Isotope Shift Anomalies and Nuclear Structure

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When observed isotope shifts in the electronic spectra of the heavy elements have been corrected for electronic structure, the resulting shifts give new insight into nuclear structure. In comparison with the shifts predicted for constant density spherical nuclei, the reduced observed shifts are anomalous (1) in magnitude, being on the average too small by a factor 0.5, (2) in trend with neutron number, showing an oscillation in magnitude, and (3) in the staggering of shifts between even and odd isotopes. The present work shows that the anomaly in magnitude largely disappears when account is taken of the compressibility and polarizability of nuclear matter, as a result of which the nuclear radius is found to vary less rapidly with change in neutron number than is predicted by the average $A^{\frac{1}{3}}$ law. The analysis of experimental data also suggests that nuclear particle density variations are slightly greater than indicated by earlier theoretical estimates. Likewise, the oscillatory trend in isotope shifts ceases to appear anomalous when associated with the regular progression in nuclear shapes as derived from configuration theory, from the analysis of quadrupole moments and the first excited states of even-even nuclei, and from other sources. The general size of the amplitude and the nodal positions of the shift variation are consistent with expectations. Finally, the even-odd staggering in isotope shifts is reasonably to be connected with the staggering to be expected in the progression of nuclear deformations through odd and even isotopes. Thus the over-all pattern of isotope shifts appears to fit together with present ideas and to provide new specific information on nuclear structure.

I. INTRODUCTION

THE atomic electron provides a useful and sensitive probe for the investigation of certain nuclear properties. In addition to the measurement of nuclear magnetic dipole and electric quadrupole moments, atomic spectra yield other information concerning the differences in the electronic interaction among the isotopes of a given element. The resulting displacements in the line spectra are appropriately called isotope shifts.

There are two general effects which give rise to atomic isotope shifts: the finite nuclear mass, and the deviation of the electron-nuclear potential from that of the $1/r$ Coulomb law for a point charge. The latter effect, with which this paper is primarily concerned, is referred to broadly as the field effect and arises from the spatial extension of the nuclear charge distribution.

In Sec. II we consider the field effects for spherical, incompressible nuclei, and present the anomalies which exist. In Secs. III and IV we show how most of the anomalies can be removed in terms of nuclear deformability and compressibility.

The theory of the mass effect has been described by Hughes and Eckart[1] and others.[2] In a coordinate system where the center of mass of an atom is at rest, the momentum of the nucleus must be equal in magnitude and opposite in direction to the resultant momentum of the electrons. The expression for the energy of an atom with a nucleus of finite mass differs from that of an atom with a nucleus of infinite mass by the addition of the term for the kinetic energy of the nucleus,

$$\frac{1}{2M}\left(\sum_i \mathbf{P}_i\right)^2=\frac{1}{2M}\sum_i P_i^2+\frac{1}{2M}\sum_{i\neq j}\mathbf{P}_i\cdot\mathbf{P}_j. \quad (1)$$

For computational purposes it has been convenient to subdivide the mass effect into the "normal" effect, due to the first term on the right of Eq. (1), and the "specific" effect, due to the second term on the right of Eq. (1). The contribution of normal mass effect to the energy levels can be determined by replacing the electronic mass m by the reduced mass μ in the Hamiltonian. The resulting energy levels are displaced by a factor μ/m with respect to the levels of an atom with an infinite nucleus. This is the only mass effect present in hydrogen and one electron ions. The specific mass effect requires the evaluation of the expectation value of the last term in Eq. (1) and depends upon the particular electron configuration.

The nuclear mass effects decrease with increasing atomic weight and are seldom observed in the spectra of heavy elements. The nuclear field effects, on the other hand, are small among the light elements and dominate among the heavy elements.

II. FIELD EFFECTS FOR SPHERICAL, INCOMPRESSIBLE NUCLEI

A. The Nuclear Volume Effect

Racah,[3] and Rosenthal and Breit[4] investigated the effect of the extended proton distribution in the nucleus.

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[1] D. S. Hughes and C. Eckart, Phys. Rev. **36**, 694 (1930).
[2] J. H. Bartlett, Jr., and J. J. Gibbons, Jr., Phys. Rev. **44**, 538 (1933); J. P. Vinti, Phys. Rev. **56**, 1120 (1939).
[3] G. Racah, Nuovo cimento **8**, 178 (1931).
[4] J. E. Rosenthal and G. Breit, Phys. Rev. **41**, 459 (1932); G. Breit, Phys. Rev. **42**, 348 (1932).

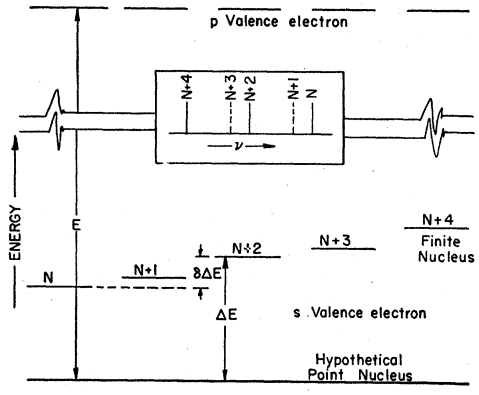

FIG. 1. Schematic representation of energy level displacements. The transition is assumed to involve the change of valence electron from a p state to an s state. The p-state levels for different isotopes are normalized to the same energy. Then the s states for a point nucleus lie also approximately at the same energy. The actual s states lie at progressively higher energies for heavier isotopes. Typically, an odd-N isotope level lies closer to its lower even-N neighbor than to its upper even-N neighbor (even-odd staggering). The energy scale is distorted. In the inset is shown the appearance on a photographic plate of the isotope shift splitting.

The probability density of an s electron in the neighborhood of a point charge as obtained from the Dirac equation is given by

$$P(r)=\frac{2(\rho+1)}{[\Gamma(2\rho+1)]^2}\psi^2(0)\left[\frac{2Zr}{a_H}\right]^{2\rho-2}, \tag{2}$$

where $\rho=(1-Z^2\alpha^2)^{\frac{1}{2}}$, $\psi(0)$ is the nonrelativistic Schrödinger wave function at $r=0$, and a_H is the Bohr radius. The $p_{\frac{1}{2}}$ electron wave function becomes infinite at $r=0$ like the $s_{\frac{1}{2}}$ wave function, but with smaller amplitude; the wave functions for all other electrons are zero at the origin. The principal effects of nuclear structure therefore appear in the energy levels of the s electrons.

If one considers a spherical nucleus of radius a, the electron potential energy is some function $eV(r)$ for $r<a$, and $-Ze^2/r$ for $r>a$. The first-order perturbation of the electronic energy is given by

$$\Delta E=4\pi e\int_0^a P(r)\left[V(r)+\frac{Ze}{r}\right]r^2dr. \tag{3}$$

The sum of these perturbation energies over all s electrons in the atom would give the first-order displacement of the energy level of the atom, but this total displacement is not directly observable. ΔE for an s valence electron gives the displacement in the energy of a photon emitted during the transition of the electron from a p state to an s state, for example. What is generally observed is the difference (shift) in ΔE between the lines of two different isotopes. An exception occurs in the case of x-ray spectra where ΔE is very large and the lines may be calculated accurately. We shall use the term displacement to mean the perturbation energy ΔE, and isotope shift to mean the difference in ΔE between two isotopes of the same element ($\delta\Delta E$). Figure 1 gives a schematic indication of the energy shifts involved.

Two extreme models which might bracket the nuclear proton distribution may be considered (see Fig. 2):

(a) The charge is distributed over the surface of the nucleus. This model is called by Breit the "top slice."

$$V(r)=-Ze^2/a, \quad r<a, \tag{4}$$

$$\Delta E=Ba^{2\rho}/2\rho(2\rho+1), \tag{5}$$

$$\delta\Delta E=\frac{Ba^{2\rho}}{2\rho+1}\frac{\delta a}{a}, \tag{6}$$

where

$$B=8\pi Ze^2\left(\frac{2Z}{a_H}\right)^{2\rho-2}\frac{(\rho+1)}{[\Gamma(2\rho+1)]^2}\psi^2(0),$$

and $\delta\Delta E$ is the energy shift between two isotopes differing in nuclear radius by δa.

(b) The charge is distributed uniformly throughout the nuclear sphere.

$$V(r)=\frac{Ze^2}{a}\left[-\frac{3}{2}+\frac{1}{3}\left(\frac{r}{a}\right)^2\right], \quad r<a, \tag{7}$$

$$\Delta E=\frac{3Ba^{2\rho}}{2\rho(2\rho+1)(2\rho+3)}, \tag{8}$$

$$\delta\Delta E=\frac{3Ba^{2\rho}}{(2\rho+1)(2\rho+3)}\frac{\delta a}{a}. \tag{9}$$

Because the deviation from the unperturbed $1/r$ potential is large inside the nucleus, the first-order perturbation theory is not sufficiently accurate in determining the energy shifts. Rosenthal and Breit[4] have subjected the perturbation treatment to analysis and have found that a factor as small as 0.5 may be required

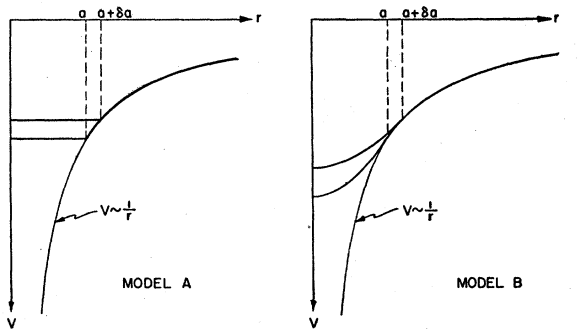

FIG. 2. Schematic diagram of electron potential energy. Model A, charge in shell on outside of nucleus ("top slice"); Model B, charge uniformly distributed through nucleus. The displacement, ΔE, is due to the difference between the actual potential (heavy line) and the $1/r$ potential (light line). The isotope shift, $\delta\Delta E$, is due to the difference in the potentials between isotopes of radius a and $a+\delta a$.

to correct the results predicted by the first-order calculations in some cases. Broch[5] has investigated the shifts by a treatment which avoids perturbation theory, and in the simple case where $V(r)=0$, $r<a$, he shows that a factor $k\approx 2\rho^2/(\rho+1)$ is needed to correct the first order results. Crawford and Schawlow[6] investigated the perturbation theory in the case of thallium ($Z=81$) for both nuclear models. Using Rosenthal and Breit wave functions for the top slice model and a power series solution for the case of uniform charge density, they found that the first-order calculations must be corrected by factors of 0.74 and 0.75. Broch's correction factor for the zero potential model is 0.72. The proximity of the correction factors for such different nuclear models is striking and indicates that Broch's factor k is probably sufficiently good to use generally.

B. Comparison with Experiment

Crawford and Schawlow[6] have compared in detail the experimental results in the isoelectronic sequence Hg II, Tl III, and Pb IV with the theory of the volume effect. They included the correction to the first-order perturbation theory and the contribution of the p electrons. They also took into account the contributions of the inner s electrons. An s valence electron partially shields the inner electrons (and in particular, the inner s electrons) while a p valence electron effectively does not. The effect of shielding on an s electron is to reduce its probability density at the nucleus and so to reduce ΔE. Thus ΔE changes for the inner electrons during a transition and they do contribute to the isotope shift, but with opposite sign than for the valence electron. Crawford and Schawlow found that the inner electrons reduced the shifts by about 20 percent.

The value of $\psi^2(0)$ was obtained from the semi-empirical formula of Fermi and Segrè,[7]

$$\psi^2(0)=\frac{Z_i Z_0^2}{\pi a_H^3 n_0^3}\left(1-\frac{d\sigma}{dn}\right), \tag{10}$$

where Z_i is the effective nuclear charge in the inner region of the orbit and may be set equal to Z, Z_0 is the effective nuclear charge in the outer regions, n_0 is the effective quantum number, and σ is the quantum defect, $n_0=n-\sigma$. The formula was originally derived for alkali metals but has been found to be valid more generally. In a few cases among the heavy elements there are isotopes whose magnetic moments have been

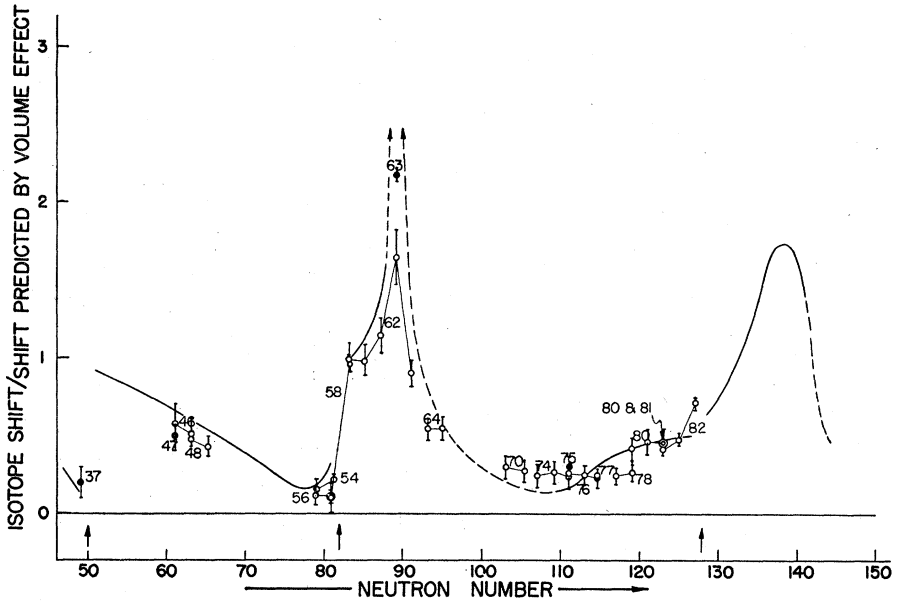

FIG. 3. Comparison of theoretical curve with experimental isotope shifts. The points are taken from Brix and Kopfermann (reference 9) with the exception of the values for Ba and the shift between Ce^{138} and Ce^{140} which were supplied by Dr. O. H. Arrøe in a kind private communication; the cerium datum has not yet been published by Dr. Arrøe. The points are plotted in units of the shifts predicted by the normal volume effect, assuming $a=1.40\times10^{-13}A^{\frac{1}{3}}$ cm. The limits of error indicated (which were generously supplied by Dr. Arrøe) refer only to the error in the spectroscopic measurements and do not include the theoretical uncertainties in the wave function normalization. Points are labeled by the Z value of the element concerned, and are plotted at the average N value of the isotopes measured. The shifts all refer to isotopes differing by two or four in neutron number. Open circles denote even-even nuclei, closed circles, odd-even nuclei. The theoretical curve is the curve of Fig. 5 increased by 0.5, rather than by 1.0, in order to give the best fit to the data. The correction to the volume effect for nuclear compressibility (Sec. IV) may account for the small average relative value ($\sim$0.5) of the experimental isotope shifts. The predicted peak near neutron number 90 is especially well verified. A somewhat broader peak is predicted just below 140 neutrons.

[5] E. K. Broch, Arch. Math. Naturvidenskab **48**, 25 (1945).
[6] M. F. Crawford and A. L. Schawlow, Phys. Rev. **76**, 1310 (1949).
[7] E. Fermi and E. Segrè, Z. Physik **82**, 729 (1933).

measured by induction methods, and the hyperfine structure splittings have been observed spectroscopically. The hfs splittings are proportional to the nuclear magnetic moment and to $\psi^2(0)$ for an s electron, while the value of the magnetic moment obtained from the induction method is independent of the electronic wave functions. The values of $\psi^2(0)$ obtained in this manner are in good agreement with the values given by the Fermi-Segrè formula.

On the basis of uniform charge distribution Crawford and Schawlow found that the observed shifts in the even isotopes of the three elements were very nearly half the values that would be expected from the empirical formula[8] for the nuclear radius, $a=1.5\times10^{-13}A^{\frac{1}{3}}$ cm. The top slice model leads to a greater discrepancy.

Brix and Kopfermann[9] and Humbach[10] have analyzed the experimental data from a large number of elements in a manner similar to that of Crawford and Schawlow. The ratio of the experimental value of the shifts to that predicted by a uniform charge distribution and $a=1.4\times10^{-13}A^{\frac{1}{3}}$ cm is shown in Fig. 3. (The theoretical curve in Fig. 3 is discussed in the next section.) The ratios are plotted as a function of the neutron number and are given in all cases for $\delta N=\delta A=2$. Brix and Kopfermann state that the theoretical values used are probably good to at least 20 percent.

It can be seen from Fig. 3 that the experimental shifts are on the average about half the values predicted by the volume effect, and that there are also considerable variations in the ratios with very large isotope shifts observed in the vicinity of the rare earths.

Schawlow and Townes[11] have investigated data from x-ray spectra where the energy displacement from the point nucleus ΔE can be observed instead of the differential shift between isotopes $\delta\Delta E$. The data are consistent with a nuclear charge distribution which is either uniform or slightly more concentrated toward the surface, and a nuclear radius given by the empirical formula. This suggests that the small value of the observed isotope shifts is a differential effect.

For those elements which have both even and odd isotopes it is observed that the centers of gravity of the energy levels of the odd isotopes are not spaced midway between the levels of the even isotopes, but are rather usually lower than midway (see Fig. 1).

The isotope shift anomalies, then, are these: (1) The observed shifts are on the average smaller by a factor of one-half than is predicted by the volume effect, using a nuclear radius given by $a=1.4\times10^{-13}A^{\frac{1}{3}}$ cm; (2) the shifts vary with the neutron number in a more or less regular manner which appears to be associated with the magic numbers; (3) the energy levels of the odd isotopes are not located midway between the levels of the even isotopes, but are staggered.

C. Nuclear Polarization

Breit, Arfken, and Clendenin[12] have investigated the possible polarization of the nucleus by the atomic electrons. The polarization acts in the direction to increase the binding of the electrons and is thus contrary in sense to the volume effect. Only the monopole effect should be large enough to contribute appreciably to the isotope shifts.

The polarization of the nucleus may be interpreted in terms of admixtures to the nuclear ground state of low-lying excited levels, in such a way as to give greater concentration of protons toward the center of the nucleus where the electron probability density is greatest. The size of the effect depends upon the number and spacing of the low-lying excited levels of the nucleus, and might thus be expected to be greater for odd nuclei than for even ones. The authors were looking for an effect half as great as the volume effect in order to account for even-odd staggering. Reasonable approximations appeared to give no more than 20 percent of the needed one-half and rather extreme departures from the central field approximations for an individual nucleon seemed necessary to account for the observed effect.

While the polarization may account in some part for the even-odd staggering, it probably is not important in explaining the small average value of the even shifts. The consistency of the data from x-ray spectra with the predictions of the volume effect indicates that the energy displacement ΔE is due mainly to that effect and that the anomalies in the even shifts may come mainly from differential effects.

D. Non-Coulomb Forces

Havens, Rainwater, and Rabi[13] have reported the observation of an electron-neutron interaction of $V=5300\pm1000$ electron volts, assuming a square well of the classical electron radius $r_e=2.8\times10^{-13}$ cm. An estimate of the effect of this interaction can be obtained by assuming the additional neutrons of a heavier isotope are found primarily near the surface of the nucleus. The ratio of the contribution from this source with that predicted by the volume effect is then given by[14]

$$\frac{\epsilon}{\delta\Delta E}=0.0122(2\rho+1)(2\rho+3)\frac{A^{\frac{1}{3}}}{Z}. \tag{11}$$

For zinc ($A\sim64$, $Z=30$), which is low in weight for field effect, the ratio is about 0.024; it decreases with increasing atomic weight. Even this is probably an

[8] See, for example, H. A. Bethe, *Elementary Nuclear Theory* (John Wiley and Sons, Inc., New York, 1947), p. 8.
[9] P. Brix and H. Kopfermann, Z. Physik **126**, 344 (1949); Festschr. Akad. Wiss. Göttingen, Math.-Physik. Kl. 17 (1951); Phys. Rev. **85**, 1050 (1952).
[10] Walter Humbach, Z. Physik **133**, 589 (1952).
[11] A. L. Schawlow and C. H. Townes, Science **115**, 284 (1952).
[12] Breit, Arfken, and Clendenin, Phys. Rev. **78**, 390 (1950); Phys. Rev. **77**, 569 (1950).
[13] Havens, Rainwater, and Rabi, Phys. Rev. **82**, 345 (1951).
[14] L. Wilets and L. C. Bradley III, Phys. Rev. **82**, 285 (1951).

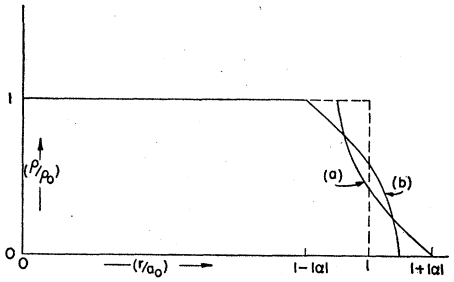

FIG. 4. Average charge density for deformed nucleus. Dashed curve: hypothetical uniform density for spherical nucleus of radius a_0. Curve (a): charge density averaged over all angles for prolate distorted nucleus with maximum radius $a_0(1+\alpha)$, $\alpha>0$. Curve (b): charge density averaged over all angles for oblate distorted nucleus with maximum radius $a_0(1-\alpha/2)$, $\alpha<0$.

overestimate, since as Foldy[15] has pointed out, the reported value of the electron-neutron interaction can be understood almost alone on the basis of the magnetic moment of the neutron. Thus the effects might be expected to cancel for an even number of neutrons and to be given equal weight (equal probability of spin orientation) in determining the center of gravity of the hyperfine structure of an odd isotope.

III. THE NUCLEAR DISTORTION EFFECT

A. Theory

The effect of nuclear distortion on the average nuclear charge distribution seen by an s electron is shown in Fig. 4. The charge distribution is extended radially and the energy displacement, ΔE, is increased for either prolate or oblate deformation. The effect on ΔE is small, but the differential effect measured by the isotope shift may be quite large, since the nuclear deformation may change by a large fraction of itself from one isotope to another, while the nuclear volume does not.[16]

Restricting consideration to cylindrically symmetric ellipsoidal deformations, we write for the nuclear radius,

$$R(\theta)=a_0[1+\alpha P_2(\cos\theta)], \tag{12}$$

where α may be >0 (prolate) or <0 (oblate). Because we deal only with s electrons, there is no loss of generality in choosing the ellipsoid axis along the Z axis. In order to isolate the distortion effect from other effects, it is assumed in the present section that the nucleus has a constant volume, $(4/3)\pi a^3$, and a uniform charge density,

$$\begin{aligned}&\rho_0(r)=Ze[4\pi a^3/3]^{-1}, \quad r<R(\theta);\\ &\rho_0(r)=0, \qquad\qquad\quad r>R(\theta).\end{aligned} \tag{13}$$

The constant volume assumption relates the parameters a_0 and a appearing in Eqs. (12) and (13):

$$a_0=a[1+\tfrac{2}{5}\alpha^2+(2/35)\alpha^3]^{-\frac{1}{3}}. \tag{14}$$

Introducing the notation $m=$minimum nuclear radius, $M=$maximum nuclear radius, one obtains for the charge density averaged over all angles in the region $m<r<M$:

$$\text{prolate:}\quad \rho(r)=\rho_0\{1-[(r-m)/(M-m)]^{\frac{1}{2}}\}; \tag{15a}$$

$$\text{oblate:}\quad \rho(r)=\rho_0[(M-r)/(M-m)]^{\frac{1}{2}}. \tag{15b}$$

It is these expressions which are plotted in Fig. 4.

The perturbing energy of interest is the difference between the electron potential energy in the field of a deformed nucleus and in the field of a spherical nucleus of the same charge and volume. Calling this perturbation ΔV, we have, for the case of prolate deformation,

$r<m$,

$$\Delta V_1=-\frac{Ze^2}{a}\left\{\frac{3}{2}[(a_0/a)^2-1]+\frac{3}{10}(a_0/a)^2\alpha^2\right\}; \tag{16a}$$

$m<r<a$,

$$\Delta V_2=-\frac{Ze^2}{a^3}\left\{\frac{4}{35}\frac{(r-m)^{5/2}}{(M-m)^{1/2}}\left[3+4\frac{m}{r}\right]\right\}; \tag{16b}$$

$a<r<M$,

$$\Delta V_3=\Delta V_2-\frac{Ze^2}{a}\left\{\frac{3}{2}-\frac{1}{2}\left(\frac{r}{a}\right)^2-\frac{a}{r}\right\}; \tag{16c}$$

and for oblate deformation,

$r<m$,

$$\Delta V_1=-\frac{Ze^2}{a}\left\{\frac{3}{2}[(a_0/a)^2-1]+\frac{3}{10}(a_0/a)^2\alpha^2\right\}; \tag{17a}$$

$m<r<a$,

$$\Delta V_2=\Delta V_3+\frac{Ze^2}{a}\left\{\frac{3}{2}-\frac{1}{2}\left(\frac{r}{a}\right)^2-\frac{a}{r}\right\}; \tag{17b}$$

$a<r<M$,

$$\Delta V_3=\frac{Ze^2}{a^3}\frac{4}{35}\frac{(M-r)^{5/2}}{(M-m)^{1/2}}\left(3+4\frac{M}{r}\right). \tag{17c}$$

The electronic energy displacement to first order is the diagonal matrix element of ΔV for an s electron, the same for either prolate or oblate deformation. If we define the electron density in the vicinity of the nucleus by

$$\bar{\psi}\psi=Br^{2\rho-2}/4\pi Ze^2 \tag{18}$$

(see Sect. II-A), the extra energy displacement due to nuclear deformation is given by

$$\Delta E_a=Ba^{2\rho}\frac{3}{10}\frac{\alpha^2}{2\rho+1}\left[1+\frac{2}{21}(2\rho+3)\alpha+\cdots\right]. \tag{19}$$

[15] L. L. Foldy, Phys. Rev. **87**, 693 (1952).

[16] Brix and Kopfermann have noted the importance of nuclear deformations in a discussion of the isotope shifts of Sm (reference 9). Shell structure effects from another point of view have been considered by G. Breit, Phys. Rev. **86**, 254 (1952).

The ratio of this displacement to that of the normal volume effect is, to order α^2,

$$\frac{\Delta E_\alpha}{\Delta E_v}=\frac{\rho(2\rho+3)}{5}\alpha^2. \tag{20}$$

If 0.25 is taken to be an upper limit to α, this ratio is less than 0.06. To the accuracy of the α^2 term in Eq. (19), the ratio of the isotope shift due to deformation to the shift due to the normal volume effect is

$$\frac{S_\alpha}{S_v}=\frac{\delta(\Delta E)_\alpha}{\delta(\Delta E)_v}=\frac{3}{10}(2\rho+3)A\left(\frac{\delta(\alpha^2)}{\delta N}\right)_Z. \tag{21}$$

Since the coefficient of α within the bracket of equation (19) is of the order of 0.5, the fractional error of Eq. (21) due to the neglect of higher orders in α is approximately $\frac{1}{2}\alpha$ (with the sign of α). As stated in Sec. II-A, corrections to the first-order perturbation energy are insensitive to the shape of the potential within the nucleus. Therefore the ratio S_α/S_v, calculated using the zero-order electron wave function, should remain very nearly the same if higher-order corrections to the perturbation theory are applied.

One can quickly estimate from Eq. (21) on the basis of known quadrupole moments that the nuclear distortion effect is substantial. Setting $Z=62$, $A=150$, $\delta(\alpha^2)=0.005$. and $\delta N=2$, one finds $S_\alpha/S_v=0.54$. It should be stressed that the α appearing throughout the above derivation is the *intrinsic* nuclear deformation,[17] not the deformation measurable from quadrupole moments. Hence the effect is expected in even-even as well as odd-even nuclei.

In order to predict actual values of isotope shifts expected from the distortion effect, it is required to know the magnitudes of nuclear distortions. There are several sources of information to which we may turn.

(1) *Theory*

The equilibrium shape of the nucleus may be calculated theoretically if the configuration of nucleons outside closed shells is known. In the static approximation (surface motion treated classically), the deformation parameter α is given by[18,19]

$$\alpha=\frac{\bar{T}}{C'}\sum_i\frac{j_i(j_i+1)-3m_i^2}{4j_i(j_i+1)}, \tag{22}$$

where $\bar{T}$ is an interaction energy parameter which is the order of the kinetic energy of the extra nucleons, and C' is the coefficient in the surface potential energy, $V_s=(1/2)C'\alpha^2$, which is around 160 Mev for the heavy elements. The coefficient on the right side of Eq. (22) has approximately the value 0.12 for the heavy elements, varying only slowly with mass number. The m_i are the projections of the vectors $\mathbf{j}_i$ along the axis of the deformed nucleus. There is some evidence[20] that deformations calculated in this way are too large, due to an overestimate of the magnitude of the particle-to-surface interaction, to a breakdown of the idea that the $\mathbf{j}_i$ are quantized along the nuclear axis, or to the approximations employed—considering the P_2 deformation only and working only to first order in α. The more serious practical difficulty preventing an accurate theoretical determination of the nuclear deformation is our ignorance of the detailed nucleon configurations. This is especially true of those nuclei where one expects large deformations and hence large contributions to the isotope shift. It is necessary therefore in our present state of knowledge of nuclear structure, to turn to empirical evidences of the nonspherical shape of the nucleus.

(2) *Quadrupole Moments*

These are generally not measured to high accuracy, so that the differences $\delta(\alpha^2)$ are rather rough. For $A>75$, quadrupole moments are known for pairs of isotopes of the following elements: $_{49}In^{113,115}$, $_{51}Sb^{121,123}$, $_{53}I^{127,129}$, $_{62}Sm^{147,149}$, $_{63}Eu^{151,153}$, $_{71}Lu^{176,177}$, and $_{75}Re^{185,187}$.

Values of α may be computed from measured quadrupole moments by the formula[20]

$$\alpha=\frac{53.5}{ZA^{\frac{2}{3}}}\frac{(I+1)(2I+3)}{I(2I-1)}Q_{\text{obs}}, \tag{23}$$

where Q_{obs} is in barns. This formula assumes that the nuclear spin is directed along the nuclear axis, and that the total spin vector is strongly coupled to the axis.[21] For this reason it gives a lower limit to the intrinsic nuclear distortion. Other orientations of the spin, or weak coupling of spin to axis, would yield smaller measured quadrupole moments than are assumed in Eq. (23), and hence lead to an underestimate of α. In addition to the few elements mentioned above with known quadrupole moments for pairs of isotopes, there are measured quadrupole moments for single isotopes which contribute information useful for isotope shifts in an indirect way by suggesting that the nuclear deformation varies in a regular way among the elements, reaching minima at the magic numbers and maxima between the magic numbers.

(3) *Energy of the First Excited State of Even-Even Nuclei*

The strong coupling approximation of the collective model of the nucleus[19] yields a simple approximate relation[17,20] between the nuclear deformation and the energy of the first excited state of even-even nuclei:

$$\alpha=10.4A^{-5/6}E_1^{-1/2}, \tag{24}$$

[17] A. Bohr and B. R. Mottelson, Phys. Rev. **89**, 316 (1953).
[18] J. Rainwater, Phys. Rev. **79**, 432 (1950).
[19] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952).
[20] K. W. Ford, Phys. Rev. **90**, 29 (1953).
[21] A. Bohr, Phys. Rev. **81**, 134 (1951).

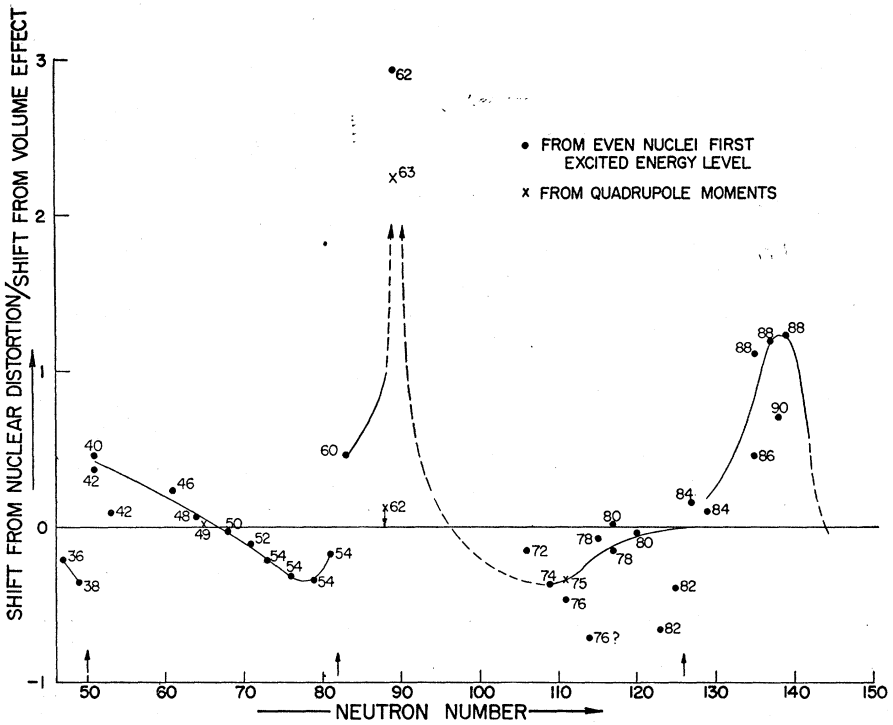

FIG. 5. Theoretical curve of isotope shifts due to nuclear deformation in units of the shift predicted by the normal volume effect. Points are predicted isotope shifts and are labeled with the Z value of the elements concerned, and are plotted at the average neutron number of the two isotopes on which each point is based. The particularly low value of the Sm point ($Z=62$) based on quadrupole moments may possibly be explained in terms of mixing of the prolate and oblate forms of deformation. Other special features of the curve are discussed in the text.

where E_1 is in Mev. This simple formula is shown in reference 20 to overestimate the nuclear distortion by a substantial amount, even if all of the parameters of the theory are correct. The error of the formula depends on the detailed nucleon configurations, and can therefore not be estimated with any accuracy.

Our procedure therefore is to normalize Eq. (24) with a single arbitrary parameter to yield agreement with deformations calculated from quadrupole moments in the vicinity of the rare earth elements. The factor of reduction required is 1.7 for α or 3 for α^2. This means of arriving at values of the nuclear deformation is theoretically more tenuous than the more direct evidence of quadrupole moments, but has several advantages: (*a*) the data on the first excited states of even-even nuclei in the region $50<N<126$ are more extensive and more accurate than the quadrupole moment data; (*b*) most of the measured isotope shifts are for even-even nuclei; and (*c*) the isotope shifts and the excited states of even-even nuclei both measure the intrinsic nuclear deformation, independent of spin orientation and independent of possible mixing of states of prolate and oblate deformation.

(4) *Electric Quadrupole Transition Rates in Even-Even Nuclei*

Bohr and Mottelson[17] have given an approximate formula relating the transition rate from first excited state to ground state in even-even nuclei to the intrinsic nuclear deformation. For the few cases of measured transition rate, mostly in the rare earth group, good agreement of calculated deformations is obtained with the trend of values obtained from quadrupole moments, but with deformations obtained from $E2$ transitions being somewhat smaller than those obtained from quadrupole moments. This is consistent with the approximation employed in their formula, which assumes that the surface vibrational part of the wave function is identical in initial and final states. This approximation should lead to an overestimate of the value of the matrix element.

Two other effects can be mentioned which may yield information on nuclear distortions.

(5) *The Regularities in Level Spacing of Single-Particle-Type Energy Levels in the Odd-Even Nuclei*[22]

The single particle energy level positions depend on the nuclear distortion,[23] and the regularities observed in energy level spacings may reflect regularities in the nuclear deformations.

(6) *Alpha Decay*

The nuclear deformation may enhance alpha decay rate[23,24] because of the decreased Coulomb barrier over part of the nuclear surface. In some cases it may also

[22] M. Goldhaber and R. D. Hill, Revs. Modern Phys. **24**, 179 (1952).
[23] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).
[24] J. J. Devaney, Ph.D. dissertation, Massachusetts Institute of Technology, 1950 (unpublished).

contribute to a non-overlap of the initial and final wave functions and thereby inhibit alpha decay.

Neither of the effects (5) and (6) has been analyzed in sufficient detail to yield useful quantitative information on nuclear deformations. Only the effects (2) and (3), and to some extent (4), are based on sufficient experimental data and adequate theoretical grounds to permit a prediction of isotope shifts expected from the nuclear distortion effect.

Figure 5, based on quadrupole moments and on the energies of the first excited states of even-even nuclei, gives the predicted differential isotope shifts caused by nuclear deformation in units of the shift due to the ordinary volume effect. Points are shown where data exist for two or more isotopes of the same element. The trend of the curve is influenced also by the data on single isotopes, e.g., the value of $N=96$ where the curve crosses the horizontal axis. Several special features of the theoretical curve may be noted. Discontinuities appear at the magic numbers, where the predicted shift changes sign. Between magic numbers, the shift also changes sign, but continuously. The positive peak expected between $N=82$ and 126 is higher and narrower than the negative valley in the same shell. Quadrupole moment points were given special weight in drawing the curve. The Pb isotopes were given very little weight because the strong coupling approximation, on which the points due to the even-even nuclei are based, is not expected to be valid in the near vicinity of a double closed shell. Because of the dependence of nuclear deformation on both N and Z, the shifts are not expected to lie accurately along a single curve as a function of N only. The data on which the theoretical curve is based are inadequate for a more detailed prediction, however. The curve should be regarded as a mean prediction along the line of stable elements.

B. Comparison with Experiment

Figure 3 gives the experimental isotope shifts in units of the shift expected from the ordinary volume effect. If only the volume effect plus the distortion effect were effective for isotope shifts, the curve of Fig. 5, increased by one unit, should lie along the experimental points. In fact, such a curve lies too high. In order to fit the data reasonably well, it is required to add to the curve of Fig. 2 a value of only 0.5. The theoretical curve, arbitrarily modified in this way, is given in Fig. 3, and is seen to fit the data rather well, both in shape and in magnitude.

It should be stressed that two arbitrary parameters have gone into the theoretical curve of Fig. 3. The first of these was the normalization of the distortions computed from the energies of the first excited states of even-even nuclei to the distortions computed from quadrupole moments in the rare earth elements. This normalization was required by inadequate knowledge of the detailed nucleon configurations and was in the expected direction—to decrease the distortion found from the even-even nuclei. The second parameter was the shift of the theoretical curve downward everywhere by 0.5. This is equivalent to assuming that all effects other than the volume effect and the distortion effect give a net contribution to the isotope shift half as great as the volume effect and in the opposite direction. It will be shown in Part IV that it may be reasonable to attribute such an effect largely to the finite compressibility of nuclear matter and the consequent variations of nuclear density among the isotopes of a single element. Note that in the theoretical curve of Fig. 3 all of the *variations* of the isotope shifts away from a mean value are attributed to the distortion effect only.

C. Even-Odd Staggering

The nuclear distortion effect may afford also an explanation of the even-odd staggering observed experimentally. There is slight, but by no means conclusive evidence, from the first excited states of even-even nuclei that the even-even nuclei are somewhat more deformed than the odd-even nuclei. Such an effect, in any case, is expected theoretically. Figure 6 shows the square of the nuclear deformation (to arbitrary scale) *vs* number of nucleons in a simple shell of $j=9/2$ particles, calculated according to Eq. (22). If one considers a given element with Z even, then the horizontal scale in Fig. 6 represents the number of neutrons in different isotopes of the element. The α^2 of odd-even isotopes is seen to be less than the average value of α^2 for the neighboring even-even isotopes, which leads to an even-odd staggering in the direction usually observed. A similar result holds for more complicated shells, involving particles in more than one subshell.

Examples of observed even-odd staggering in Ba^{25} and in Pb^{26} and the possible connection of the staggering

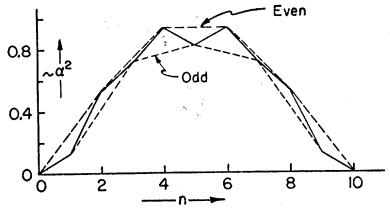

FIG. 6. Nuclear distortions in simple shell of $j=9/2$ particles. Plotted vertically is the quantity

$$\left[\sum_i \frac{j(j+1)-3m_i^2}{4j(j+1)}\right]^2,$$

which in the static approximation is proportional to the square of the nuclear deformation, α^2. Plotted horizontally is the number of nucleons (of a given type) in the shell. The value of α^2 for $n=$ odd is less than the average value of α^2 for the neighboring $n=$ even points. This may afford an explanation of the observed even-odd staggering.

[25] O. H. Arrøe, Ph.D. dissertation, University of Copenhagen, 1951 (Kobenhavn Nordisk Bogtrykkeri, 1951).

[26] Manning, Anderson, and Watson, Phys. Rev. **78**, 417 (1950); K. Murakawa and S. Suwa, J. Phys. Soc. Japan **5**, 382 (1950); F. Geiger, Phys. Rev. **79**, 212 (1950); Brix, von Buttlar, Houtermans, and Kopfermann, Z. Physik **133**, 192 (1952).

with nuclear deformation are illustrated in Fig. 7. Each of these elements possesses an isotope with closed neutron shell. Pb has in addition a closed proton shell, and Ba may have a closed proton subshell ($g_{7/2}$). The deformations, $|\alpha|$, shown in the figure are calculated from the observed isotope shifts assuming (a) that the shift due to the volume effect alone is half of that for an incompressible nucleus with $R=1.4\times10^{-13}A^{\frac{1}{3}}$ cm, and (b) that the deformations for the isotopes with closed neutron shells are negligibly small. The plotted deformations are therefore only qualitatively correct. (In addition, the observed shifts in Ba are subject to considerable uncertainty[25] because of their small value.) To be noted are: (1) The occurrence together of large anomalies in the isotope shift and large even-odd staggering is in agreement with theory. (2) The nuclear deformation effect may reasonably account for much of the observed even-odd staggering, even in the extreme case of barium. There is room, however, for an additional small effect caused by nuclear polarization.[12] It may incidentally be remarked that the large observed anomaly in Ba speaks against a closed proton subshell for all of the Ba isotopes.

IV. FIELD EFFECTS IN THE COMPRESSIBLE NUCLEAR MODEL

In the previous sections we have, for simplicity, discussed the isotope shifts which would follow from nuclei of sharply defined boundaries enclosing uniform charge density. In the present section we shall retain

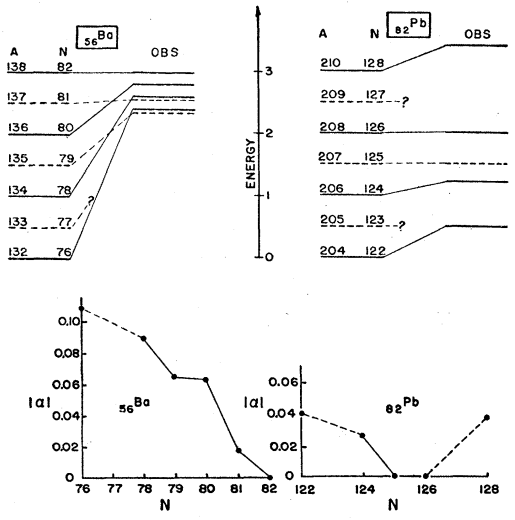

FIG. 7. Nuclear deformations calculated from observed isotope shifts. The energy levels are plotted in units of the energy difference predicted for incompressible nuclei with standard radius $1.4\cdot10^{-13}A^{\frac{1}{3}}$ cm and differing by one neutron. To the left of each diagram are the *presumed* levels in the absence of deformation (half of the standard theoretical value), and to the right are the observed levels, normalized to leave unchanged the level of the nucleus with closed neutron shell. Values of the nuclear deformation plotted below the diagrams are calculated assuming that all of the changes from the left half to the right half of the diagrams are due to the nuclear distortion effect.

the fiction of sharply defined nuclear surface, but subject to this constraint we shall refine our description of the charge distribution. Discarding the assumption of constant mass and charge densities within nuclei, and restricting our discussion initially to spherical nuclei, we shall (1) examine more closely the effect of adding one or more neutrons upon the potential in which the atomic electrons are bound; and so (2) ascertain the isotope shifts to be expected when consideration is given to current estimates of nuclear compressibility and polarizability.

A. Neutron and Proton Densities

The Coulomb repulsion augments the proton density near the nuclear surface relative to the central density; a similar but less pronounced displacement occurs for the neutrons. Wigner,[27] Feenberg,[28] and Swiatecki[29] have computed these particle densities by somewhat different procedures derived from the principle that, for a chosen nuclear radius, the equilibrium neutron and proton densities, n_n and n_p, are given by the variational statement,

$$\delta\int\epsilon(n_n, n_p)d(\text{volume})=0, \tag{25}$$

in which the energy density $\epsilon(n_n, n_p)$ is to be integrated over the nuclear volume. The analytically simple expression for the energy density used by Swiatecki results in particle densities (see Fig. 8) which ascend with monotonically increasing slope toward the nuclear surface, where the densities are truncated to zero value. A similar truncation is used by Feenberg, but in his work the boundary condition is imposed that the particle densities shall intercept the surface at zero slope. The latter method approaches somewhat more closely what is perhaps the actual situation for a heavy nucleus; namely, each particle density rises initially to a maximum, and then falls continuously to reach a negligible value slightly beyond the apparent radius. No satisfactory treatment of particle densities closely resembling this "actuality" exists at present in a form to yield reliable estimates of the small differential effects pertinent to the prediction of isotope shifts. We therefore shall obtain an order of magnitude estimate of how isotope shifts depend on nuclear compressibility, following Feenberg in the treatment of particle densities. The result of this treatment is that the addition of two neutrons increases the nuclear radius by a substantially smaller amount than the increase predicted by the average $A^{\frac{1}{3}}$ law.

The particle densities, which are assumed to vary but slightly from the mean values, $\bar{n}_N$ and $\bar{n}_P$, may be written

$$n_N(r)=\bar{n}_N[1+v_N(x)]; \quad n_P(r)=\bar{n}_P[1+v_P(x)]. \tag{26}$$

[27] E. Wigner, Bicentennial Symposium, University of Pennsylvania, 1940 (unpublished).
[28] E. Feenberg, Phys. Rev. **59**, 593 (1941).
[29] W. J. Swiatecki, Proc. Phys. Soc. (London) **A63**, 1208 (1950).

Here the deviations $v_N(x)$ and $v_P(x)$ are functions of the relative radius $x=r/R$ and may be expanded in terms of the flat-edge polynomials[30] of Feenberg:

$$v_N(x)=a_N v_1(x)+b_N v_2(x),$$
$$v_P(x)=a_P v_1(x)+b_P v_2(x), \tag{27}$$

where the coefficients of expansion are to be determined from (25). In this section we conform to Feenberg's notation, denoting the nuclear radius as R.

B. Isotope Shift

The evaluation of the isotope shift predicted for a nonuniform charge density is simplified by employing an alternate form of Eq. (3) that is an integration of the nuclear charge density, $\rho(r)=en_P(r)$, which determintes $V(r)$, over the effective potential due to the s electron. Then, for a quite general nuclear charge distribution, the first-order perturbation from the field of the finite nucleus is given by

$$\Delta E=[4\pi B/2\rho(2\rho+1)Z]\int_0^\infty n_P(r)r^{2\rho+2}dr. \tag{28}$$

In deriving this expression we have assumed the asymptotic formula (2) to be valid out to a value of r for which the proton density, $n_P(r)$, has fallen to a negligible value. The error introduced by this assumption in both ΔE and $\delta\Delta E$ is less than 0.3 percent for heavy nuclei.

Upon specializing this formula to the present case, with $n_P(r)$ defined by Eqs. (13), (26), and (27), we have

$$\Delta E=\frac{3BR^{2\rho}}{2\rho(2\rho+1)(2\rho+3)}[1+C_a a_P+C_b b_P], \tag{29}$$

with

$$C_a=(2\rho+3)\int_0^1 v_1(x)x^{2\rho+2}dx,$$

$$C_b=(2\rho+3)\int_0^1 v_2(x)x^{2\rho+2}dx.$$

Equation (29) reduces to Eq. (8), as it must, when $v_P(x)$ goes to zero. The isotope shift is now given by

$$\delta(\Delta E)=\frac{3BR^{2\rho}}{(2\rho+1)(2\rho+3)}\left\{[1+C_a a_P+C_b b_P]\frac{\delta R}{R}+\frac{C_a}{2\rho}\delta a_P+\frac{C_b}{2\rho}\delta b_P\right\}, \tag{30}$$

where, as before, the operator δ signifies the difference in the operand values for two isotopes. The term in (30) containing δR as a factor is, as we shall see, the dominant term, representing the contribution to the isotope shift both from the change in average charge density and from the change in surface position. The other two terms give the effect of the relative redistribution of the protons. We now turn to the evaluation of the quantities within the brace of Eq. (30).

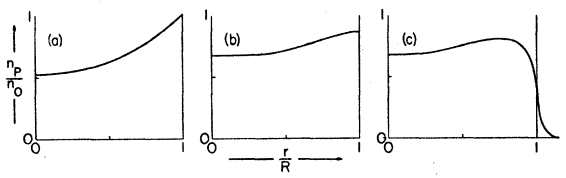

FIG. 8. Proton densities in the nuclear interior, in units of $n_0=3A/8\pi R^3$, plotted for three different models. (a) A rough estimate of the density variation is easily obtained by the method of Swiatecki (reference 29). (b) A somewhat more refined calculation by Feenberg (reference 28) adjusts the density to zero slope at the nuclear surface. (c) A self-consistent treatment of the density variation through the nuclear surface would presumably lead to such a curve as here plotted. In contrast with this estimate, however, is the result deduced by Gombas from a statistical theory of the nucleus (reference 36).

C. Relative Isotopic Radii

Just as, by Eq. (25), the particle distributions are determined which minimize the energy for a fixed radius, so the radius itself is chosen to minimize the total energy. When the potential and kinetic effects of the nuclear surface are ignored, the equilibrium radius is readily found[31] to have the value

$$R=R_0(1+E_c/E_0''). \tag{31}$$

The total Coulomb energy is, in units of $mc^2=0.511$ Mev, adequately given for our purposes by the expression

$$E_c=1.2Z^2A^{-\frac{1}{3}}, \tag{32}$$

and the compressibility coefficient may be written

$$E_0''=KA. \tag{33}$$

The factor K is poorly known, but order of magnitude estimates suggest[31] a value near 100 mc^2 or 150 mc^2. Present information[32] on nuclear radii is not adequate to fix precisely the average dependence on A, but if we accept the conventional estimate that R varies as $A^{\frac{1}{3}}$, then it would follow that R_0 varies as $A^{m/3}$, where m is a number near unity to be found below. R_0 is to be interpreted physically as the radius which the nucleus would take in the absence of Coulomb forces. It then follows (ignoring the A dependence of m) that

$$\frac{1}{R}\frac{\partial R}{\partial N}=\frac{1}{3A}\left[m-\frac{4E_c}{E_0''+E_c}\right],$$
$$\frac{1}{R}\frac{\partial R}{\partial Z}=\frac{1}{3A}\left[m-\frac{4E_c}{E_0''+E_c}\left(1-\frac{3}{2}\frac{A}{Z}\right)\right], \tag{34}$$

[30] As shown in reference 28, the two lowest-order polynomials (above the constant) which are orthonormal in the interval 0 to 1 for the weight factor x^2 and which meet the condition $dv(x)/dx=0$ for $x=0$ and 1, are $v_1(x)=6.1675(1-3.75x^2+2.5x^3)$ and $v_2(x)=11.028(1-13.7879x^2+24.1819x^3-11.2424x^4)$.

[31] E. Feenberg, Phys. Rev. **59**, 149 (1941).

[32] See, for example, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 15.

of which the former provides the value of $\delta R/R$ required in Eq. (30).

The dependence of nuclear radius on particle number here displayed for the compressible nucleus contrasts with the dependence in the incompressible nucleus:

$$\frac{1}{R}\frac{\partial R}{\partial N}=\frac{1}{R}\frac{\partial R}{\partial Z}=\frac{1}{R}\frac{dR}{dA}=\frac{1}{3A}. \tag{35}$$

Yet the last equality of this chain must, on the average, be satisfied in order to accord with empirical estimates on the average trend of nuclear radii with mass number. This requirement provides a determination of the index m appearing in Eqs. (34). We put

$$\frac{1}{R}\frac{dR}{dA}=\frac{1}{R}\frac{\partial R}{\partial N}\frac{dN}{dA}+\frac{1}{R}\frac{\partial R}{\partial Z}\frac{dZ}{dA}, \tag{36}$$

with dZ/dA and $dN/dA=1-(dZ/dA)$ found from the charge-to-mass stability curve[33] of the nuclides:

$$Z_A=A\,(1.981+0.01496A^{\frac{2}{3}})^{-1}. \tag{37}$$

It then follows that

$$\frac{3A}{R}\frac{dR}{dA}=1=m-\frac{4E_c}{E_0''+E_c}\left(1-\frac{3}{2}\frac{Z}{A}\frac{dZ}{dA}\right),$$

from which m is computed. The resultant[34] values of the radial derivatives given by Eqs. (34) are listed in Table I, for two different values of the compressibility coefficient E_0''.

TABLE I. The relative differential coefficients for the nuclear radius R as given by the compressible model for two different values of the compressibility coefficient. Also listed are the Coulomb energy E_c in units of mc^2, without correction for the small influence of nonuniform charge density; the index m of the relation $R_0=r_0A^{m/3}$; and the slope dZ/dA from Eq. (37). In an incompressible nuclear model both relative differential ratios are unity.

Z	A	E_c	E_0''	m	dZ/dA	$\frac{3A}{R}\frac{\partial R}{\partial Z}$	$\frac{3A}{R}\frac{\partial R}{\partial N}$
			(a) $E_0''=100Amc^2$				
24	50	179.8	5 000	0.953	0.430	1.248	0.814
44	100	489.1	10 000	0.936	0.394	1.386	0.750
62	150	854.2	15 000	0.928	0.367	1.495	0.713
80	200	1296.9	20 000	0.927	0.346	1.597	0.684
97	250	1773.8	25 000	0.928	0.329	1.688	0.663
			(b) $E_0''=50Amc^2$				
24	50	179.8	2 500	0.908	0.430	1.479	0.640
44	100	489.1	5 000	0.878	0.394	1.737	0.522
62	150	854.2	7 500	0.864	0.367	1.939	0.455
80	200	1296.9	10 000	0.863	0.346	2.126	0.404
97	250	1773.8	12 500	0.865	0.329	2.290	0.368

D. Evaluation of Particle Densities

The evaluation of the expansion coefficients of Eqs. (27) from Eq. (25) then is carried out by a procedure closely resembling Feenberg's.[28] Intermediate to this task is the calculation of partial derivatives of the standard nucleus energy, $E_0(R, N, Z)$, employing the relations (34) and the energy formula

$$E_0/mc^2=2A+[96|N-Z|+38(N-Z)^2/A], \tag{38}$$

in which the coefficient of the quadratic term has been chosen to accord with the same term in the semi-empirical mass formula.[33] The resulting expansion coefficients for the neutron and proton densities are listed in Table II, along with the densities near the surface of neutrons and protons, relative to their densities in the nuclear center.

E. Relevance of Nuclear Compressibility to Isotope Shifts

Having the particle densities we can now compare the isotope shifts predicted by the compressible and the incompressible nuclear model. For the latter, the isotope shift for two nuclides differing by one in mass number is given by (30) to be

$$\left(\frac{\delta(\Delta E)}{\delta N}\right)_{\text{incomp}}=\frac{BR^{2\rho}}{(2\rho+1)(2\rho+3)A}, \tag{39}$$

and the relative size of the isotope shifts in the two models is given by

$$\gamma=\left(\frac{\delta(\Delta E)}{\delta N}\right)_{\text{comp}}\Big/\left(\frac{\delta(\Delta E)}{\delta N}\right)_{\text{incomp}}$$
$$=3A\left\{[1+C_aa_P+C_bb_P]\frac{1}{R}\frac{\delta R}{\delta N}+\frac{C_a}{2\rho}\frac{\delta a_p}{\delta N}+\frac{C_b}{2\rho}\frac{\delta b_p}{\delta N}\right\}. \tag{40}$$

The evaluation of γ outlined in Table III makes it evident that the terms involving a_p, b_p and their derivatives are small and largely canceling in effect, so that the ratio γ is available in good approximation

[33] See, e.g., E. Fermi, *Nuclear Physics* (Notes by Orear, Rosenfeld, and Schluter) (University of Chicago Press, Chicago, 1950).

[34] We note that a simple attempt to introduce surface effects into our theory is of dubious value. If one makes the arbitrary assumption that the effects of the nuclear surface on the kinetic and potential energies of the system may be separated out of the nuclear Hamiltonian, the term proportional to $A^{\frac{2}{3}}$ in the semi-empirical mass formula (reference 33) may be chosen to represent the energy shift involved. Carrying through as before (reference 31) the calculation of how the actual radius is related to the standard radius R_0, one finds

$$R=R_0[1+(E_c-2E_s)/(E_0''+6E_s)], \tag{A}$$

with E_c and E_0'' as above, and $E_s=26A^{\frac{2}{3}}$ in units of mc^2. For nuclei to be stable against fission it is required [N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939)] that $E_c<2E_s$, and accordingly by (A) the actual radius is less than the "standard radius," R_0, the contraction by surface tension dominating the expansive Coulomb repulsion. The latter alone was considered in obtaining Eq. (31). On the other hand, the differential ratios derived from (A) have signs and magnitudes similar to the ratios given by (34), as derived from (31). In the present paper we have evaluated the simpler expressions (34) to obtain the differential coefficients required in (25) and (30).

TABLE II. Expansion coefficients for particle densities, and ratios of (density at surface) to (density at center) for protons and neutrons in the compressible nuclear model. Also listed are values (in units $mc^2=0.511$ Mev) of ΔE, the difference in energy between the compressible and incompressible models, a bilinear function of a_P, b_P, a_N, and b_N, which is minimized to determine these coefficients.

Z	N	a_P	b_P	a_N	b_N	ΔE	$n_P(R)/n_P(0)$	$n_N(R)/n_N(0)$
				(a) $E_0''=100Amc^2$				
24	50	−0.01785	0.00274	−0.00977	0.00079	−0.601	1.1217	1.0716
44	100	−0.02833	0.00519	−0.01612	0.00186	−2.618	1.1925	1.1160
62	150	−0.03502	0.00714	−0.02000	0.00277	−5.684	1.2355	1.1414
80	200	−0.04046	0.00887	−0.02322	0.00363	−10.014	1.2699	1.1617
97	250	−0.04457	0.01030	−0.02551	0.00434	−15.140	1.2948	1.1752
				(b) $E_0''=50Amc^2$				
24	50	−0.02075	0.00282	−0.01312	0.00088	−0.695	1.1479	1.1000
44	100	−0.03370	0.00548	−0.02263	0.00221	−3.096	1.2446	1.1738
62	150	−0.04188	0.00764	−0.02874	0.00342	−6.758	1.3044	1.2203
80	200	−0.04835	0.00959	−0.03376	0.00461	−11.897	1.3505	1.2577
97	250	−0.05310	0.01121	−0.03743	0.00563	−17.935	1.3825	1.2838

in the simple form

$$\gamma\simeq\frac{3A}{R}\frac{\delta R}{\delta N}. \tag{41}$$

The values of γ listed in Table III for two possible values of the compressibility reveal how directly the magnitude of the predicted isotope shift depends upon the assumed value of the compressibility coefficient. The trend of γ with A in Table III would be altered to yield more nearly constant values of γ in a theory in which surface effects were properly included, for the effective surface tension is relatively more important in light nuclei. As cited earlier[6,9] and as emphasized by Humbach,[10] the magnitudes of observed isotope shifts are about one-half the values predicted for an incompressible nucleus. In view of the present results, it seems likely that a major part, and perhaps all, of this discrepancy depends upon the neglect of nuclear compressibility.

A reduction in the nuclear radius would of course also reduce the isotope shifts predicted by Eqs. (6), (9), or (30). Recent studies of radiations from the μ-mesonic atom suggest[35] that the nuclear radius has been overestimated; however, similar studies by Schawlow and Townes[11] on radiations from inner electron transitions have confirmed the radius estimates employed in the present paper and seem more reliable as a guide for our analysis of the isotope shifts in *electronic* spectra. This information on x-ray levels also casts doubt on the reality of the Gaussian particle density distribution for heavy elements derived[36] by Gombas in a detailed statistical treatment of the nucleus.

Note added in proof:—Recent observations (to be published) by Hofstadter, Fechter, McIntyre and analyses (to be published) by Schiff of nuclear scattering of fast electrons suggest, however, a proton distribution peaked in the nuclear center, in qualitative agreement with the results of Gombas.

[35] L. N. Cooper and E. M. Henley, Bull. Am. Phys. Soc. **28**, No. 3, 56 (1953).
[36] P. Gombas, Acta Phys. Acad. Sci. Hung. **2**, 223 (1952).

The value of E_0'' required to reduce γ to 0.5 is smaller by a factor of 1.5 or 2 than the values of E_0'' estimated[31] with the help of the virial theorem, which, however, are only order-of-magnitude estimates, uncertain by a factor of this size. If we have employed a value for the nuclear radius which is 10 or 15 percent too large, then Feenberg's values for E_0'' are more nearly consistent with the values required by the present analysis of isotope shifts. We should emphasize, moreover, that the artificial method of handling the nuclear surface which we have here adopted renders our treatment of the variations in nuclear radii and particle densities so crude that we can expect nothing better than order-of-magnitude estimates of nuclear compressibility to follow at present from our examination of isotope shifts. The present work does, nevertheless, suggest that the compressibility coefficient E_0'' is smaller than heretofore assumed, and indicates that when a self-consistent treatment of nuclear particle densities is available, the study of isotope shifts will

TABLE III. Calculation of the ratio, γ, of isotope shifts predicted according to Eq. (40) by the compressible and by the incompressible nuclear models. The perturbation coefficients, C_a and C_b, are defined in Eq. (29); and the isotopic variations of the expansion coefficients are defined by such statements as

$$\delta a_P/\delta N=a_P(Z,A+1)-a_P(Z,A).$$

Z	A	C_a	C_b	$10^4\frac{\delta a_P}{\delta N}$	$10^4\frac{\delta b_P}{\delta N}$	$\frac{3A}{R}\frac{\delta R}{\delta N}$	γ
				(a) $E_0''=100Amc^2$			
24	50	−0.7087	0.213	1.543	−0.058	0.814	0.816
44	100	−0.6917	0.200	1.243	−0.095	0.750	0.751
62	150	−0.6659	0.181	0.978	−0.098	0.713	0.713
80	200	−0.6256	0.154	0.809	−0.097	0.684	0.683
97	250	−0.5684	0.120	0.676	−0.092	0.663	0.660
				(b) $E_0''=50Amc^2$			
24	50	−0.7087	0.213	1.710	−0.057	0.640	0.640
44	100	−0.6917	0.200	1.389	−0.097	0.522	0.519
62	150	−0.6659	0.181	1.095	−0.103	0.455	0.449
80	200	−0.6256	0.154	0.899	−0.104	0.404	0.395
97	250	−0.5684	0.120	0.748	−0.099	0.368	0.357

provide one of the most direct and precise methods of measuring nuclear compressibility.

F. Aspherical Nuclei

In aspherical nuclei of deformation α (see Part III) the proton density variation from the average density is enhanced by the approximate factor $(1+\alpha)$ along the principal axis of distortion. This aspherical variation gives a negligible displacement in the *mean* value of predicted isotope shifts. The amplitude of the periodic variation in the ratio of predicted shifts to shifts predicted by the simple volume effect, discussed in Part III, will be augmented by roughly $(1+\frac{1}{3}\langle|\alpha|\rangle_{Av})$. As the average value of the magnitude of intrinsic deformation, $\langle|\alpha|\rangle_{Av}$, is of the order of 0.1, the correction to the amplitude of periodic variation is likewise unimportant.

V. CONCLUSION

In Part II we have noted that when the observed isotope shifts have been adjusted for electronic screening and compared with the isotope shifts predicted for constant density nuclei, after these have been corrected for the deficiency of first-order perturbation theory, three anomalies appear: (1) The average magnitudes of the observed shifts are about one-half the size of the predicted shifts; (2) the observed shifts show a somewhat periodic variation about the predicted shifts, and (3) the shifts between even and odd isotopes are by staggered rather than by equal steps.

In Part III the regular progression of nuclear deformations was reviewed, as deduced mainly from individual particle configuration theory, from observed quadrupole moments, and from the analysis of the first excited states of even-even nuclei. The dependence of isotope shift on nuclear shape was also derived, and the result was then at hand that the estimated progression of nuclear shapes apparently accounts for the regular fluctuations in size of isotope shifts, both as to the amplitude of the fluctuation and in the location of the nodal points. On this basis a prediction, as yet untested, is also made of isotope shift behavior beyond $_{82}$Pb (Fig. 3).

An examination of how nuclear shapes should change between odd and even isotopes likewise provides a reasonable explanation of the observed even-odd staggering of isotope shifts.

In Part IV the influence of nuclear compressibility and polarizability on the size of isotope shifts was investigated and found to alter predictions in a direction to account for the apparent discrepancy in average size of the isotope shifts. The analysis and comparison with experiment suggests that earlier estimates of nuclear compressibility have been too small, although the present treatment is inadequate to provide a reliable measure of the compressibility. A reduction in nuclear radius would also reduce the predicted isotope shift magnitudes but would conflict with x-ray measurements of the inner electron energy levels.

It therefore appears that the general pattern of isotope shifts is consistent with present ideas of nuclear structure, and provides an important source of specific information on the sizes, shapes, and charge distributions of nuclei.

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