The Energy Levels of Odd-Odd Nuclei*

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A derivation of a closed formula for the energy levels of odd-odd nuclei is described, and the plausibility of the Nordheim rule is discussed.

INTRODUCTION

ODD-EVEN nuclei probably constitute the simplest group of nuclei with which the ideas of the shell model can be checked. The reason for this is that the energies involved in exciting the even-even core are believed to be bigger than the separation between adjacent single-particle levels, so that it might be a good approximation to assume that the core remains in its lowest state of total angular momentum zero for the ground states of odd-even nuclei, as well as for their first few excited states.

There is some hope that the situation in odd-odd nuclei is not much more complex than in the odd-even ones. The interaction between a proton and a neutron in nonequivalent orbits is expected to be considerably smaller than the one between two equivalent particles, and may, therefore, be of the order of the separation between single-particle levels or even smaller. Accepting the current interpretation of the odd-even level structure, one is tempted to assume that a state of an odd-odd nucleus is characterized by the angular momenta and parities of the separate proton and neutron groups in addition, of course, to the total angular momentum and parity. One also naturally assigns to the proton and neutron groups those spins and parities as are observed in close-lying odd-even nuclei.

Under these assumptions it was found by Nordheim[1] that the following empirical rules apply to the spin J of the ground states of odd-odd nuclei:

$$J=|j_p-j_n| \text{ if } l_p+j_p+l_n+j_n \text{ is even,}$$
$$|j_p-j_p|<J\lesssim j_p+j_n \text{ if } l_p+j_p+l_n+j_n \text{ is odd,}$$

where $(-1)^l$ and j are, respectively, the parity and spin of the proton and neutron groups (indices p and n).

The present work is an attempt to explain Nordheim's rule by adopting the single-particle model for each of the "subgroups" of neutrons and protons, and assuming further that the interaction between the odd proton and the odd neutron, which is considered as a perturbation on the central field in which they move, is given by the expression:

$$V=[a+b\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_p]\delta(\mathbf{r}_n-\mathbf{r}_p).$$

That a perturbation which depends on the relative coordinates only will probably not yield Nordheim's rule is evident from Racah's results on the energy levels of two nucleons in the jj coupling.[2] It is clear that Nordheim's rule would result for a potential of the type $\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_pV(|\mathbf{r}_n-\mathbf{r}_p|)$ without too severe limitations on $V(x)$. It is, however, still interesting to see how much spin dependent force we should introduce, and whether the different character of the two rules, namely the definite answer of one of them and the rather vague answer of the other, can be understood.

NOTATION

We shall find it very convenient to use a notation introduced by Wigner[3] which exhibits most clearly the symmetry properties of the different coefficients involved in the addition of angular momenta. For convenience we also reproduce here, without proof, some of the relations between these coefficients.[4]

The vector-addition coefficient is denoted by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}. \tag{1}$$

It is related to the usual Clebsch-Gordan coefficient[5] by the relation:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}=\frac{(-1)^{j_2-j_1+\mu_3}}{(2j_3+1)^{\frac{1}{2}}}(j_1j_2j_3-\mu_3|j_1\mu_1j_2\mu_2). \tag{2}$$

To prevent confusion we shall call this coefficient a Wigner coefficient. A Wigner coefficient vanishes unless $\mathbf{j}_1+\mathbf{j}_2+\mathbf{j}_3=0$. (This equation should be understood as meaning that the three numbers j_1, j_2, and j_3 satisfy the triangular inequalities, and that in addition $\mu_1+\mu_2+\mu_3=0$.)

The symmetry properties of the Wigner coefficients are given by:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}=\epsilon\begin{pmatrix} j_k & j_l & j_m \\ \mu_k & \mu_l & \mu_m \end{pmatrix}, \tag{3}$$

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[1] L. W. Nordheim, Phys. Rev. **78**, 294 (1950).

[2] G. Racah, Phys. Rev. **62**, 438 (1942).

[3] E. P. Wigner, "On the Matrices which Reduce the Kronecker Products of Representations of Simply-Reducible Groups" (unpublished). I am indebted to Professor Wigner for making a copy of his manuscript available to me.

[4] See also: Biedenharn, Blatt, and Rose, Revs. Modern Phys. **24**, 249 (1952).

[5] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1951).

where ϵ is $+1$ or $(-1)^{j_1+j_2+j_3}$ according to whether (k, l, m) is an even or odd permutation of $(1, 2, 3)$. Also:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -\mu_1 & -\mu_2 & -\mu_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}. \quad (4)$$

The orthogonality relations take the forms:

$$\sum_{j_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1' & \mu_2' & \mu_3 \end{pmatrix} = \delta(\mu_1, \mu_1')\delta(\mu_2, \mu_2'), \quad (5a)$$

$$(2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ \mu_1 & \mu_2 & \mu_3' \end{pmatrix} = \delta(\mu_3, \mu_3')\delta(j_3, j_3'), \quad (5b)$$

where the summation convention is applied to double Greek indices.

The Racah coefficients will be denoted by

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}. \quad (6)$$

They are related to Racah's W^2 by the relation

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = (-1)^{j_1+l_1+j_2+l_2} \cdot W(j_1 j_2 l_2 l_1; j_3 l_3). \quad (7)$$

A Racah coefficient vanishes unless:

$$\mathbf{j}_1+\mathbf{j}_2+\mathbf{j}_3=0, \quad \mathbf{l}_1-\mathbf{l}_2+\mathbf{j}_3=0, \quad \mathbf{j}_1+\mathbf{l}_2-\mathbf{l}_3=0, \quad -\mathbf{l}_1+\mathbf{j}_2+\mathbf{l}_3=0.$$

The symmetry properties of the Racah coefficients are

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_k & j_l & j_m \\ l_k & l_l & l_m \end{Bmatrix} = \begin{Bmatrix} l_k & l_l & j_m \\ j_k & j_l & l_m \end{Bmatrix}, \quad (8)$$

where $(k\,l\,m)$ is any permutation of $(1, 2, 3)$. The orthogonality relation for the Racah coefficients is

$$\sum_{j_3} (2j_3+1)(2l_3+1) \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3' \end{Bmatrix} = \delta(l_3, l_3'). \quad (9)$$

In addition they satisfy the following relations:

$$\sum_{l_3} (-1)^{j_3+j_3'+l_3}(2l_3+1) \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \begin{Bmatrix} j_1 & l_1 & j_3' \\ j_2 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & l_1 & j_3' \end{Bmatrix}, \quad (10a)$$

$$\sum_{k} (-1)^{j_1+j_2+j_3+l_1+l_2+l_3+k_1+k_2+k_3+k}(2k+1) \begin{Bmatrix} j_1 & k & l_1 \\ l_2 & k_3 & j_2 \end{Bmatrix} \begin{Bmatrix} j_2 & k & l_2 \\ l_3 & k_1 & j_3 \end{Bmatrix} \begin{Bmatrix} j_3 & k & l_3 \\ l_1 & k_2 & j_1 \end{Bmatrix} = \begin{Bmatrix} j_1 & k_2 & j_3 \\ k_1 & j_2 & k_3 \end{Bmatrix} \begin{Bmatrix} l_1 & k_2 & l_3 \\ k_1 & l_2 & k_3 \end{Bmatrix}, \quad (10b)$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ 0 & l_2 & l_3 \end{Bmatrix} = (-1)^{j_1+j_2+j_3}[(2j_2+1)(2j_3+1)]^{-\frac{1}{2}} \times \delta(l_2, j_3)\delta(l_3, j_2). \quad (11)$$

The following relations are satisfied by combinations of Racah and Wigner coefficients:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ \lambda_1 & \lambda_2 & -\mu_3 \end{pmatrix} = \sum_{l_3} (-1)^{j_3+l_3+\mu_1+\lambda_1} \times (2l_3+1) \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ \lambda_1 & \mu_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} j_1 & l_2 & l_3 \\ \mu_1 & \lambda_2 & -\lambda_3 \end{pmatrix}, \quad (12)$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} = (-1)^{l_1+l_2+l_3+\lambda_1+\lambda_2+\lambda_3} \times \begin{pmatrix} j_1 & l_2 & l_3 \\ \mu_1 & \lambda_2 & -\lambda_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -\lambda_1 & \mu_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ \lambda_1 & -\lambda_2 & \mu_3 \end{pmatrix}, \quad (13)$$

$$\begin{pmatrix} j_{11} & j_{12} & j_{13} \\ \lambda_{11} & \lambda_{12} & \lambda_{13} \end{pmatrix} \begin{pmatrix} j_{21} & j_{22} & j_{23} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{pmatrix} \begin{pmatrix} j_{31} & j_{32} & j_{33} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \times \begin{pmatrix} j_{11} & j_{21} & j_{31} \\ \lambda_{11} & \lambda_{21} & \lambda_{31} \end{pmatrix} \begin{pmatrix} j_{12} & j_{22} & j_{32} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} \end{pmatrix} \begin{pmatrix} j_{13} & j_{23} & j_{33} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix}$$

$$= \sum_{j} (-1)^{2j}(2j+1) \begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{23} & j_{33} & j \end{Bmatrix} \begin{Bmatrix} j_{21} & j_{22} & j_{23} \\ j_{12} & j & j_{32} \end{Bmatrix} \times \begin{Bmatrix} j_{31} & j_{32} & j_{33} \\ j & j_{11} & j_{21} \end{Bmatrix} = \begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix}. \quad (14)$$

(14) defines Wigner's nine-j symbol, which is also called an X coefficient (Racah and Fano) or a Schwinger coefficient.[6] It is easy to see from the symmetry properties of the Wigner coefficients that the interchange of any two rows or two columns in a Schwinger coefficient multiples it by the factor

$$(-1)^{j_{11}+j_{12}+j_{13}+j_{21}+j_{22}+j_{23}+j_{31}+j_{32}+j_{33}}.$$

OUTLINE OF THE COMPUTATION

We shall assume that in the zeroth-order approximation the particles are moving in a central field with no interaction between them. The central field, ac-

[6] J. Schwinger, Nuclear Development Associates Report NYO-3071 (unpublished).

cording to this approach, takes care of the major part of the mutual interaction between the nucleons, the remaining part of the interaction being treated as a small perturbation. Limiting our considerations to two particles in the jj-coupling scheme (l_1j_1) and (l_2j_2), the degenerate states of the configuration j_1j_2 will be split by the perturbation, and the first-order correction to the energy of a state with a total angular momentum J will be given by

$$E(j_1j_2, J)=\int\psi^*(j_1j_2JM)V\psi(j_1j_2JM)d\tau. \quad (15)$$

We shall take as our wave function for the state (j_1j_2, JM):

$$\Psi(j_1j_2JM)=(-1)^{j_1-j_2+M}(2J+1)^{\frac{1}{2}} \times\begin{pmatrix} j_1 & j_2 & J \\ \mu_1 & \mu_2 & -M\end{pmatrix}\psi_1(j_1\mu_1)\psi_2(j_2\mu_2). \quad (16)$$

For particles in equivalent states $(j_1=j_2)$ this gives antisymmetric states for even J and symmetric ones for odd J, etc. No τ-dependence is assumed, so that this wave function is good only for two distinct particles such as a proton and a neutron, provided they are not considered as two different states of the same entity, i.e., provided the Pauli principle should not be applied to them. This assumption is probably good for the heavier odd-odd nuclei.

Introducing the notation

$$\sigma_0^{(0)}=1, \quad \sigma_{-1}^{(1)}=-\tfrac{1}{2}(\sigma_x-i\sigma_y),$$
$$\sigma_0^{(1)}=\sigma_z, \quad \sigma_{+1}^{(1)}=\tfrac{1}{2}(\sigma_x+i\sigma_y),$$

and expanding $V(|\mathbf{r}_1-\mathbf{r}_2|)$ in Legendre polynomials of the angle ω between $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$V(|\mathbf{r}_1-\mathbf{r}_2|)=\sum v_k(|r_1|, |r_2|)P_k(\cos\omega), \quad (17)$$

we find that the first-order correction to the energy of the state (j_1j_2, JM) due to a perturbation $\boldsymbol{\sigma}_1^{(n)}\cdot\boldsymbol{\sigma}_2^{(n)}V(|\mathbf{r}_1-\mathbf{r}_2|)$ can be written in the form

$$E(j_1j_2J)=\sum\nolimits_k f_{nk}(j_1j_2J)F_k(n_1l_1n_2l_2), \quad (18)$$

where

$$f_{nk}=(2k+1)(j_1j_2JM|\boldsymbol{\sigma}_1^{(n)}\cdot\boldsymbol{\sigma}_2^{(n)}P_k(\cos\omega)|j_1j_2JM), \quad (19)$$

and

$$F_k=1/(2k+1)\int R_1^2(n_1l_1)R_2^2(n_2l_2)v_k(r_1, r_2)dr_1dr_2, \quad (20)$$

$R(nl)/r$ being the radial part of the single-particle wave function $\psi(nljm)$.

The integrals f_{nk} can be evaluated exactly since they do not involve any unknown parameters of the central field. This is most easily done by the tensor-operator algebra developed by Racah,[2] and in fact the explicit expression for f_{0k} was given by Racah as early as 1942. We shall here follow a slightly different way which is more convenient for the summations involved in the special case of delta forces.[7]

Following Racah we decompose $P_k(\cos\omega)$ by the addition theorem for spherical harmonics,

$$P_k(\cos\omega)=\sum\nolimits_\kappa(-1)^\kappa C_\kappa^{(k)}(1)C_{-\kappa}^{(k)}(2),$$
$$C_\kappa^{(k)}(i)=[4\pi/(2k+1)]^{\frac{1}{2}}Y_\kappa^{(k)}(\theta_i\varphi_i),$$

and get for (19) the expression:

$$f_{nk}=\sum_{\kappa,\gamma}(2k+1)(j_1j_2JM|\sigma_\gamma^{(n)}(1)C_\kappa^{(k)}(1) \cdot\sigma_{-\gamma}^{(n)}(2)C_{-\kappa}^{(k)}(2)|j_1j_2JM)\cdot(-1)^{\gamma+\kappa}. \quad (21)$$

Since both $\sigma^{(n)}$ and $C^{(k)}$ are reduced tensor operators, their product can be reduced into a sum of tensor operators of orders $r(|k-n|\leqslant r\leqslant k+n)$ which are given by

$$T_\rho^{(r)}=(-1)^{k+n+\rho}(2r+1)^{\frac{1}{2}}\begin{pmatrix} n & k & r \\ \gamma & \kappa & -\rho\end{pmatrix}\sigma_\gamma^{(n)}C_\kappa^{(k)}. \quad (22)$$

If we use (5), we obtain

$$f_{nk}=(-1)^{n+k}(2k+1) \times(j_1j_2JM|\sum\nolimits_r(-1)^r\mathbf{T}_1^{(r)}\cdot\mathbf{T}_2^{(r)}|j_1j_2JM), \quad (23)$$

where $\mathbf{A}\cdot\mathbf{B}$ stands for the scalar product of the tensor operators $\mathbf{A}$ and $\mathbf{B}$.

The matrix elements for the scalar product of tensor operators were given by Racah,[2] and in the present notation one obtains

$$f_{nk}=(-1)^{j_1+j_2+J}\sum_r(-1)^{n+k+r}(2k+1)\begin{Bmatrix} j_1 & j_2 & J \\ j_2 & j_1 & r\end{Bmatrix} \times(j_1\|T_1^{(r)}\|j_1)(j_2\|T_2^{(r)}\|j_2). \quad (24)$$

By definition,

$$(-1)^{j-\mu}(slj\|T^{(r)}\|s'l'j')\begin{pmatrix} j & r & j' \\ -\mu & \rho & \mu'\end{pmatrix} =(slj\mu|T_\rho^{(r)}|s'l'j'\mu')=(-1)^{k+n+\rho}(2r+1)^{\frac{1}{2}} \times\begin{pmatrix} n & k & r \\ \gamma & \kappa & -\rho\end{pmatrix}(slj\mu|\sigma_\gamma^{(n)}C_\kappa^{(k)}|s'l'j'\mu'). \quad (25)$$

If we transform the last matrix element from the $(slj\mu)$ scheme to the $(s\mu_s l\mu_l)$ scheme, and remember that both $\boldsymbol{\sigma}^{(n)}$ and $\mathbf{C}^{(k)}$ are tensor operators with respect to $\mathbf{s}$ and $\mathbf{l}$, respectively, we find that

$$(slj\|T^{(r)}\|s'l'j')=[(2r+1)(2j+1)(2j'+1)]^{\frac{1}{2}} \times(s\|\sigma^{(n)}\|s')(l\|C^{(k)}\|l')\begin{Bmatrix} n & k & r \\ s & l & j \\ s' & l' & j'\end{Bmatrix}. \quad (26)$$

We may note here that due to the symmetry properties of the Schwinger coefficient the diagonal matrix elements of $T^{(r)}$ vanish unless $k+n+r$ is even.

[7] Compare also similar calculations by: M. H. L. Pryce, Proc. Phys. Soc. (London) **A65**, 773 (1952); D. Kurath, Phys. Rev. **87**, 218 (1952); B. Feld and L. Marks (private communication); I. Talmi, Phys. Rev. **90**, 1001 (1953).

From (26) we obtain for (19) the expression

$$f_{nk}=(-1)^{j_1+j_2+J}(2j_1+1)(2j_2+1)(2k+1)\times(s_1\|\sigma^{(n)}\|s_1)(s_2\|\sigma^{(n)}\|s_2)(l_1\|C^{(k)}\|l_1)\times(l_2\|C^{(k)}\|l_2)\sum_r(-1)^{k+n+r}\times\begin{Bmatrix} j_1 & j_2 & J \\ j_2 & j_1 & r\end{Bmatrix}\begin{Bmatrix} n & k & r \\ s_1 & l_1 & j_1 \\ s_1 & l_1 & j_1\end{Bmatrix}\begin{Bmatrix} n & k & r \\ s_2 & l_2 & j_2 \\ s_2 & l_2 & j_2\end{Bmatrix}. \quad (27)$$

The sum over r can be performed yielding a relatively simple expression. Since, however, we are interested in the values of the energies for delta interaction, we shall find it easier to evaluate the sum over k first. We also notice that n can assume only two values (since $s=\frac{1}{2}$): 0 and 1; for $n=0$ the summation over k and r reduces to a very simple expression. It is therefore very convenient to sum over n too and obtain the value of $E(j_1j_2,J)$ for $n=1$ by subtracting the contribution of $n=0$.

To proceed with the calculation we note that for

$$V(|\mathbf{r}_1-\mathbf{r}_2|)=\delta(r_1-r_2)\delta(\cos\omega-1)_{r_1r_2},\quad v_k(|r_1|,|r_2|)=(2k+1)_{r_1r_2}, \quad (28)$$

so that $F_k=F_0$ for every k. $(l\|C^{(k)}\|l)$ was computed by Racah:[2]

$$(l\|C^{(k)}\|l)=(-1)^l(2l+1)\begin{pmatrix} l & l & k \\ 0 & 0 & 0\end{pmatrix}.$$

It is also easy to see that $(s\|\sigma^{(n)}\|s)=[2(2n+1)]^{\frac{1}{2}}$.

If we combine all these results we obtain (expressing E in units of F_0)

$$\frac{1}{2}\sum_{n=0,1}E_n(j_1j_2J)=\sum_{n,k,r}(-1)^{j_1+j_2+J+n+2k+r}(2j_1+1)(2j_2+1)\times(2l_1+1)(2l_2+1)(2n+1)(2k+1)(2r+1)\times\begin{pmatrix} l_1 & l_1 & k \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_2 & l_2 & k \\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} j_1 & j_2 & J \\ j_2 & j_1 & r\end{Bmatrix}\times\begin{Bmatrix} n & k & r \\ s_1 & l_1 & j_1 \\ s_1 & l_1 & j_1\end{Bmatrix}\begin{Bmatrix} n & k & r \\ s_2 & l_2 & j_2 \\ s_2 & l_2 & j_2\end{Bmatrix}=\sum_{nrk_1k_2}(-1)^{j_1+j_2+J+n+r}\times(2j_1+1)(2j_2+1)(2l_1+1)(2l_2+1)(2n+1)\times(2r+1)\cdot\begin{Bmatrix} j_1 & j_2 & J \\ j_2 & j_1 & r\end{Bmatrix}(-1)^{k_1}(2k_1+1)\times\begin{pmatrix} n & k_1 & r \\ \gamma & 0 & \rho\end{pmatrix}\begin{pmatrix} l_1 & l_1 & k_1 \\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} n & k_1 & r \\ s_1 & l_1 & j_1 \\ s_1 & l_1 & j_1\end{Bmatrix}(-1)^{k_2}\times(2k_2+1)\begin{pmatrix} n & k_2 & r \\ \gamma & 0 & \rho\end{pmatrix}\begin{pmatrix} l_2 & l_2 & k_2 \\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} n & k_2 & r \\ s_2 & l_2 & j_2 \\ s_2 & l_2 & j_2\end{Bmatrix}. \quad (29)$$

The sums over k_1 and k_2 can be carried out separately by using the following relation which is easy to verify:

$$\sum_r(2r+1)\begin{pmatrix} n & k & r \\ \gamma & \kappa & \rho\end{pmatrix}\begin{pmatrix} r & j & j' \\ \rho & \iota & \iota'\end{pmatrix}\begin{Bmatrix} n & k & r \\ s & l & j \\ s' & l' & j'\end{Bmatrix}=\begin{pmatrix} n & s & s' \\ \gamma & \sigma & \sigma'\end{pmatrix}\begin{pmatrix} k & l & l' \\ \kappa & \lambda & \lambda'\end{pmatrix}\begin{pmatrix} s & l & j \\ \sigma & \lambda & \iota\end{pmatrix}\begin{pmatrix} s' & l' & j' \\ \sigma' & \lambda' & \iota'\end{pmatrix}. \quad (30)$$

The sum over n can then be carried out by applying directly the orthogonality relations of the Wigner coefficients, and the sum over r results immediately by the use of the relations involving the Racah coefficients.

The final expression is

$$\frac{1}{2}\sum_n E_n(j_1j_2J)=(-1)^{l_1+l_2+j+1}(2j_1+1)(2j_2+1)(2l_1+1)(2l_2+1)\times\begin{pmatrix} j_1 & j_2 & J \\ \alpha_1 & -\alpha_2 & \alpha_2-\alpha_1\end{pmatrix}\begin{pmatrix} j_1 & j_2 & J \\ \alpha_2 & -\alpha_1 & \alpha_1-\alpha_2\end{pmatrix}\times\begin{pmatrix} s_1 & l_1 & j_1 \\ \alpha_1 & 0 & -\alpha_1\end{pmatrix}\begin{pmatrix} s_1 & l_1 & j_1 \\ \alpha_2 & 0 & -\alpha_2\end{pmatrix}\times\begin{pmatrix} s_2 & l_2 & j_2 \\ \alpha_1 & 0 & -\alpha_1\end{pmatrix}\begin{pmatrix} s_2 & l_2 & j_2 \\ \alpha_2 & 0 & -\alpha_2\end{pmatrix}. \quad (31)$$

Since both α_1 and α_2 can have only the values $\pm\frac{1}{2}$ and since

$$\begin{pmatrix} \frac{1}{2} & l & j \\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}^2=\frac{1}{2(2l+1)},$$

we get

$$\frac{1}{2}\sum E_n(j_1j_2J)=\frac{1}{2}(2j_1+1)(2j_2+1)\times\left\{\begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}^2-(-1)^{l_1+l_2+J}\begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}^2\right\}. \quad (32)$$

This expression can be further simplified by means of the relation

$$\begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & \frac{1}{2} & -1\end{pmatrix}^2=\frac{1}{4J(J+1)}\{(2j_1+1)+(-1)^{j_1+j_2+J}\times(2j_2+1)\}^2\begin{pmatrix} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0\end{pmatrix}^2. \quad (33)$$

Noting that

$$\begin{Bmatrix} 0 & k & r \\ s & l & j \\ s & l & j\end{Bmatrix}=(-1)^{k+s+l+j}[2(2k+1)]^{-\frac{1}{2}}\cdot\begin{Bmatrix} k & j & j \\ s & l & l\end{Bmatrix}\delta(k,r), \quad (34)$$

we get, for $n=0$,

$$E_0(j_1j_2J)=\tfrac{1}{2}(2j_1+1)(2j_2+1)\begin{pmatrix} j_1 & j_2 & J \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{pmatrix}^2 \left[1+\frac{[(2j_1+1)+(-1)^{j_1+j_2+J}(2j_2+1)]^2}{4J(J+1)}\right], \quad (35a)$$

and therefore

$$E_1(j_1j_2J)=\tfrac{1}{2}(2j_1+1)(2j_2+1)\begin{pmatrix} j_1 & j_2 & J \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{pmatrix}^2 \times\left[\frac{[(2j_1+1)+(-1)^{j_1+j_2+J}(2j_2+1)]^2}{4J(J+1)} -(1+2(-1)^{l_1+l_2+J})\right]. \quad (35b)$$

The last two formulas obtain an especially simple form when $j_1=j_2$ and $l_1=l_2$:

$$E_0(j^2J)=\tfrac{1}{2}(2j+1)^2\begin{pmatrix} j & j & J \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{pmatrix}^2 \times\begin{cases} 1 & \text{for even } J, \\ [1+(2j+1)^2/J(J+1)] & \text{for odd } J; \end{cases} \quad (36a)$$

$$E_1(j^2J)=\tfrac{1}{2}(2j+1)^2\begin{pmatrix} j & j & J \\ \tfrac{1}{2} & -\tfrac{1}{2} & 0 \end{pmatrix}^2 \times\begin{cases} -3 & \text{for even } J, \\ [1+(2j+1)^2/J(J+1)] & \text{for odd } J. \end{cases} \quad (36b)$$

For equivalent nucleons only even-J states are allowed (because of the Pauli principle) and one sees that a potential of the type $[3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]\delta(\mathbf{r}_1-\mathbf{r}_2)$ yields no interaction in these states corresponding to the fact that $3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ vanishes for states symmetric in the coordinates (and therefore antisymmetric in the spins); whereas, $\delta(\mathbf{r}_1-\mathbf{r}_2)$ vanishes for states antisymmetric in the coordinates. For similar reasons, one can see that for these states the Wigner force and the Majorana force $[\tfrac{1}{4}(1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ for particles of the same charge$]$ give the same results for the energy levels in the delta limit.

NORDHEIM'S RULE

On the basis of the analysis of β transitions in even-A nuclei, Nordheim has deduced the empirical rule for the spins of the ground states of odd-odd nuclei.[1] When the states of the two odd nucleons are l_1j_1 and l_2j_2, this rule can be stated with the help of the Nordheim number $N=l_1+j_1+l_2+j_2$ in the following way:

For even N, $J=|j_1-j_2|$; for odd N, $J>|j_1-j_2|$ and is usually of the order of j_1+j_2.

TABLE I. $|(j_1\tfrac{1}{2}\, j_2-\tfrac{1}{2}|j_1j_2\, j0)|^2$. The Clebsch-Gordan coefficient is minus or plus the square root of the entry in the table according to whether this entry is, or is not, preceded by an asterisk.

	$j_2\diagdown j_1$	1/2	3/2	5/2	7/2	9/2
$J=0$	1/2	1/2	0	0	0	0
	3/2	0	*1/4	0	0	0
	5/2	0	0	1/6	0	0
	7/2	0	0	0	*1/8	0
	9/2	0	0	0	0	1/10
$J=1$	1/2	1/2	1/2	0	0	0
	3/2	1/2	*1/20	*3/10	0	0
	5/2	0	*3/10	1/70	3/14	0
	7/2	0	0	3/14	*1/168	*1/6
	9/2	0	0	0	*1/6	1/330
$J=2$	1/2	0	1/2	1/2	0	0
	3/2	1/2	1/4	*1/14	*9/28	0
	5/2	1/2	*1/14	*4/21	1/42	5/21
	7/2	0	*9/28	1/42	25/168	*5/462
	9/2	0	0	5/21	*5/462	*4/33
$J=3$	1/2	0	0	1/2	1/2	0
	3/2	0	9/20	1/5	*1/12	*1/3
	5/2	1/2	1/5	*4/45	*1/6	1/33
	7/2	1/2	*1/12	*1/6	3/88	3/22
	9/2	0	*1/3	1/33	3/22	12/385
$J=4$	1/2	0	0	0	1/2	1/2
	3/2	0	0	3/7	5/28	*1/11
	5/2	0	3/7	1/7	*15/154	*12/77
	7/2	1/2	5/28	*15/154	*81/616	81/2002
	9/2	1/2	*1/11	*12/77	81/2002	81/385
$J=5$	1/2	0	0	0	0	1/2
	3/2	0	0	0	5/12	1/6
	5/2	0	0	25/63	5/42	*4/39
	7/2	0	5/12	5/42	*75/728	*3/26
	9/2	1/2	1/6	*4/39	*3/26	3/65
$J=6$	1/2	0	0	0	0	0
	3/2	0	0	0	0	9/22
	5/2	0	0	0	25/66	7/66
	7/2	0	0	25/66	25/264	*7/66
	9/2	0	9/22	7/66	*7/66	*16/165
$J=7$	1/2	0	0	0	0	0
	3/2	0	0	0	0	0
	5/2	0	0	0	0	105/286
	7/2	0	0	0	1225/3432	35/429
	9/2	0	0	105/286	35/429	*784/7293
$J=8$	1/2	0	0	0	0	0
	3/2	0	0	0	0	0
	5/2	0	0	0	0	0
	7/2	0	0	0	0	49/143
	9/2	0	0	0	49/143	49/715
$J=9$	1/2	0	0	0	0	0
	3/2	0	0	0	0	0
	5/2	0	0	0	0	0
	7/2	0	0	0	0	0
	9/2	0	0	0	0	3969/12155

We shall now proceed to investigate our results in order to see whether they offer any way of explaining these regularities. It should be emphasized that we do not know as yet any reason to believe that the delta forces give best description of the interactions between nucleons in a nucleus, although they might represent not too bad an approximation.

For pure Wigner forces one gets a dependence of the energy levels on the j's of the interacting particles only

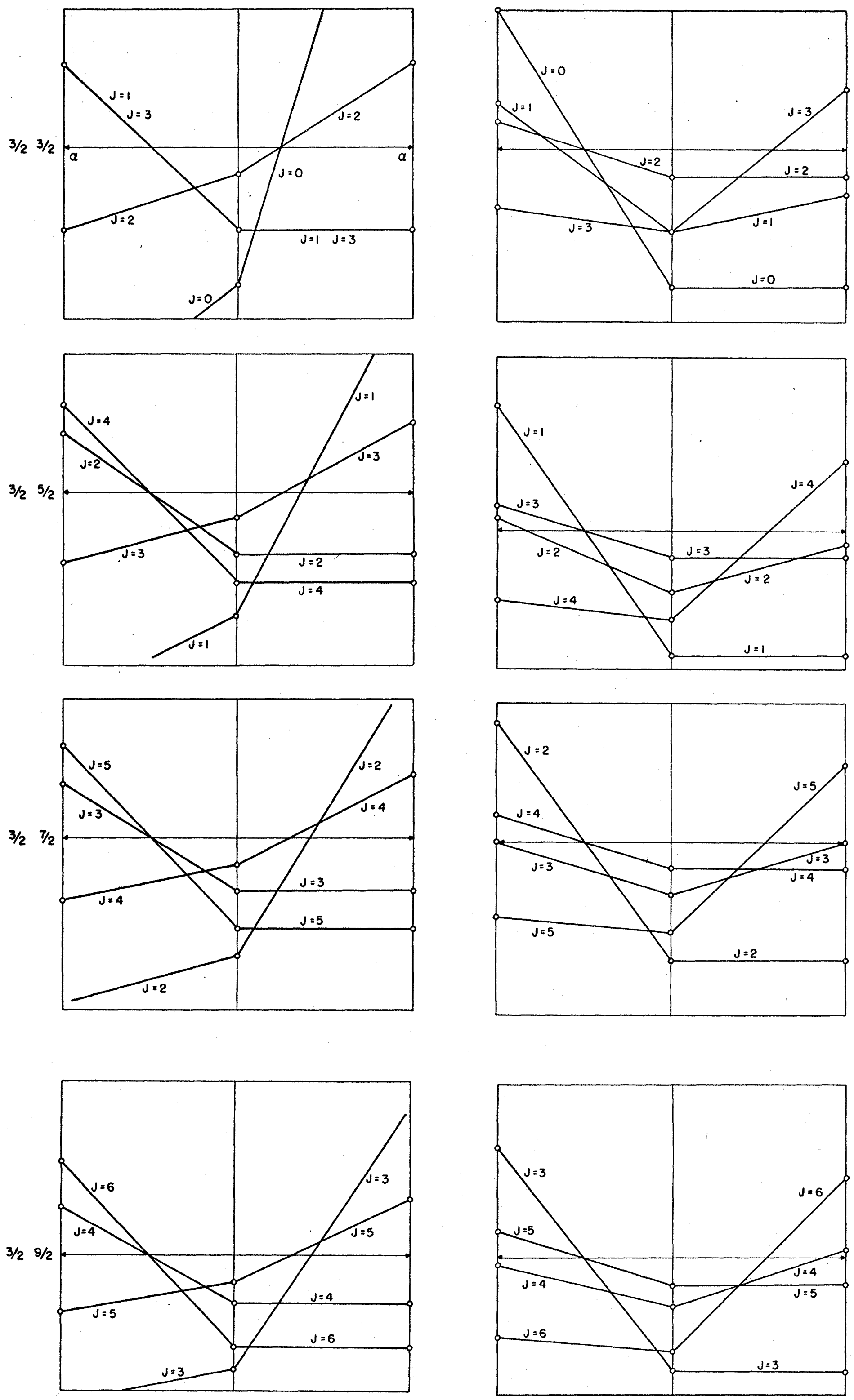

FIG. 1. Energy levels of odd-odd nuclei for a potential $[(1-\alpha)+\epsilon\alpha\sigma_1\cdot\sigma_2]\delta(r_1-r_2)$. Left column—odd Nordheim numbers; right column—even Nordheim numbers. In each figure the right half corresponds to $\epsilon=+1$ and the left half to $\epsilon=-1$. α varies from 0 to 1. The energy scale is arbitrary.

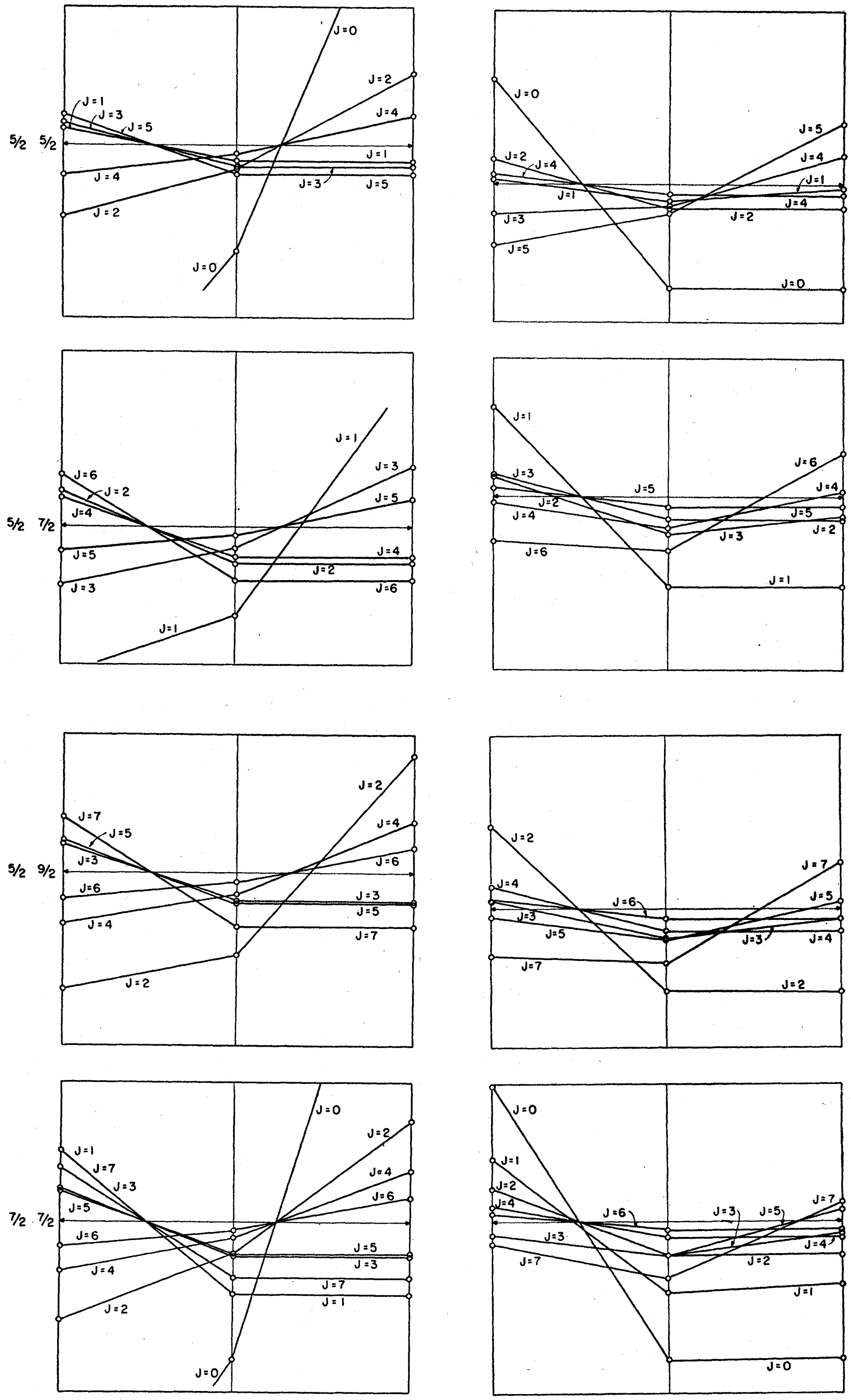

FIG. 1.—*Continued*

(in the jj coupling) and not on their l's. This result is strictly valid for the case of delta interactions, as can be seen from the results of the previous section, but also in other ranges and forms of the potential the dependence on the l's enters only through the usually weak dependence of the F_k on the l's.[2] We, therefore conclude that Nordheim's rule probably cannot be explained on the basis of the Wigner forces alone. A potential which evidently might give results in accordance with Nordheim's rule is $[a+b\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]\delta(\mathbf{r}_1-\mathbf{r}_2)$. The energy levels for this potential can be deduced from the results of the previous section. For this purpose we calculated the Clebsch-Gordan coefficients which are involved in the determination of the energy levels. These are given in Table I. In Fig. 1 we have plotted these energy levels for a potential of the form $[(1-\alpha)+\epsilon\alpha\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]\delta(\mathbf{r}_1-\mathbf{r}_2)$ with $0\leqslant\alpha\leqslant 1$ and $\epsilon=\pm 1$.

The similarity between the graphs so obtained for different j's is striking, and it is immediately evident that the configurations in question can be divided into two classes according to whether the Nordheim number is odd or even. For $\epsilon=+1$ and starting from $\alpha=0$ (pure Wigner forces) we find that for even N, $J=|j_1-j_2|$ always remains the lowest state even as α increases, and that it is an *isolated* lowest state, i.e., all the other states are clustered at relatively high excitations. For odd N, $J=|j_1-j_2|$ is still the lowest state for $\alpha=0$, but it rises very steeply with α; and even for small values of α, $J=j_1+j_2$ becomes the lowest state. The state $J=j_1+j_2$, however, is no longer isolated as was the state $J=|j_1-j_2|$ for even N, and is relatively close to j_1+j_2-2 and j_1+j_2-4, etc. Thus a weak configuration interaction or a transition to a potential of a finite range could bring one of these close-lying states below $J=j_1+j_2$. We thus see that this simple model can explain Nordheim's rule and its decomposition into "strong" and "weak" rules. It goes a little bit beyond the original formulation of the rule, in as much as it predicts that in the "weak" case the ground state will have an even or odd spin according to whether j_1+j_2 is even or odd.

The spins of very few odd-odd nuclei have been measured directly and the spin assignment for most of them is usually made on the basis of β-decay analysis. This analysis, however, is less reliable in the case of odd-odd nuclei than in the case of odd-even nuclei. Thus, if an odd-odd nucleus with odd neutron in j_n and an odd proton in j_p and total angular momentum J_i undergoes a β decay to the state J_f of the configuration $j_p{}^2$ in the daughter nucleus, the matrix element for an allowed transition will be reduced by a factor[8]

$$(2j_n+1)(2J_f+1)|W(j_nJ_ij_pJ_f;\,j_p1)|^2,$$

compared to that of the corresponding single-particle transition. This factor may be quite small and may make an allowed transition look like a first-forbidden one. Also the order of the transition l is determined by the inequalities:

$$|l-1|\leqslant J\leqslant l+1,\quad \pi_i=(-1)^l\pi_f,$$
$$\max(|j_n-j_p|,\,|J_i-J_f|)\leqslant J\leqslant\min(j_n+j_p,\,J_i+J_f);$$

so that if one observes for instance the "unique forbidden" shape one knows only that $\max(|j_n-j_p|, |J_i-J_f|)=2$, and one cannot make a definite statement about $|J_i-J_f|$ without some assumption about j_n and j_p, unless $J_f=0$.

Even if one adopts the order of levels for the odd particles from the shell model, an ambiguity arises from an effect similar to that of the pairing energy. It is very probable that the situation in nuclear spectroscopy is such that the energy difference between two adjacent single-particle states (computed for a central field which takes into account most of the mutual interactions of the particles) is small compared to the "extra" interaction between the particles (which is considered as a perturbation upon the central-field approximation). Thus, if the order of levels for the single particles is $j_nj_n'j_n''\cdots j_pj_p'j_p''\cdots$, it does not necessarily mean that the ground state of the configuration (j_nj_p) is lower than that of (j_nj_p') or $(j_n'j_p)$ or even $(j_n'j_p')$, and the assignment of the right configurations for the states of the odd-odd nuclei is not always unique, even if the total spin and parity are known. In principle, the measurements of transition probabilities involving these states could determine their configuration, but so far it has been impossible to get a good enough agreement between calculated and experimental values for transition probabilities even in the "simpler" cases of odd-even nuclei. It seems that the only practical way of obtaining more information concerning the configuration of a certain state in odd-odd nuclei is the measurement of its magnetic moment and eventually its quadrupole moment. This would also enable a check of the refinement proposed here for the Nordheim rule.

[8] Henry Brysk, Phys. Rev. **90**, 365 (1953); A. de-Shalit and M. Goldhaber (unpublished).