf{k})^2}\psi(\mathbf{k}^2). \tag{30}$$

This equation may be solved by a method suggested by the work of Rubinowicz;[16] the solution is obtained in Appendix B. It is shown there that the only solution of Eq. (30) corresponds to $4\pi\lambda=1$; that solution is given by

$$\psi(\mathbf{p}^2)=[(\mathbf{p}^2+m^2)^{\frac{1}{2}}-m]^{\frac{1}{2}}/[\mathbf{p}^2(\mathbf{p}^2+m^2)]^{\frac{1}{4}}. \tag{31}$$

This example serves to demonstrate that it is extremely plausible to suppose that the continuum actually does arise from the coincidence of the two singularities in question; and if this is the case, it may be expected that the continuum will arise for any value of E, as long as no noncovariant approximations are made.

In the next section we shall examine the effect on the spectrum of reducing the singularity of the function $S(x)$ in a covariant way. We shall find that this procedure enables us to reduce the continuous spectrum to a single physically acceptable solution.

VI. REDUCTION OF THE CONTINUUM

The program to be carried out in this section is the following. We will introduce a factor into Eq. (1) which cuts off the contribution of momenta large compared to some cutoff parameter, say Ω. We will then solve the problem for $E=\kappa=0$ again, and afterwards allow Ω to become arbitrarily large. We will then examine the resulting spectrum.

If we introduce an invariant cutoff factor, $C(p^2-\Omega)$, into Eq. (1), this factor will appear in the integrand of Eq. (3), and multiplying the integral on the right-hand side of Eq. (4), and finally, it will appear in Eq. (14) as follows:

$$s(s-1)\Phi^\Omega(s)$$
$$=\lambda C(s-\Omega)\left\{\int_0^s t\Phi^\Omega(t)dt+s\int_s^\infty\Phi^\Omega(t)dt\right\}. \tag{32}$$

The superscript on the function Φ^Ω is used to indicate that Φ^Ω differs from Φ in that the former depends on Ω.

We require of the function $C(s-\Omega)$ that it have the property

$$\begin{aligned} &C(s-\Omega)=1, && s\ll\Omega;\\ &C(s-\Omega)=0, && s\gg\Omega. \end{aligned} \tag{33}$$

In order to construct a problem which may be solved, we shall take $C(s-\Omega)$ to be a step function. It seems likely that the form of the cutoff should be unimportant in the limit, and the results do not appear to depend on the sharpness of the cutoff, provided it is sufficiently sharp. Consequently, we write for $C(s-\Omega)$:

$$C(s-\Omega)=\theta(\Omega-s)=\begin{cases}1, & s<\Omega;\\ 0, & s>\Omega.\end{cases} \tag{34}$$

Consider now the function $\xi(s)$ which is defined as the solution of

$$\begin{aligned} s(s-1)\xi(s)&=\lambda\int_0^s t\xi(t)dt+\lambda s\int_s^\Omega\xi(t)dt, && s<\Omega;\\ &=\lambda\int_0^\Omega t\xi(t)dt, && s>\Omega. \end{aligned} \tag{35}$$

It is clear that $\Phi^\Omega(s)$ is related to $\xi(s)$ by

$$\Phi^\Omega(s)=\theta(\Omega-s)\xi(s), \tag{36}$$

and therefore we need only solve Eq. (35).

The function $\xi(s)$ satisfies the differential equation (15) for $s<\Omega$, but the boundary condition (16b) now applies at $s=\Omega$ rather than at $s=\infty$. Thus, $\xi(s)$ must satisfy

$$\xi'(\Omega)/\xi(\Omega)=-(2\Omega-1)/\Omega(\Omega-1), \tag{37}$$

as the new boundary condition. Furthermore, for $s<\Omega$, $\xi(s)=\Phi(s)$ as defined by Eqs. (17); and if $\Omega\gg1$, we have

$$\xi(\Omega)=-\frac{1}{\alpha}\left\{\frac{\Gamma(1+\alpha)\Gamma(2\alpha-1)}{\Gamma(\alpha-1)}(-\Omega)^{-\alpha-1}+\frac{\Gamma(2-\alpha)\Gamma(1-2\alpha)}{\Gamma(-\alpha)}(-\Omega)^{\alpha-2}\right.$$
$$=-\frac{1}{\alpha}\{C_1(\alpha)\Omega^{-\alpha-1}+C_2(\alpha)\Omega^{\alpha-2}\}. \tag{38}$$

The definitions of $C_1(\alpha)$ and $C_2(\alpha)$ are apparent. The condition (37) then makes α a function of Ω, as follows (for $\Omega\gg1$):

$$\frac{(\alpha+1)C_1(\alpha)\Omega^{-\alpha-1}-(\alpha-2)C_2(\alpha)\Omega^{\alpha-2}}{C_1(\alpha)\Omega^{-\alpha-1}+C_2(\alpha)\Omega^{\alpha-2}}=2, \tag{39}$$

or

$$\frac{C_1(\alpha)}{C_2(\alpha)}=(-1)^{-2\alpha}\frac{\Gamma(2\alpha)}{\Gamma(2-2\alpha)}=\Omega^{2\alpha-1}\frac{\alpha}{\alpha-1}. \tag{40}$$

The only solution of this equation which is independent of Ω is $\alpha=\frac{1}{2}$; hence we conclude that the only solution which is insensitive to the cut-off procedure is the one for $\lambda=\frac{1}{4}$, i.e., the least divergent (in configuration space, on the relative light cone) of all the solutions.

VII. CONCLUSIONS

We may summarize this work as follows. The SB equation has been considered in the ladder approximation for a relativistic Coulomb interaction, with the

[16] A. Rubinowicz, Prace Mat.-Fiz. 47, 41 (1949).

following results. Exact solutions have been found corresponding to the binding energy equal to the total mass, ($E=0$), for all positive values of the coupling constant. It is suggested that the existence of a continuous spectrum in the coupling constant is due to the possibility of coincidence of the singularities of the interaction and the particle propagators; to make this suggestion plausible, the SB equation is also considered for $E=0$ with an instantaneous interaction, which removes the possibility of coincidence of the singularities. It is shown that a single-point spectrum results.

It is further shown that if the singularity of the particle propagators is reduced by the introduction of an invariant high momentum cutoff, a single-point spectrum results even with the relativistic Coulomb interaction; this spectrum does not approach a continuum in the limit as the cut-off parameter becomes infinite.

Finally, it is shown that the requirement that the solutions be normalizable on a space-like surface does not suffice to reduce the continuum to a point spectrum; but the solution remaining after the cut-off procedure is carried out is normalizable on a space-like surface.

It has not been conclusively shown, of course, that if Eq. (1) were solved for $E>0$, a continuous spectrum in λ would result; however, the evidence does seem to suggest that the character of the spectrum is determined by the high degree of singularity of the kernel of the equation, and these singularities are independent of E.

This result is of immediate importance if variational procedures are to be used to determine the energy spectrum. For example, we have shown that any analysis which is designed to obtain the lowest-lying ^1S energy level will give $E=0$ as the result, unless extra conditions are imposed which are not already contained in the SB equation. Such a condition has been suggested in this paper; namely, that the solution must be "stable" under the cut-off procedure. It is admittedly very unsatisfactory to introduce such an *ad hoc* procedure into the problem; but the fact that this procedure selects the solution which is least divergent on the light cone suggests, perhaps, that the procedure might be replaced by an additional boundary condition on the light cone. However, any such condition must come directly from the theory when the complete physical interpretation of the solutions is available from the conservation laws.

An attempt has been made, which is not reported here, to obtain approximate solutions for small E (large binding energy) using the solutions for $E=0$ as the starting point of an iteration or perturbation procedure. These methods yield expansions of the wave function in powers of E; but if one may draw an analogy to the Dirac one-particle equation for the hydrogen atom [Eq. (30) suggests that such an analogy may be useful], an expansion of the wave function in powers of E will not be valid near the light cone. Since the eigenvalue problem seems to depend critically on the behavior at small proper distances, it seems plausible to suppose that any method yielding an expansion in powers of E will not be useful in obtaining the spectrum.

The equations describing higher angular momentum states for $E=0$ may be obtained from Eq. (9) as follows. Since Eq. (9) is invariant under Lorentz transformations, one must seek solutions also having that property; thus, one may require that the 4×4 matrix $\Phi(p)$ transform like a scalar, pseudoscalar, vector, etc., corresponding to ^1S, $^3P_0$, ^3S, etc.

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APPENDIX

A. Normalization Integrals

We wish to calculate the integral defined by Eq. (25), which is

$$N=\int d^3r^{-2}H_n^{(1)}(ir)H_n^{(2)}(-ir)$$

$$=\frac{16}{\pi}\int_0^\infty K_n^2(r)dr. \tag{A1}$$

This integration may be performed with the help of the following integral representation:[17]

$$K_n^2(r)=\frac{1}{2}\int_0^\infty \exp\left[-\left(\frac{x}{2}+\frac{r^2}{x}\right)\right]K_n(r^2/x)\frac{dx}{x}. \tag{A2}$$

Let $u=r^2/x$, and insert (A2) into (A1):

$$N=\frac{8}{\pi}\int_0^\infty \frac{du}{u}e^{-u}K_n(u)\int_0^\infty dre^{-r^2/2u}$$

$$=\left(\frac{32}{\pi}\right)^{\frac{1}{2}}\int_0^\infty u^{-\frac{1}{2}}due^{-u}K_n(u). \tag{A3}$$

We now make use of the following integral representation for $K_n(u)$:

$$K_n(u)=\frac{\pi^{\frac{1}{2}}}{\Gamma(n+\frac{1}{2})2^n}u^n\int_0^\infty e^{-u\cosh t}(\sinh t)^{2n}dt, \tag{A4}$$

which gives

$$N=\frac{8}{2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}\int_0^\infty dt(\sinh t)^{2n}\int_0^\infty u^{n-\frac{1}{2}}$$

$$e^{-u(1+\cosh t)}du. \tag{A5}$$

[17] G. N. Watson, *Bessel Functions* (Macmillan and Company, New York, 1944).

The inner integral may be found in tables of Laplace transforms;[18] the result is

$$N=\frac{8}{2^{n+\frac{1}{2}}}\int_0^\infty dt\frac{(\sinh t)^{2n}}{(1+\cosh t)^{n+\frac{1}{2}}}. \quad \text{(A6)}$$

The transformation $\cosh t=2x+1$ transforms this integral into the standard beta-function integral, giving

$$N=4\int_0^\infty \frac{x^{n-\frac{1}{2}}}{x+1}=4\Gamma(\tfrac{1}{2}+n)\Gamma(\tfrac{1}{2}-n)=\frac{4\pi}{\cos n\pi}. \quad \text{(A7)}$$

The integral is clearly finite only if $\mathrm{Re}(n)<\frac{1}{2}$.

The integral for the radius of the state may also be performed. It is defined by

$$\langle r\rangle=N^{-1}\int d^3rH_n^{(1)}(ir)rH_n^{(2)}(-ir)$$

$$=\frac{4\cos n\pi}{\pi^2}\int_0^\infty rdrK_n^2(r). \quad \text{(A8)}$$

Using the integral representation of Eq. (A2), we obtain

$$\langle r\rangle=\frac{2\cos n\pi}{\pi^2}\int_0^\infty\frac{du}{u}e^{-u}K_n(u)\int_0^\infty rdr\exp(-r^2/2u)$$

$$=\frac{2\cos n\pi}{\pi^2}\int_0^\infty due^{-u}K_n(u). \quad \text{(A9)}$$

This integral is also to be found in tables of Laplace transforms. The result is

$$\langle r\rangle=(4n/\pi)\cot(n\pi). \quad \text{(A10)}$$

B. Solution for Instantaneous Interaction

The equation which is to be solved [Eq. (30) of the text] is

$$(\mathbf{p}^2+m^2)^{\frac{1}{2}}\psi(\mathbf{p}^2)=\lambda\pi^{-1}\int d^3k\frac{1}{(\mathbf{p}-\mathbf{k})^2}\psi(\mathbf{k}^2). \quad \text{(B1)}$$

We use a method suggested by Rubinowicz' momentum-space treatment of the Dirac equation for the hydrogen atom.[16] Carrying out the integration over angles in Eq. (1), we obtain (with $m=1$)

$$(1+p^2)^{\frac{1}{2}}\psi(p)=\lambda\int_0^\infty\frac{k}{p}dk\psi(k)\int_{-1}^1\frac{dx}{t-x}$$

$$=2\lambda\int_0^\infty\frac{k}{p}dk\psi(k)Q_0(t), \quad \text{(B2)}$$

where $t=(p^2+k^2)/2pk$, and where we have written p for $|\mathbf{p}|$. Q_0 is the zero-order Legendre function of the second kind. We use the following integral representation of that function:[17]

$$Q_0\left(\frac{p^2+k^2}{2pk}\right)=\pi(pk)^{\frac{1}{2}}\int_0^\infty J_{\frac{1}{2}}(sp)J_{\frac{1}{2}}(sk)ds, \quad \text{(B3)}$$

which gives

$$(1+p^2)^{\frac{1}{2}}\psi(p)=2\pi\lambda p^{-\frac{1}{2}}\int_0^\infty J_{\frac{1}{2}}(sp)ds$$

$$\cdot\int_0^\infty k^{\frac{3}{2}}dkJ_{\frac{1}{2}}(sk)\psi(k). \quad \text{(B4)}$$

We then assume that $\psi(p)$ may be written

$$\psi(p)=p^{-\frac{1}{2}}\int_0^\infty f(x)J_{\frac{1}{2}}(px)xdx, \quad \text{(B5)}$$

which is equivalent to assuming that $\psi(p)$ has a Fourier transform. If we insert (B5) into (B4), we obtain

$$(1+p^2)^{\frac{1}{2}}\int_0^\infty f(x)J_{\frac{1}{2}}(px)xdx=2\pi\lambda\int_0^\infty J_{\frac{1}{2}}(sp)ds$$

$$\times\int_0^\infty kdkJ_{\frac{1}{2}}(sk)\cdot\int_0^\infty f(x)J_{\frac{1}{2}}(kx)xdx. \quad \text{(B6)}$$

Because of the Hankel inversion theorem, we may write

$$\int_0^\infty kJ_{\frac{1}{2}}(sk)dk\int_0^\infty xf(x)J_{\frac{1}{2}}(xk)dx=f(s), \quad \text{(B7)}$$

which yields

$$(1+p^2)^{\frac{1}{2}}\int_0^\infty xf(x)J_{\frac{1}{2}}(px)dx=2\pi\lambda\int_0^\infty f(x)J_{\frac{1}{2}}(px)dx. \quad \text{(B8)}$$

We now make use of the following two integral formulas:[18]

$$\int_0^\infty e^{-t}J_{\frac{1}{2}}(pt)dt=[(1+p^2)^{\frac{1}{2}}-1]^{\frac{1}{2}}p^{-\frac{1}{2}}(1+p^2)^{-\frac{1}{2}},$$

$$\int_0^\infty e^{-t}J_{\frac{1}{2}}(pt)\frac{dt}{t}=2[(1+p^2)^{\frac{1}{2}}-1]^{\frac{1}{2}}p^{-\frac{1}{2}}, \quad \text{(B9)}$$

which enables us to write

$$(1+p^2)^{\frac{1}{2}}=\frac{1}{2}\int_0^\infty e^{-t}J_{\frac{1}{2}}(pt)\frac{dt}{t}\Big/\int_0^\infty e^{-t}J_{\frac{1}{2}}(pt)dt, \quad \text{(B10)}$$

or

$$\int_0^\infty e^{-t}J_{\frac{1}{2}}(pt)\frac{dt}{t}\int_0^\infty J_{\frac{1}{2}}(xp)xf(x)dx$$

$$=4\pi\lambda\int_0^\infty e^{-t}J_{\frac{1}{2}}(pt)dt\int_0^\infty J_{\frac{1}{2}}(xp)f(x)dx. \quad \text{(B11)}$$

[18] W. Magnus and F. Oberhettinger, *Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949).

Multiply both sides of Eq. (B11) by pdp and integrate over p,

$$\int_0^\infty e^{-t}\frac{dt}{t}\int_0^\infty pdpJ_{\frac{1}{2}}(pt)\int_0^\infty xf(x)J_{\frac{1}{2}}(px)dx$$

$$=4\pi\lambda\int_0^\infty e^{-t}dt\int_0^\infty pdpJ_{\frac{1}{2}}(pt)\int_0^\infty f(x)J_{\frac{1}{2}}(px)dx. \quad \text{(B12)}$$

If we use the Hankel inversion formula once more, we obtain

$$\int_0^\infty e^{-t}\frac{dt}{t}f(t)=4\pi\lambda\int_0^\infty e^{-t}\frac{dt}{t}f(t),$$

or

$$4\pi\lambda=1. \quad \text{(B13)}$$

This solution is evidently given by

$$\psi(p)=p^{-\frac{1}{2}}\int_0^\infty e^{-t}dtJ_{\frac{1}{2}}(pt)$$

$$=[(1+p^2)^{\frac{1}{2}}-1]^{\frac{1}{2}}p^{-1}(1+p^2)^{-\frac{1}{2}}, \quad \text{(B14)}$$

and apart from possible degenerate solutions also belonging to $4\pi\lambda=1$, is the only solution having a Fourier transform.

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The Fredholm Theory of the S Matrix

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The equation for the interaction representation transformation operator is put into integral form and the Fredholm theory of integral equations is used to give an explicit expression for the solution. No extension of the circle of convergence of the S matrix seems possible in the general case.

INTRODUCTION

THE Fredholm theory of linear integral equations has recently been used by Salam and Matthews[1] to study the convergence of the S-matrix expansion for the scattering of an electron in an external time dependant electromagnetic field. The Fredholm solution is expressed as the ratio of two infinite series, both of which have an infinite radius of convergence in terms of the coupling constant, provided that the square of the kernel of the equation is integrable. Salam and Matthews deduce that in general the iterative expansion of the S matrix is convergent, when the quantum fluctuations of the electromagnetic field are ignored.

The Fredholm solution of an integral equation (with a suitably bounded kernel) is identical with the iterative solution in the region in which the latter converges, and if the iterative solution has a finite radius of convergence the Fredholm method gives a continuation of the solution outside this radius of convergence. It seems reasonable to enquire to what extent the Fredholm method can be used to extend the convergence of the S matrix when both the electron and the photon fields are quantized and no external electromagnetic field is present. In view of the good reasons for believing that the iterative solution has zero radius of convergence in terms of the coupling constant,[2] any extension would be of considerable value.

It is necessary to write the equation giving the interaction representation transformation operator $S(t)$ in integral form, and to apply the Fredholm method to this equation. As the integral equation has a q-number kernel it is immediately obvious that the usual theory of the convergence of the Fredholm solution[3] need not be valid. However, it is possible to give an explicit expression for the Fredholm solution. This expression shows that the integral equation method actually has no advantage in the general S-matrix case; and it is easy to see that this fact is closely related to the unitary character of the matrix $S(t)$. It seems to be impossible to derive any benefit from using the integral-equation method once the quantum fluctuations of the electromagnetic field are included.

THE INTEGRAL EQUATION

The operator $S(t)$ satisfies the differential equation

$$i\hbar\partial S(t)/\partial t=\lambda H_R(t)S(t),$$

[1] A. Salam and P. T. Matthews, Phys. Rev. **90**, 690 (1953). The author is grateful for being able to see the typescript of this paper.

[2] F. J. Dyson, Phys. Rev. **85**, 631 (1952); C. A. Hurst, Proc. Cambridge Phil. Soc. **48**, 625 (1952); W. Thirring, Helv. Phys. Acta **26**, 33 (1953); A. Petermann, Arch. Sci. **6**, 5 (1953).

[3] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1940), 4th edition, Chap. XI.