$=+2$, $B_2=B_3=-C_2=-C_3=D_2=D_3=-1$ with normalizing factor $(27)^{-\frac{1}{2}}$.

If one does not set coefficients equal to zero as in the approximate case, the I^+ and T^+ operators will lead to equations reducing the number of arbitrary constants from 16 to 3. This indicates that there are three states with $I=7/2$, $T=1/2$, which can only be resolved by solving a secular determinant of the matrix elements for nuclear interactions. Three arbitrary wave functions are constructed, consistent with the $I=7/2$, $T=1/2$ requirements and mutually orthogonal. The secular determinant is solved for the case of a two-particle delta-function interaction

$$V_{12}=A_\delta(0.8+0.2Q_{12})\delta(r_{12}), \qquad \text{(A14)}$$

where Q_{12} is spin exchange. The various physical quantities calculated with the resulting wave functions are given in Table XXI and compared with values obtained by using the approximate wave function.

TABLE XXI. Expectation values with the approximate and exact wave functions for Sc^{43}.

Wave function	$1f_{7/2}$ shell energy	Mag. mom. nm	$\lvert M\rvert_{GT}^2$ for $Ti^{43}(\beta^+)Sc^{43}$	$\lvert M\rvert_{GT}^2$ for $Sc^{43}(\beta^+)Ca^{43}$
ψ (approximate)	2.400 G	+4.08	0.40	0.57
ψ_1	2.516 G	+4.82	0.72	0.24
ψ_2	1.313 G	+1.03		
ψ_3	0.981 G	+0.02		

From Table XXI it is evident that the lowest energy solution of the secular determinant ψ_1 represents a state with about the same energy as the approximate state. However, while the energies agree within 5 percent, the magnetic moments and Gamow-Teller matrix elements for beta decay are considerably different. This indicates the degree of validity in using the approximate wave functions and is satisfactory for energies.

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The Linear Combination in β Decay*

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Empirical evidence on the shapes of first- and second-forbidden spectra is surveyed to determine the linear combination of invariants in the β-decay interaction. Absence of $1/W$ terms in allowed shape first-forbidden spectra excludes combinations SA and VT. Spectrum shapes for $\Delta I=2$, no, transitions exclude VA and indicate that for negatron emission the relative algebraic sign of the terms is $(S-T)$. These spectra also indicate the need for considerable correction of nuclear matrix element estimates because of the presence of pseudoscalar-coupled forces in the nucleus. Extrapolation of these empirical corrections to the case of RaE shows that the full linear combination should be $[S-T+(1/\delta)P]$, where δ is positive and of order unity. Arguments based on symmetry principles indicate that the correct interaction should in fact be $(S\pm T+P)$, where $\pm$ refers to $\beta^\pm$ emission. The effects of the difference in sign of the T term should in principle be observable. This law for β decay is compared with the decay of the μ meson, and it is found that if the linear combinations are the same, the coupling constants also have identical absolute values. The mean coupling constant is $|f|=1.44\times0.04\times10^{-49}$ erg cm^3. The chief experimental doubt concerns the μ-meson spectrum, which is expected to go through zero at its end point if the β and μ interactions are identical. It appears possible to deduce the combination $(S-T+P)$ uniquely from the postulates that the neutrino field is (1) unique; (2) massless; (3) part of a universal four-particle interaction.

I. INTRODUCTION AND SUMMARY

THE shapes of forbidden β spectra are analyzed to provide information about the linear combination of invariants, $SVTAP$, in the β-decay interaction. It is assumed as a basis throughout that (1) the interaction contains equal parts of Fermi (S, V) and Gamow-Teller (T, A) invariants; and (2) that there is no appreciable ($\lesssim$10 percent) mixture of S and V or of T and A in the interaction. These assumptions are indicated by the most recent analyses of allowed spectra.[1,2]

In forbidden transitions only those spectra involving a mixture of different nuclear matrix elements can give information about the linear combination; by assumption (2) this implies that the only useful spectra are those with a spin change $\Delta I<n+1$, where n is the order of forbiddenness. To extract the parameters λ of the linear combination, it is necessary to have some independent estimate of the ratios of nuclear matrix elements, which always occur in combination with λ. Section II attempts to form such estimates for the general matrix element, extending a procedure previously given[3] for first-forbidden matrix elements and

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[2] J. P. Davidson and D. C. Peaslee, Phys. Rev. **91**, 1232 (1953).

[3] T. Ahrens and E. Feenberg, Phys. Rev. **86**, 64 (1952).

making some allowance for pseudoscalar-coupled nuclear forces.[4]

In Sec. III are given simplified expressions for the spectrum shapes of first- and second-forbidden transitions. The low-Z approximation is used in order to obtain perspicuous algebraic forms for the spectrum shapes; since the analysis relies only on order-of-magnitude arguments, the low-Z approximation should not generally vitiate the conclusions. It is found possible to simplify the expressions for the spectrum shape even further by taking only the large terms of order $\alpha Z/2R$. The combinations ST and VA are observed to have the same generic shapes in this lowest-order approximation. They can be differentiated only by using independent estimates of matrix element ratios for second and higher forbidden spectra, or by comparison with spectra like RaE, where the usually dominant terms of order $\alpha Z/2R$ happen to cancel.

Section IV discusses the comparison with empirical data. The allowed shape of most first-forbidden spectra with $\Delta I<2$ clearly arises from the dominance of energy-independent Coulomb terms in the electron wave function, as has been observed.[5] The absence of $1/W$ terms in these first-forbidden transitions of allowed shape definitely rules out the combination SA and provides some evidence against VT. Nothing more can be deduced from these spectra because their allowed shape does not provide any indication of nuclear matrix element ratios.

The second-forbidden spectra with $\Delta I=2$ have nonunique shapes for which only the combinations VA and ST are available on the basis of previous evidence. The combination VA is found to be at variance with the spectrum of Cl^{36}, so that the only remaining possibility is ST. This conclusion is in harmony with recent $\beta-\nu$ correlation experiments[6] in He^6. Further comparison with the $\Delta I=2$, no, shapes indicates that (1) the relative algebraic sign of the terms in negatron decay is given by $S-T$; (2) a reasonable fit cannot be obtained without a considerable correction of the nuclear matrix elements for pseudoscalar-coupled forces.

The spectrum of RaE is considered as the only case that gives evidence on the amount of pseudoscalar interaction P in the linear combination. General arguments based on the rarity of the RaE type spectrum indicate that it should be considered a $0^-\rightarrow 0^+$ transition. The unique shape indicates cancellation of the usually dominant, energy-independent terms. These conclusions are the same as those obtained by a detailed analysis of the spectrum.[7] Extrapolation of the empirical pseudoscalar-coupled force corrections from the second-forbidden spectra to the case of RaE indicates that the linear combination active in this transition is $T-(1/\delta)P$, where δ is a positive number of order unity.

In Sec. V are reviewed the symmetry arguments that bear on the linear combination. The only combination consistent with any symmetry principle and the empirical evidence is $(S-T+P)$. This is taken as the correct form for negatron decay; for positron emission the combination becomes $S+T+P$, and the difference in sign of T should lead to observable effects. This linear combination is uniquely selected by the conditions that (1) the interaction should be antisymmetric in exchange of pairs of equivalent particles, where this does not include all pairs of particles in the interaction; (2) the interaction should be invariant on mass reversal of the massless neutrino field, in the case of μ-meson decay into two neutrinos; and (3) the β-decay interaction is identical with that for μ-meson decay.

In Sec. VI the decay of the isolated μ meson is compared with β decay. It is found that if the linear combination is assumed to be the same for both cases, then the coupling constants have the same magnitudes within allowances for experimental errors. The mean value of the coupling constant in $f(S-T+P)$ is $|f|=1.44\pm0.04$ erg cm^3. The chief point of present experimental doubt is the shape of the μ-meson decay spectrum, which should go to zero at its end point if the β- and μ-decay interactions are indeed identical.

II. NUCLEAR MATRIX ELEMENT RATIOS

The shapes of forbidden spectra with $\Delta I<n+1$ can be used to determine the linear combination only if the ratios of certain nuclear matrix elements can be estimated *a priori*. Arguments of invariance under rotation and reflection show that general relationships exist only between matrix elements with identical selection rules. Fortunately, only such ratios are required in fitting the spectrum shapes. Ratios of matrix elements with different selection rules depend on specific details of the wave functions involved.

A procedure has been given[3] for estimating the ratios of first-forbidden matrix elements. It leads to conclusions at variance with the fit[7] of the RaE spectrum, unless one assumes an unconscionable proportion of P in the linear combination. This dilemma can be resolved[4] in the case of RaE by considering the pseudoscalar nature of nuclear forces. In this section we attempt to obtain approximate ratios for first- and second-forbidden matrix elements, taking pseudoscalar forces into account but otherwise following the spirit of reference 3.

In writing the β-decay operators, a summation $\sum_k$ over all the nucleons of the nucleus will be understood but not written explicitly unless necessary. The β-decay operators concerned are all contained among the

[4] M. Ruderman, Phys. Rev. **89**, 1227 (1953).
[5] H. M. Mahmoud and E. J. Konopinski, Phys. Rev. **88**, 1266 (1952).
[6] B. M. Rustad and S. L. Ruby, Phys. Rev. **89**, 880 (1953); J. S. Allen and W. K. Jentschke, Phys. Rev. **89**, 902 (1953).
[7] A. G. Petschek and R. E. Marshak, Phys. Rev. **85**, 698 (1952).

forms

$$Q\begin{Bmatrix}1\\ \beta\end{Bmatrix}\begin{Bmatrix}1\\ \gamma_5\end{Bmatrix}\begin{Bmatrix}1\\ \boldsymbol{\sigma}\cdot\mathbf{r}\\ R_n\\ \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}R_n\\ \boldsymbol{\sigma}\times\mathbf{r}\cdot\boldsymbol{\nabla}R_n\end{Bmatrix}, \tag{1}$$

where Q changes a neutron into a proton and R_n is a solid harmonic $[r^nY_n(\theta)]$ or equivalent coordinate tensor[8] with $(2n+1)$ independent components. The "relativistic" operators contain the factor γ_5 and depend strongly on the presence or absence of the factor β. The nonrelativistic operators are not much affected by β and will be written without it. The most general relations are those between relativistic and nonrelativistic operators, but there is one fairly general relation between two nonrelativistic operators for an nth-forbidden transition:

$$\int\Psi_f^*Q\boldsymbol{\sigma}\times\mathbf{r}\cdot\boldsymbol{\nabla}R_n\Psi_i=i/h\int\Psi_f^*Q[\boldsymbol{\sigma}\cdot\mathbf{r}\times\mathbf{p},R_n]\Psi_i \tag{2}$$

$$=it\int\Psi_f^*QR_n\Psi_i,$$

where the square bracket is a commutator and the subscripts i and f indicate initial and final nuclear wave functions. For multiparticle transitions or cases where there is considerable Russell-Saunders coupling, the coefficient t can be evaluated only from detailed knowledge of the nuclear wave functions. For the most useful first approximation of transitions between the same subshells in $j-j$ coupling, however,

$$t=(1/\hbar)\{(\boldsymbol{\sigma}\cdot\mathbf{L})_f-(\boldsymbol{\sigma}\cdot\mathbf{L})_i\} \tag{3}$$

$$=\pm n \text{ for odd } A. \tag{3a}$$

The relativistic matrix elements not containing β can be related directly to nonrelativistic elements without explicit introduction of the nuclear forces. For example,

$$\int Q\gamma_5\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}R_n=\frac{-i}{\hbar c}\int[H_0,QR_n]$$

$$=\frac{-i}{\hbar c}\int[H-H_c-H_\nu,QR_n]$$

$$\approx\frac{-i}{\hbar c}\left\{\Delta E-1.2\frac{\mathrm{Ze}^2}{R}(1-f_\nu)\right\}\int QR_n, \tag{4}$$

where $H_0=-\boldsymbol{\alpha}\cdot\mathbf{p}c-\beta Mc^2$, again without writing explicitly the summation $\sum_k$. The argument follows that of reference 3: the total energy difference of the two nuclear states is $\Delta E=E_f-E_i$, the Coulomb energy difference for a uniform charge distribution is $1.2\,\mathrm{Ze}^2/R$ with R the nuclear radius, and $0<f_\nu<1$ is a fraction representing the extent to which the change in Coulomb energy is counterbalanced by an opposite change in the nuclear symmetry energy. Reference 3 takes $f_\nu\approx\frac{3}{5}$, while a value equivalent to $f_\nu\approx\frac{1}{6}$ is obtained from more explicit calculations[9] with single-particle wave functions and a specific Hamiltonian operator. With atomic units $\hbar/m_0c$ for length, $m_0c^2=\frac{1}{2}$ Mev for energy, Eq. (4) becomes

$$\int Q\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}R_n\approx i\chi\int QR_n,\quad \chi=2.4B(1-f_\nu)-\Delta W, \tag{5}$$

where B is the Coulomb barrier height of the (final) nucleus in Mev and $\Delta W=\Delta E/m_0c^2$. For negatron emission in elements heavy enough to have forbidden spectra, ΔW is frequently negligible relative to the term in B. Equation (5) for $n=1$ is the first Eq. (19) of reference 3.

The only other relativistic operator of possible importance in β decay that does not involve β is $Q\gamma_5$; consider

$$\int[H_0,Q\boldsymbol{\sigma}\cdot\mathbf{r}]=ihc\int Q\gamma_5[\boldsymbol{\sigma}\cdot\boldsymbol{\nabla},Q\boldsymbol{\sigma}\cdot\mathbf{r}]$$

$$=ihc\int Q\gamma_5(3+2i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\times\mathbf{r}) \tag{6}$$

$$=ihc\int Q\gamma_5\left(3+\frac{2}{\hbar}\boldsymbol{\sigma}\cdot\mathbf{L}\right).$$

For the operator $\boldsymbol{\sigma}\cdot\mathbf{L}$ in this matrix element we must take, for any individual nucleon that makes a transition, the average value of $\boldsymbol{\sigma}\cdot\mathbf{L}$ for fixed j and $l=j\pm\frac{1}{2}$; this value is $-\hbar$, independent of j and l, so that in general

$$\int Q\gamma_5=\frac{-i}{\hbar c}\int[H_0,Q\boldsymbol{\sigma}\cdot\mathbf{r}]=i\chi\int Q\boldsymbol{\sigma}\cdot\mathbf{r}. \tag{7}$$

This is the second Eq. (19) of reference 3.

In contrast to Eqs. (5) and (7), relativistic operators containing β can be related to nonrelativistic operators only by writing out the matrix elements in nonrelativistic approximation. At this point certain characteristic terms will arise if the nuclear forces are pseudoscalar with pseudoscalar coupling. Consider the operator $\beta\gamma_5$:

$$\int\Psi_f^*Q\beta\gamma_5\Psi_i\approx\int\{\psi_{1f}^*Q\psi_{0i}-\psi_{0f}^*Q\psi_{1i}\}. \tag{8}$$

Here ψ_0 is the product of the large components of the wave function for all nucleons in the nucleus, and ψ_1 differs only by the substitution of the small component for the particle on which $Q\beta\gamma_5$ operates. A nonrela-

[8] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).

[9] D. L. Pursey, Phil. Mag. **42**, 1193 (1951).

tivistic approximation for ψ_1 is to be obtained from the equation of motion:

$$\sum_k(-\boldsymbol{\alpha}^k\cdot\mathbf{p}^kc-\beta^kMc^2)\Psi=(E-V-V_p)\Psi, \quad (9)$$

where the pseudoscalar potential is

$$V_p=\tfrac{1}{2}\sum_{j\neq k}T^{jk}V^{jk}(\beta\gamma_5)^j(\beta\gamma_5)^k,$$
$$T^{jk}=\boldsymbol{\tau}^j\cdot\boldsymbol{\tau}^k, \quad V^{jk}=V_0\exp(-\kappa|\mathbf{r}^j-\mathbf{r}^k|)/|\mathbf{r}^j-\mathbf{r}^k|. \quad (10)$$

The "ordinary" potential V does not contain any $\beta\gamma_5$ factors. We write out the component of (9) that corresponds to $(E-V)\psi_1{}^k$:

$$-\boldsymbol{\sigma}^k\cdot\mathbf{p}^kc\psi_0-\sum_{j\neq k}\boldsymbol{\sigma}^j\cdot\mathbf{p}^jc\psi_2{}^{jk}+(A-2)Mc^2\psi_1{}^k=(E-V)\psi_1{}^k$$
$$+\sum_{j\neq k}T^{jk}V^{jk}\psi_1{}^j-\tfrac{1}{2}\sum_{k\neq i\neq j\neq k}T^{ij}V^{ij}\psi_3{}^{ijk}, \quad (11)$$

where no summation is implied on the index k. Here $\psi_n{}^{ijk\cdots}$ is the product of the small component wave functions φ for the n nucleons $ijk\cdots$, times the product of the large components Φ for all other nucleons. Equation (11) reduces to first order in φ by dropping the second and sixth terms, which are small of order φ^2 relative to the otherwise similar first and fifth terms, respectively. Taking $E-V\approx AMc^2$ yields

$$\psi_1{}^k\approx-\frac{1}{2Mc^2}[\boldsymbol{\sigma}^k\cdot\mathbf{p}^kc\psi_0+\sum_{j\neq k}T^{jk}V^{jk}\psi_1{}^j]. \quad (12)$$

Since only terms of first order in φ are preserved, it is consistent to retain only those of first order in V and substitute $\psi_1{}^j\approx(-1/2Mc)\boldsymbol{\sigma}^j\cdot\mathbf{p}^j\psi_0$ on the right-hand side of (12), so that

$$\psi_1{}^k\approx\frac{-1}{2Mc}[\boldsymbol{\sigma}^k\cdot\mathbf{p}^k-\sum_{j\neq k}T^{jk}U^{jk}\boldsymbol{\sigma}^j\cdot\mathbf{p}^j]\psi_0, \quad (13)$$

where $U=V/2Mc^2$.

The matrix element (8) is now, writing the $\sum_k$ explicitly,

$$\langle Q\beta\gamma_5\rangle\approx\frac{-1}{2Mc}\sum_k\int[\boldsymbol{\sigma}^k\cdot\mathbf{p}^k, Q^k]$$
$$+\frac{1}{2Mc}\sum_k\sum_{j\neq k}\int[T^{jk}\boldsymbol{\sigma}^j\cdot\mathbf{p}^j, Q^kU^{jk}]. \quad (14)$$

The first term of (14) is just that obtained without considering pseudoscalar forces and to this approximation vanishes identically. The second term arises from the pseudoscalar potential. Using the fact that $Q=\frac{1}{2}(\tau_x+i\tau_y)$, we obtain

$$[T^{jk}\boldsymbol{\sigma}^j\cdot\mathbf{p}^j, Q^kU^{jk}]=Q^j[\boldsymbol{\sigma}^j\cdot\mathbf{p}^j, U^{jk}]$$
$$+(Q^k\tau_z{}^j-Q^j\tau_z{}^k)[\boldsymbol{\sigma}^j\cdot\mathbf{p}^j, U^{jk}]_+, \quad (15)$$

where the + bracket is an anticommutator. The second term in (15) will be dropped because a summation over τ_z is expected to give a term of order $(N-Z)/A\lesssim 10$ percent relative to the first. The first term of (15) gives a contribution to $\langle Q\beta\gamma_5\rangle$ of

$$\frac{1}{2Mc}\sum_i\sum_{k\neq j}\int[Q^j\boldsymbol{\sigma}^j\cdot\mathbf{p}^j, U^{jk}]. \quad (16)$$

The order-of-magnitude relation between V^{jk} and the nonrelativistic approximation to the two-body pseudoscalar potential is $V^{jk}\sim-(2Mc/\hbar\kappa)^2H^{jk}$. Now $\sum_{k\neq j}H^{jk}$ is that part of the nuclear force on the jth nucleon which is contributed by the two-body pseudoscalar potential. We accordingly write

$$\sum_{k\neq j}U^{jk}=\rho\left(\frac{2M}{\mu}\right)^2\frac{H_\nu{}^j}{2Mc^2}, \quad (17)$$

where μ is approximately the π-meson mass and ρ is an undetermined factor. Here ρ takes care of the fact that the effective potential in the nucleus may not all arise from two-body pseudoscalar forces and also includes the uncertainty in the relation of V^{jk} to H^{jk}. This uncertainty is aggravated because the above discussion depends on the first term of a power series expansion in V_p, and higher terms may be important. We do not expect $|\rho|$ to differ by an order of magnitude from unity, but not much more can be said; it may show appreciable fluctuation from one nucleus to the next, and under these fluctuations may be a gradual trend with A.

Lumping our ignorance in the factor ρ, we have

$$\langle Q\beta\gamma_5\rangle\approx\frac{\rho}{\mu^2c^3}\int[Q\boldsymbol{\sigma}\cdot\mathbf{p}, H_\nu]$$
$$=f_\nu\left(\frac{1.2Ze^2}{R}\right)\frac{\rho}{\mu^2c^3}\int Q\boldsymbol{\sigma}\cdot\mathbf{p}. \quad (18)$$

Since we have already taken the pseudoscalar forces into account once, we can evaluate $\int Q\boldsymbol{\sigma}\cdot\mathbf{p}$ to this order by taking for H_0 the nonrelativistic approximation without pseudoscalar forces, $H_0\approx p^2/2M$. Then

$$\int Q\boldsymbol{\sigma}\cdot\mathbf{p}\approx\frac{iM}{\hbar}\int[H_0, Q\boldsymbol{\sigma}\cdot\mathbf{r}]$$
$$\approx\frac{iM}{\hbar}\left\{\Delta E-1.2\frac{Ze^2}{R}(1-f_\nu)\right\}\int Q\boldsymbol{\sigma}\cdot\mathbf{r}, \quad (19)$$

so that, in atomic units,

$$\langle Q\beta\gamma_5\rangle\approx-i\rho\left(\frac{m_0M}{\mu^2}\right)\chi f_\nu 2.4B\int Q\boldsymbol{\sigma}\cdot\mathbf{r}. \quad (20)$$

The procedure leading to (12) may be immediately applied to show that nuclear forces with pseudovector coupling will make a negligible correction to the matrix elements as compared with that from pseudoscalar

coupling. In this case, (13) becomes

$$\psi_1{}^k \approx -(1/2Mc^2)[\boldsymbol{\sigma}^k\cdot\mathbf{p}^k c\psi_0 - \rho H_\nu \psi_1{}^k] \approx -(1/2Mc^2)[1+\rho H_\nu/2Mc^2]\boldsymbol{\sigma}^k\cdot\mathbf{p}^k\psi_0, \quad (21)$$

so that

$$\langle Q\beta\gamma_5\rangle \approx -(1/2Mc)\int[\boldsymbol{\sigma}\cdot\mathbf{p}, Q] - (\rho/2Mc)\int[\boldsymbol{\sigma}\cdot\mathbf{p}, QH_\nu]/2Mc^2. \quad (22)$$

In this case the second term is smaller than (18) by a factor of order $(\mu/2M)^2\sim 5\times 10^{-3}$. Thus pseudovector coupled forces behave like any "ordinary" forces in making no appreciable correction to the lowest order approximation for relativistic matrix elements.

Repeating the above arguments for the operator $Q\beta\gamma_5\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}R_n$ leads to

$$\langle Q\beta\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}R_n\rangle \approx -\frac{1}{2Mc}\sum_k\int Q^k[\boldsymbol{\sigma}^k\cdot\mathbf{p}^k, \boldsymbol{\sigma}^k\cdot\boldsymbol{\nabla}^k R_n{}^k] + \frac{1}{2Mc}\sum_i\sum_{k\neq j}\int[Q^i\boldsymbol{\sigma}^i\cdot\mathbf{p}^i, \boldsymbol{\sigma}^k\cdot\boldsymbol{\nabla}^k R_n{}^k U^{ik}]. \quad (23)$$

Here again the first term is obtained under any nuclear force assumption, and the second is peculiar to pseudoscalar coupling. For the first term we have

$$\int[\boldsymbol{\sigma}\cdot\mathbf{p}, \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}R_n] = 2i\int(\boldsymbol{\sigma}\times\mathbf{p})\cdot\boldsymbol{\nabla}R_n. \quad (24)$$

Consider the quantities

$$a = (-i\hbar/M)(\boldsymbol{\sigma}\times\mathbf{p})\cdot\boldsymbol{\nabla}R_n$$
$$b_m = (\boldsymbol{\sigma}\times\mathbf{r})\cdot\boldsymbol{\nabla}(-(i\hbar/M)\mathbf{p}\cdot\boldsymbol{\nabla})^m R_n. \quad (25)$$

Now using $H_0\approx p^2/2M$ and atomic units, we have

$$\int[H_0, b_m] = -\chi\int b_m = \int\left(\frac{-i\hbar}{M}\boldsymbol{\sigma}\times\mathbf{p}\cdot\boldsymbol{\nabla}\right)\left(\frac{-i\hbar}{M}\mathbf{p}\cdot\boldsymbol{\nabla}\right)^m R_n + \int b_{m+1}. \quad (26)$$

The first term of (26) can be obtained by taking the commutator of a with H_0 m successive times, so that (26) reads

$$-\chi\int b_m = (-\chi)^m\int a + \int b_{m+1}. \quad (27)$$

Equation (27) is a recursion relation in $\int b_m$ subject to the boundary condition that $b_n=0$, since R_n is of degree n in the coordinate on which $\boldsymbol{\nabla}$ operates. The solution is

$$\int b_m = (n-m)(-\chi)^{m-1}\int a, \quad (28)$$

and for the particular case $m=0$,

$$\int(\boldsymbol{\sigma}\times\mathbf{p})\cdot\boldsymbol{\nabla}R_n = -\left(\frac{iM}{\hbar}\right)\left(\frac{\chi}{n}\right)\int(\boldsymbol{\sigma}\times\mathbf{r})\cdot\boldsymbol{\nabla}R_n. \quad (29)$$

The first term of (23) is, accordingly,

$$\langle Q\beta\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}R_n\rangle_0 \approx -\left(\frac{\chi}{n}\right)\int(\boldsymbol{\sigma}\times\mathbf{r})\cdot\boldsymbol{\nabla}R_n = -i\left(\frac{t}{n}\right)\chi\int R_n, \quad (30)$$

by use of (2). For $n=1$ this is the third Eq. (19) of reference 3.

For the second term of (23) we write

$$[\boldsymbol{\sigma}^i\cdot\mathbf{p}^i, \boldsymbol{\sigma}^k\cdot\boldsymbol{\nabla}^k R_n{}^k U^{ik}] = [\tfrac{1}{2}[\boldsymbol{\sigma}^i\cdot\mathbf{p}^i, \times\boldsymbol{\sigma}^k\cdot\boldsymbol{\nabla}^k R_n{}^k]_+, U^{ik}], \quad (31)$$

and then capitalize on the fact that U^{ik} is almost a δ function. Under this approximation, $\boldsymbol{\nabla}^k R_n{}^k = -\boldsymbol{\nabla}^i R_n{}^i$ and the $\sum_{k\neq j}\boldsymbol{\sigma}^k = \mathbf{S}-\boldsymbol{\sigma}^j$, where S is the total spin. Assuming no correlation between $\mathbf{S}$ and $\boldsymbol{\nabla}^i$, we drop the term $\mathbf{S}\cdot\boldsymbol{\nabla}^i$ and have for (31):

$$\sum_{k\neq j}[\boldsymbol{\sigma}^i\cdot\mathbf{p}^i, \boldsymbol{\sigma}^k\cdot\boldsymbol{\nabla}^k R_n{}^k U^{ik}] \approx \sum_{k\neq j}[\mathbf{p}^i\cdot\boldsymbol{\nabla}^i R_n{}^i, U^{ik}]. \quad (32)$$

With (17) the second term of (23) becomes, in atomic units,

$$\frac{\rho}{\mu^2 c^3}\int Q[\mathbf{p}\cdot\boldsymbol{\nabla}R_n, H_\nu] = f_\nu\left(\frac{1.2\,Ze^2}{R}\right)\frac{\rho}{\mu^2c^3}\int Q\mathbf{p}\cdot\boldsymbol{\nabla}R_n = -i\rho\left(\frac{m_0 M}{\mu^2}\right)\chi f_\nu(2.4B)\int QR_n, \quad (33)$$

since

$$\int\mathbf{p}\cdot\boldsymbol{\nabla}R_n = \frac{iM}{\hbar}\int[H_0, R_n] = \frac{-iM\chi}{\hbar}\int R_n. \quad (34)$$

Thus, finally,

$$\langle Q\beta\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}R_n\rangle \approx -i\chi\left[\frac{t}{n}+\rho\left(\frac{m_0 M}{\mu^2}\right)f_\nu(2.4B)\right]\int QR_n. \quad (35)$$

Relations (2), (4), (7), (20), and (35) will be used in the analysis of the shapes of forbidden spectra. From this analysis it will appear that the general form of the pseudoscalar corrections is adequate, but that the specific expressions for the coefficients of the pseudoscalar terms cannot be taken too seriously. This is not surprising in view of the many crude approximations made in obtaining these coefficients; they are perhaps better replaced by a single phenomenological parameter with absolute magnitude of order unity.

III. ALGEBRAIC FORMS OF FORBIDDEN SPECTRA

To obtain some insight into the shapes of forbidden spectra that are possible under various linear combinations, we write out the correction factors in low-Z approximation. This procedure has the advantage of showing at once which terms are large and what cancellations must occur among these large terms in order to produce unusual spectral shapes among the remaining small terms. Use of the low-Z formulas means that the cancellation conditions and the spectral shapes resulting from cancellation are given only approximately. Any conclusions based only on order-of-magnitude arguments ought still to be valid, however.

The features of all linear combinations not containing SV or TA mixtures are illustrated by the four choices STP, VA, SAP, VT. The real coefficients λ in these equations are limited by the result from the mirror transitions that $\lambda_F=\pm\lambda_{GT}$. This condition is assumed throughout. To specify the first-forbidden correction factors,[8,10] it is convenient to define the following functions:

$$
\begin{aligned}
&12x_1=(p^2+K^2),\\
&y_1(K,a)=[\{B(\lambda+t)-a\}+\tfrac{1}{3}\{(p^2/W)(\lambda+t)\\
&\quad+K(\lambda-t)\}]^2+(1/18)[p^2(2\lambda-t)^2+K^2(2\lambda+t)^2]\\
&\qquad+\tfrac{1}{9}(p^2/W^2)(\lambda+t)^2,\\
&y_1(K,a,a')=[Bt-a+\tfrac{1}{3}t(p^2/W+K)]^2\\
&\quad+[B-a'+\tfrac{1}{3}(p^2/W+K)]^2+(2\lambda/W)[(Bt-a\\
&\quad+\tfrac{1}{3}Kt)(B-a'+\tfrac{1}{3}K)-\tfrac{1}{9}K^2t]+(1/18)(p^2+K^2)\\
&\qquad\times(4+t^2)+\tfrac{1}{9}(p^2/W^2)(1+t^2),\\
&z_1(K,g,g')=[B-g+\tfrac{1}{3}(p^2/W+K)]^2+g'^2\\
&\qquad-(2/W)g'(B-g+\tfrac{1}{3}K)+\tfrac{1}{9}(p^2/W^2).
\end{aligned}
\tag{36}
$$

Here $B=Ze^2/2R$; and if R is in atomic units, B is numerically equal to the Coulomb barrier height in Mev of the final nucleus, $B\approx Z/A^{\frac{1}{3}}$. The quantities p, W and K are the electron momentum, electron energy, and neutrino energy in m_0c^2 units; and

$$\lambda=\lambda_F/\lambda_{GT}=\pm1. \tag{37}$$

The other parameters are all real numbers and represent ratios of nuclear matrix elements as follows:

$$
\begin{gathered}
\int\boldsymbol{\sigma}\times\mathbf{r}=it\int\mathbf{r},\quad \int\beta\boldsymbol{\alpha}=ia\int\mathbf{r},\quad \int\boldsymbol{\alpha}=ia'\int\mathbf{r},\\
(\lambda_P/\lambda_{GT})\int\beta\gamma_5=ig\int\boldsymbol{\sigma}\cdot\mathbf{r},\quad \int\gamma_5=ig'\int\boldsymbol{\sigma}\cdot\mathbf{r}.
\end{gathered}
\tag{38}
$$

The factor g includes in its definition the ratio λ_P/λ_{GT}, which is real but of undetermined magnitude. According to the results of the preceding section, the coefficients a, a', g, g' are all of order B.

The first-forbidden correction factors for the four combinations listed above now take the form

$$C_1=\left|\int B_{ij}\right|^2x_1+\left|\int\mathbf{r}\right|^2y_1+\left|\int\boldsymbol{\sigma}\cdot\mathbf{r}\right|^2z_1. \tag{39}$$

Here x is independent of the combination assumed, and y and z are given by the following scheme:

$$
\begin{aligned}
&STP:\ y_1(-K,a);\ z_1(K,g,0),\\
&VA:\ y_1(K,\lambda a');\ z_1(k,g',0),\\
&SAP:\ y_1(K,0,0);\ z_1(K,g',g),\\
&VT:\ y_1(K,a,a');\ z_1(-K,0,0).
\end{aligned}
\tag{40}
$$

For the second-forbidden correction factors it is convenient to define the functions

$$
\begin{aligned}
&1080x_2=[(p^2+K^2)^2+(4/3)p^2K^2],\\
&3y_2(K,a)=p^2\{[\tfrac{1}{2}B(\lambda+\tfrac{1}{2}t)-\tfrac{1}{2}a]+\tfrac{1}{5}(p^2/W)(\lambda+\tfrac{1}{2}t)\\
&\quad+\tfrac{1}{3}K(\lambda-\tfrac{1}{2}t)\}^2+K^2\{[B(\lambda+\tfrac{1}{2}t)-\tfrac{1}{2}a]+\tfrac{1}{3}(p^2/W)\\
&\quad\times(\lambda+\tfrac{1}{2}t)+\tfrac{1}{5}K(\lambda-\tfrac{1}{2}t)\}^2+(1/150)p^4(3\lambda-t)^2\\
&\quad+(1/150)K^4(3\lambda+t)^2+\tfrac{1}{9}p^2K^2+(p^2/W^2)(\lambda+\tfrac{1}{2}t)^2\\
&\qquad\times[(1/25)p^2+\tfrac{1}{9}K^2],\\
&3y_2(K,a,a')=p^2\{\eta^2(a';\tfrac{1}{5},\tfrac{1}{3})+(\tfrac{1}{2}t)^2\eta^2(2a/t;\tfrac{1}{5},\tfrac{1}{3})\\
&\quad+(2\lambda t/W)[\zeta(a';\tfrac{1}{3})\zeta(2a/t;\tfrac{1}{3})-\tfrac{1}{9}K^2]\}\\
&\quad+K^2\{\eta^2(a';\tfrac{1}{3},\tfrac{1}{5})+(\tfrac{1}{2}t)^2\eta^2(2a/t;\tfrac{1}{3},\tfrac{1}{5})\\
&\quad+(2\lambda t/W)[\zeta(a';\tfrac{1}{5})\zeta(2a/t;\tfrac{1}{5})-(1/25)K^2]\}\\
&\quad+(p^4+K^4)[(9+t^2)/150]+\tfrac{1}{9}p^2K^2+(p^2/W^2)\\
&\qquad\times(1+t^2/4)[(1/25)p^2+\tfrac{1}{9}K^2].\\
&\rho(a;k)=B-\tfrac{1}{2}a+kK,\\
&\eta(a;j,k)=\zeta(a;k)+jp^2/W,
\end{aligned}
\tag{41}
$$

where

$$
\int T_{ij}=it\int R_{ij},\quad \int\beta A_{ij}=ia\int R_{ij},\quad \int A_{ij}=ia'\int R_{ij}, \tag{42}
$$

and the T_{ij}, A_{ij}, R_{ij} are defined in reference 8. The coefficients a, a' are again of order B. In low-Z approximation, the second-forbidden correction factors are then

$$C_2=\left|\int S_{ijk}\right|^2x_2+\left|\int R_{ij}\right|^2y_2, \tag{43}$$

where x_2 is associated with the GT part of any linear combination, and y_2 is given by the following scheme:

$$
\begin{aligned}
&ST:\ y_2(-K,a),\\
&VA:\ y_2(K,\lambda a'),\\
&SA:\ y_2(-K,0,0),\\
&VT:\ y_2(K,a,a').
\end{aligned}
\tag{44}
$$

Since the matrix elements of P contribute only a negligible amount to any second-forbidden decay, the

[10] A. M. Smith, Phys. Rev. **82**, 955 (1951).

schemes (44) are independent of P. This was not true of first-forbidden decays, where P contributes appreciably to $\Delta I=0$ transitions. In the applications of (43) below, we consider only transitions with $\Delta I=2$ and drop the term in x_2 as negligibly small.

Some general features of the y_1 and y_2 are at once apparent. For medium-to-heavy nuclei where most forbidden transitions occur, the terms in (36) and (41) involving B are an order of magnitude larger than the remaining terms. Hence to a first approximation the spectrum shapes are given by the terms in B alone. This makes for great simplification in classifying the spectrum shapes. In particular, this simplification shows why most first-forbidden transitions with $\Delta I<2$ have an apparent allowed shape, as has already been pointed out:[5] the dominant terms containing B in (36) have no energy-dependent factors. The same approximation for the second-forbidden spectra leads to a general shape of the form:

$$C_2\sim p^2+v^2K^2,$$

$$v^2=\left[\frac{2B\left(\lambda+\frac{t}{2}\right)-b}{B\left(\lambda+\frac{t}{2}\right)-b}\right]^2, \tag{45}$$

$$\begin{aligned} b&=a && ST,\\ &=\lambda a' && VA,\\ &\approx 0 && SA,\\ &\approx a+\lambda a' && VT. \end{aligned}$$

For the combinations SA and VT the value of b is obtained under the approximation $1/W\approx 1$, which is not unsuitable because all second-forbidden transitions with $\Delta I=2$ have low-energy end points.

In this first approximation where only the leading terms in $\alpha Z/2R$ are retained, certain general features of the combinations are independent of the order of forbiddenness. The $ST(P)$ and VA combinations are indistinguishable in generic form: each contains two constants representing matrix element ratios (t and a or a'), and if we have no prejudices about the values of these parameters, any spectrum can be fitted equally well with $ST(P)$ or VA. The combination $SA(P)$, on the other hand, has only one such parameter (t), and is therefore relatively inflexible for fitting observed spectrum shapes. It will be seen below that this inflexibility is sufficient to rule out the combination $SA(P)$. The combination VT is the most flexible of all, containing three independent parameters (t, a, a'); it would therefore be very difficult in general to exclude this combination by comparison with observed spectral shapes. A fortunate circumstance in the first-forbidden spectra makes it possible to make a fair statistical though not absolute argument against this combination.

The higher terms that do not involve B in the spectrum shapes must be considered only in the event that the terms in B happen just to cancel. In this relatively rare circumstance, the combinations $ST(P)$ and VA will show a real difference in shape, arising from the opposite signs of K in the two cases. Apparently the only known spectrum of this type is the RaE spectrum, where analysis[7] shows that $ST(P)$ is a better fit than VA. It is of interest to note that this necessity of invoking higher terms in the spectrum shape is practically peculiar to first-forbidden transitions. For second- (and presumably for higher) forbidden transitions, the quantity B occurs in several coefficients that multiply simple functions of the energy like p^2 and K^2. These coefficients are not identical in form, so that if one of them vanishes, the others are very likely not to vanish. For example, in the first-forbidden transitions we must use the higher terms in $y_1(K, a)$ if $B(\lambda+t)\approx a$, which is certainly conceivable; in the second-forbidden $y_2(K, a)$ the higher terms will be necessary only if $\lambda+t/2\approx b\approx 0$, which is a much less likely occurrence.

These spectrum shapes, particularly their approximate forms preserving only the large terms, are applied in the next section to observed spectra. The $\Delta I=2$, no, spectra have previously been fitted by using T alone, and it is easy to see why this was possible. This choice corresponds to taking $b=a, \lambda=0$ in (45); and for any value of a that yields a certain v^2 when $\lambda=\pm 1$, we can find another value of a that yields the same v^2 with $\lambda=0$.

IV. COMPARISON WITH OBSERVED SHAPES

First-forbidden spectra with $\Delta I<2$ all have allowed shapes with the exception of RaE, which will be discussed below. These allowed shapes attest to the validity of the first approximation, in which only terms of order B are retained in fitting the spectral shapes. It is impossible from these allowed-shape spectra to distinguish between $ST(P)$ and VA, but the presence of $1/W$ terms in the expression (36) for $SA(P)$ and VT allows us to argue against these combinations.

A number of first-forbidden spectra of apparent allowed shape have been analyzed[2] with a correction factor $C_1=(1+r/W)$. The values of $|r|$ average on the order of 7 percent and are generally within the estimated errors of the determination. The algebraic sign and magnitude of r seem to fluctuate randomly among independent measurements of the same spectrum. We may conclude that there is no evidence at variance with the statement that $r=0$.

This conclusion immediately rules out the combination SA. For the $\Delta I=1$ transitions, only $y_1(K, 0, 0)$ is effective, and

$$r=2\lambda t/(1+t^2)\approx\pm 1, \tag{46}$$

since $\lambda=\pm 1$, and $t=\pm 1$ is the most likely value for a first-forbidden transition with $\Delta I=1$. Of the first-forbidden transitions analyzed in reference 2, the shell model indicates the Pr^{143}, Pm^{147}, Re^{186}, and Au^{198} are

most likely to have $\Delta I=1$; while Be^{139} and Hf^{181} are somewhat uncertain and may or may not have $\Delta I=1$. It is clear, however, that none of the cases most likely to have $\Delta I=1$ give anything approaching $|r|=1$, so the combination SA seems definitely excluded.

The argument is somewhat less positive for VT, the other combination that gives $1/W$ terms in the allowed-shape spectra. Again taking only the $\Delta I=1$ transitions and retaining only terms of order B in (36), we have

$$r=\frac{2\lambda(Bt-a)(B-a')}{[(Bt-a)^2+(B-a')^2]}. \tag{47}$$

This can give $r=0$ in agreement with observation if $a'=B$ or $a=Bt$. The matrix element estimate (35) indicates that $a\neq Bt$; but the estimate (5) shows that $a'\approx\chi$, which is at least of order B. In fact, if $f_v=0.6$ as assumed in reference 3, we have $\chi\approx B$. It is therefore apparent that the nuclear parameters can conspire to give $r\approx 0$ by a fortuitous cancellation in the VT case, while still retaining the allowed shape.[11]

The matrix element estimate (5) is certainly not to be regarded as accurate to the last percent, however: fluctuations are expected in nuclear parameters like f_v and Ze^2/R, and the term ΔW in χ will vary in any case. For example, variations in ΔW alone cause a standard deviation of about 5 percent around a mean value of $\chi/B=1$ for the four $\Delta I=1$ cases of reference 2. If we assume an equivalent, independent deviation in χ/B from other sources and include the factor 2 in (47), we have an expected standard deviation of $\sigma_r\sim 15$ percent about a mean value of $r=0$.

The $\Delta I=1$ transitions in reference 2, weighted according to their errors Δ_r, show a $\sigma_r'\sim 5$ percent about a mean value of -0.02. The probability that four independent elements which should have a fluctuation σ_r actually show by chance a fluctuation $\leq\sigma_r'$ is roughly $p\approx\frac{2}{3}(\sigma_r'/\sigma_r)^{4/3}$ for $\sigma_r'/\sigma_r\ll 1$. If we take $\sigma_r'/\sigma_r\sim 1/3$ as is suggested by the figures in the preceding paragraph, the value obtained for p is on the order of 0.2. We may therefore make a statistical argument that it is quite unlikely (expected chances of order 20 percent) for the linear combination to be VT.

For the second-forbidden transition we again make the simplification of retaining only the terms in (41) of order B^2, since this approximation proves to be so well followed in the first-forbidden cases. We fit the $\Delta I=2$, no, spectra by first plotting $C(W)=[n/(W_0-W)]^2$, where n gives the conventional Kurie plot. According to (45), this $C(W)$ should have a minimum at W_m, where $v^2=W_m/(W_0-W_m)$. Although the experimental minimum is not always very pronounced, it seems more reliable to estimate v^2 in this way than by a least-squares fit, which tends to overemphasize the ends of the spectrum where the data are poorest. Table I lists values† of v^2.

TABLE I. Fits of $\Delta I=2$, no, spectra.

Decay	v^2	Transition	t
Cl^{36} [a]	0.6	$d_{3/2}\rightarrow d_{3/2}$	0
Tc^{99} [a,b]	~ 2	$d_{5/2}\rightarrow g_{9/2}$	+2
I^{129} [c]	$\gtrsim 10$		
Cs^{135} [d]	$\lesssim 10$	$d_{3/2}\rightarrow g_{7/2}$	−2
Cs^{137} [e]	$\gg 10$		

[a] L. Feldman and C. S. Wu, Phys. Rev. **87**, 1091 (1952).
[b] S. I. Taimuty, Phys. Rev. **81**, 461 (1951).
[c] C. S. Wu and E. der Mateosian (private communication).
[d] Lidofsky, Alperovitch, and Wu, Phys. Rev. **90**, 387 (1953).
[e] L. M. Langer and R. J. D. Moffat, Phys. Rev. **82**, 635 (1951).

Table I also indicates the individual particle transitions according to the shell model and the associated values of the parameter t. As far as the crude arguments below are concerned, Table I contains just three independent cases: Cl^{36} with $t=0$ and $Z=18$ has $v^2<1$; Tc^{99} with $t=+2$ and $Z=44$ has $v^2\gtrsim 1$; and I^{129}, Cs^{135}, Cs^{137} (high-energy transition) all have $t=-2$, $Z\approx 55$, and $v^2\gg 1$. We shall use just these three groups in applying (45) to determine the linear combination. The ideal values of t given in Table I will also be assumed; they are presumably subject to at least the same sort of deviations as found between the magnetic moments and the Schmidt limits, but this should not vitiate the order-of-magnitude considerations below.

The results for first-forbidden spectra indicate that we have only to consider the combinations VA and ST. In allowed-shape, first-forbidden spectra it is impossible to distinguish between these combinations; and the distinction would remain impossible in any order of forbiddenness, if we had no independent means of estimating nuclear matrix element ratios. Because of estimates like those of Sec. II, however, we are able to distinguish VA and ST in the second-forbidden spectra. Consider first VA: Eq. (5) indicates that $a'\sim B$, and inserting this relation in (45), we have

$$v^2\sim 4[(\lambda+t)/t]^2. \tag{48}$$

This should be of order 10 and 1 for $t=2\lambda(=\pm 2)$ and $t=-2\lambda$, respectively, which might agree with Table I if $\lambda=-1$; on the other hand, v^2 should be very large for $t=0$, which is in disagreement with Table I. Hence the spectral shapes provide evidence against VA, corroborating the direct recoil measurements.[6]

For the ST combination we use the matrix element estimate (35), writing it in the form

$$a=-\chi[\tfrac{1}{2}t+\delta], \tag{49}$$

[11] This is just the opposite of the conclusion reached in reference 6 by means of the same arguments. The difference is that reference 6 assumes not only $a'\approx B$ but also $a\approx Bt$, in which case all the first-order terms of $y_1(K,a,a')$ vanish, leaving only the strongly energy-dependent second-order terms. Actually, the matrix element estimate should be $a\approx -Bt$; the error appears to arise from the notation in reference 6, which rather suppresses the roles of λ and t and thus makes it possible to overlook a minus sign.

† *Note added in proof:*—The second-forbidden spectrum of the ground-state β decay of Fe^{59} has been measured by F. R. Metzger (private communication). The spectrum shape indicates $v^2\approx 2$. The most likely shell model assignment is a $p_{3/2}\rightarrow f_{7/2}$ transition with $t=+2$. This case therefore appears to be strictly analogous to Tc^{99}.

where the pseudoscalar correction is regarded as an empirical parameter. The specific form given for δ in (35) depends on too many approximations to be very reliable. From (45) it is clear that $v^2\approx 1$ when $t\approx -2\lambda$, regardless of the value of a. It is therefore only a question of whether Tc^{99} or the I–Cs group has v^2 closest to unity; and Tc^{99} is obviously the choice. This implies that $\lambda=-\frac{1}{2}t=-1$, or that the combination involves $(S-T)$ for negatron decay.

From (45) we see that v^2 is large (I−Cs) when the denominator approximately vanishes and small (Cl^{36}) when the numerator approximately vanishes. Taking $\lambda=-1$ and the appropriate values of t, these conditions lead to the requirement that

$$a\approx -2B \tag{50}$$

in each case. Comparing with (49) and taking $\chi/B=1-2$, we have

$$\begin{aligned}&\delta=1-2 \qquad Cl^{36}\\ &\delta=2-3 \qquad \text{I}-\text{Cs}.\end{aligned} \tag{51}$$

The fact that $\delta\neq 0$ is empirical evidence for the necessity of pseudoscalar-coupling corrections to the β-decay matrix elements. It would be impossible to approach cancellation of numerator or denominator in (45) if $\delta=0$.

The second-forbidden spectrum shapes thus lead to the conclusions that (1) the combination VA is excluded in favor of ST; (2) the relative sign of the terms is $(S-T)$ for negatron emission; (3) nuclear matrix element estimates require substantial corrections for pseudoscalar-coupled forces in the nucleus.

We now return to the first-forbidden spectrum of RaE. This transition has a comparatively long half-life and a non-unique forbidden shape, both of which indicate a cancellation among the large, energy-independent terms in (36). We may argue the choice of the particular expression in (36) as follows: the transition is most likely to be one involving y_1 alone or z_1 alone, because a simultaneous cancellation of the large terms in both y_1 and z_1 is even less likely than the relatively rare cancellation in either one. The quantity y_1 appears alone in all $\Delta I=1$ transitions, while z_1 appears alone only in $0^-\rightarrow 0^+$ transitions. These latter occur very infrequently,[12] which implies that the rare RaE-type spectrum should be associated with z_1. The previous analysis limits the choice of interaction to STP, and cancellation of the large terms in z_1 implies that

$$g\approx B. \tag{52}$$

This is just the result obtained by detailed analysis[7] of the RaE spectrum. To compare this with the matrix element estimate (20), we write (20) in the form

$$g\approx -(\lambda_P/\lambda_T)\chi\delta. \tag{53}$$

Taking δ to be the same as in (51) and $\chi\approx B$, we obtain

$$-\lambda_P/\lambda_T=1/\delta\approx(\tfrac{2}{3}\pm\tfrac{1}{3}). \tag{54}$$

The entire empirical argument based on spectrum shapes may now be summarized as follows: the absence of $1/W$ terms in allowed spectra excludes SV and TA; the absence of $1/W$ terms in allowed-shape first-forbidden spectra excludes SA and VT; the non-unique second-forbidden spectra exclude VA and indicate a relative sign $(S-T)$ for negatron emission; and the RaE spectrum indicates that the full linear combination for negatron emission should be

$$[S-T+(1/\delta)P], \tag{55}$$

where δ is a positive number of order unity.

V. SYMMETRY PRINCIPLES

The empirical evidence on the linear combination in β decay is summarized in (55). Because of the difficulty in estimating precisely the contribution of pseudoscalar forces to the nuclear matrix elements, we cannot hope to obtain more than an order of magnitude value of δ from the data. Since it appears impossible to proceed further on an empirical basis, we consider what combinations like (55) may be in accord with various simple symmetry principles.

The general interaction of two Fermi-Dirac fields is

$$\bar{\psi}_1 O\psi_2\bar{\psi}_3 O\psi_4, \tag{56}$$

where $\bar{\psi}=\psi^{T*}\beta$ is the adjoint function and $O\cdots O$ is a scalar product of any of the five Hermitian relativistic invariants, or a linear combination of these products. The behavior of (56) on permutation of $\bar{\psi}_1$ and $\bar{\psi}_3$ or ψ_2 and ψ_4 has been given.[13] It is convenient to write (56) in terms of the "normal" linear combinations that are eigenvectors under the exchange operation $P=p_{13}$ or p_{24},

$$\begin{aligned}&\Omega_1=S-T+P, && p=-1;\\ &\Omega_2=A-V, && p=-1;\\ &\Omega_3=2(S-P)-(A+V), && p=-1;\\ &\Omega_4=2(S-P)+(A+V), && p=+1;\\ &\Omega_5=3S+T+3P, && p=+1.\end{aligned} \tag{57}$$

The eigenvalue of P for each Ω is given in (57).

To generate all possible permutations of (56), it would be necessary to add to P just one further type of permutation operator $P'=p_{12}$ or p_{34}. We cannot find general eigenvalues for P', however, because

$$P'[\bar{\psi}_1 O\psi_2]=\psi_2{}^T O\bar{\psi}_1{}^T=(\bar{\psi}_1 O^T\psi_2)^T=\bar{\psi}_1 O^T\psi_2, \tag{58}$$

where the superscript T indicates transpose. There is no general linear relation between O and O^T that is independent of the representation of the Dirac matrices; hence the eigenvalues of P' are dependent on the Dirac

[12] R. W. King (private communication).

[13] M. Fierz, Z. Physik **104**, 553 (1937).

representation, implying that P' has no physical significance.

The only operator like P' for which general eigenvalues exist is P^c, where

$$P^c[\bar{\psi}_1 O\psi_2]=\bar{\psi}_2{}^c O\psi_1{}^c, \tag{59}$$

and ψ^c is the charge conjugate solution[14] defined by

$$\psi^c=C\bar{\psi}^T, \quad \bar{\psi}^c=-\psi^T C^{-1}, \quad C^{-1}\gamma_\mu C=-\gamma_\mu{}^T. \tag{60}$$

This operator is physically suitable because $\bar{\psi}_1 O\psi_2$ and $\bar{\psi}_2{}^c O\psi_1{}^c$ both describe the same physical process. Moreover, the eigenvalues of P^c are independent of the representation, being

$$\begin{aligned} p^c&=-1, \quad O=S, A, P \\ &=+1 \quad O=V, T. \end{aligned} \tag{61}$$

Equations (57) and (61) give the eigenfunctions of P and P^c. There is only one linear combination that is a simultaneous eigenfunction of both operators:[15]

$$\tfrac{1}{2}(\Omega_3-\Omega_2)=S-A-P, \quad p=p^c=-1. \tag{62}$$

Since the combination (62) is excluded for β decay by the experimental data, we may inquire which of the operators P and P^c to abandon first. There is some precedent for requiring the eigenvalue $p=-1$, if the interaction (56) is to hold for equivalent particles in the positions (1,3) or (2,4), since these particles are known to obey anticommutative statistics. On the other hand, the experimental evidence favoring P^c as a valid physical operator is all obtained from interactions with the electromagnetic field, which is not the case described by (56). On this basis the operator P seems to be the more fundamental for (56); the only eigenfunction of P resembling the empirical (55) is Ω_1. We henceforth assume that the true combination in β decay is in fact Ω_1, so that in (55) $\delta\equiv 1$.

The linear combination Ω_1 describes β^- decay when (56) is written in the conventional order:

$$(\bar{\psi}_P O\psi_N)(\bar{\psi}_e O\psi_\nu). \tag{63}$$

To find the corresponding combination Ω_1' that is appropriate the describe β^+ decay, first take the charge conjugate of the right-hand term,

$$\bar{\psi}_e O\psi_\nu=\bar{\psi}_\nu{}^c O'\psi_{e^+}, \tag{64}$$

where O' differs from O by the associated eigenvalue $p^c=\pm 1$. Then insert (64) in (63) and take the complex conjugate of the entire expression, obtaining

$$(\bar{\psi}_N O\psi_P)(\bar{\psi}_{e^+} O'\psi_\nu{}^c). \tag{65}$$

According to the eigenvalues (61), the linear combination Ω_1' in (65) that corresponds to Ω_1 in (63) is

$$-\Omega_1'=S+T+P. \tag{66}$$

The minus sign prefixing Ω_1' is unobservable, so that the net result is a change in the relative sign of the T and $(S+P)$ terms in going from β^- to β^+ emission. The effects of this change should in principle be observable.

VI. DECAY OF THE μ MESON

It is of interest to compare the decay of the isolated μ meson into an electron and two neutrinos[16,17] with the law (55) for β decay. In fact, it appears that we can select the combination Ω_1 on the basis of a couple of plausible postulates concerning the neutrino field. In the interest of formal simplicity we consider the equivalent process of annihilation of a μ^+ meson by an electron with the production of two neutrinos. The interaction operator has the form

$$(\bar{\psi}_\nu O\psi_\mu)(\bar{\psi}_\nu O\psi_e). \tag{67}$$

Now if both ψ_ν represent quanta of the same field, we require $p=-1$ to satisfy their anticommutative statistics. This limits the choice of linear combinations for (67) to Ω_1, Ω_2, Ω_3.

We obtain a further selection of the linear combinations by assuming that the unique neutrino field in (67) is strictly massless. This means that (67) should be invariant under mass reversal of the neutrino field. This mass reversal is effected by $\bar{\psi}_\nu\rightarrow\bar{\psi}_\nu\gamma_5$, or hence $O\rightarrow\gamma_5 O$ in (67). This transformation will at most change the sign of (67). The sign changes for the five combinations (57) are, respectively, $m=+, -, -, -, +$. The only combination that has $p=-1$, $m=+1$ is Ω_1.

We can therefore deduce the unique choice Ω_1 for β decay from three postulates:

(1) there is only one type of neutrino field;
(2) this field is massless; (68)
(3) the interaction of four spin $\frac{1}{2}$ fields is "universal."

In the last postulate "universal" means only that β- and μ-meson decay have the same interactions, with proper replacement of equivalent fields. If later developments prove that other four-particle interactions have the same form, we can remove the quotation marks. We can make certain tests of this third postulate by comparing the half-lives for β decay and μ-meson decay, and also by examining the shape of the electron spectrum for μ-meson decay.

The electron spectrum for μ-meson decay has been calculated.[17] It is convenient to express this spectrum in terms of the linear combinations Ω' associated with (65) by writing

$$\sum g_j O_j=\sum f_j\Omega_j', \tag{69}$$

where the g and f are coupling constants. A substitution $g_2, g_5\rightarrow -g_2, -g_5$ is necessary to convert the sign conventions of reference 17 to the usual ones. Neglect-

[14] A. Pais and R. Jost, Phys. Rev. **87**, 871 (1952); earlier references are given here.
[15] C. L. Critchfield, Phys. Rev. **63**, 417 (1943).
[16] Tiomno, Wheeler, and Rau, Revs. Modern Phys. **21**, 144 (1949).
[17] L. Michel, Proc. Phys. Soc. (London) **63**, 514 (1950).

ing the electron rest mass, we have for the spectrum formula (45) of reference 17:

$$P(E)dE=\frac{4}{3}\frac{\mu E^2 dE}{\hbar(2\pi\hbar^2c^2)^3}[a\mu-bE],$$
$$a=3F^2=3(f_1^2+f_2^2+f_3^2+f_4^2+f_5^2), \quad (70)$$
$$b=6f_1^2+6f_2^2+4f_3^2+4f_4^2+2f_5^2,$$

where E is the electron energy and μ is the rest mass energy of the μ meson. Use of the linear combinations Ω' has the advantage of eliminating cross terms in f_if_j from (70). The shape parameter ρ of reference 17 is

$$\rho=\tfrac{3}{2}(1-b/2a)=[f_5^2+\tfrac{1}{2}(f_4^2+f_3^2)]/F^2. \quad (71)$$

The mean life τ for μ-meson decay is given by $1/\tau=\int_0^{\mu/2}P(E)dE$, or hence:

$$f_1^2+f_2^2+2(f_3^2+f_4^2)+3f_5^2 = (24\hbar\mu/\tau)[2\pi\hbar^2c^2/\mu^2]^3 = 2.00\times10^{-98}\ (\text{erg cm}^3)^2, \quad (72)$$

when the experimental values[18,19] $\tau=2.09\times10^{-6}$ sec and $\mu=(209.8\pm2.2)m_0c^2$ are used. We may compare (72) with the value of the interaction constant deduced from β decay. The most detailed estimate published to date[1] is based on mirror-image transitions in light nuclei where the interaction P plays no part. A re-analysis of this problem, using the most recent data, selecting only those cases closest to the ideal one-particle model, and assuming the interaction Ω_1, leads to a constant[12]

$$B_0=(ft)|\Omega_1|^2=2\pi^3\hbar7\ \ln2/(f_1^2m_0^5c^4)=5600\ \text{sec}, \quad (73)$$

where (ft) is the empirical parameter determined in measurements of β-decay lifetimes and m_0 is the electron rest mass. An extra factor of 2 appears in this value for B_0 because the empirical fit in reference 1 was so normalized that $B_0/(ft)=\frac{1}{2}|\Omega_1|^2$. The corresponding value of f_1^2 is

$$f_1^2=2.19\times10^{-98}\ (\text{erg cm}^3)^2. \quad (74)$$

Although the numerical values in (72) and (74) are not identical, the uncertainty in the μ-meson mass introduces an error of about 5 percent into (72), and the scatter of individual (ft) values in determining B makes it reasonable to assign an error on the order of 10 percent to (74). Therefore, if we assume the same linear combination Ω_1 for β and μ decay, the coupling constants are the same within the experimental errors specified. The weighted mean value of the coupling constant is

$$|f_1|=1.44\pm0.04\times10^{-49}\ \text{erg cm}^3. \quad (75)$$

If the μ-meson decay interaction is really Ω_1', the shape factor (71) should be $\rho=0$, indicating a decay electron spectrum that goes to zero at its end point. Present experimental evidence on this question is somewhat conflicting,[20] and an accurate determination of the spectrum near its end point would be of crucial significance in determining whether the β- and μ-decay interactions are identical. It would also be desirable, if possible, to determine the capture rate of μ mesons by individual nucleons to an accuracy comparable with that of (72) and (74), say 10 percent.

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