Application of the *jj*-Coupling Model to Moderately Light Nuclei*

DIETER KURATH†
Department of Physics, University of Chicago, Chicago, Illinois, and Argonne National Laboratory, Lemont, Illinois
(Received June 8, 1953)

An investigation based on the strong spin-orbit coupling model is made for nuclei in the $1d_{3/2}$ and $1f_{7/2}$ shells. Hartree-Fock wave functions and central forces are used in calculating ground state spins and binding energy differences. Coupling rules are found for configurations of identical nucleons and for the simplest odd-odd nuclei. Approximate wave functions are constructed for odd-A and even-even nuclei having both neutrons and protons in the $1f_{7/2}$ shell. Binding energy differences, magnetic moments, and beta decay matrix elements are calculated with these functions. The result of comparing binding energy differences with experiment is interpreted as favoring strong spin-orbit coupling over weak spin-orbit coupling.

I. INTRODUCTION

THE hypothesis of strong spin-orbit coupling,[1] the basis of the *jj* model, has been quite successful in explaining the shell structure of nuclei, and in accounting for observed spins and magnetic moments. For this reason a more detailed investigation has been made to see whether there are other experimental features of nuclear structure contained in the model.

In an earlier paper,[2] the model was applied to the $1p$ shell which is filled in between $_2$He^4 and $_8$O^{16}. The results were compared with those of a previous calculation[3] based on the hypothesis of weak spin-orbit coupling, the *LS* model. Neither model presents a consistently better picture of the experimental results than the other, but there is a tendency for the *LS* model to be preferable for the light nuclei in the shell, while the *jj* model is better for the heavier nuclei.

In going to heavier nuclei, the next region where the filling of shells is unambiguous is the $1d_{3/2}$ shell from $_{16}$S^{32} to $_{20}$Ca^{40} and the $1f_{7/2}$ shell from $_{20}$Ca^{40} to $_{28}$Ni^{56}. The first part of the paper treats the order of nuclear energy levels in this region for configurations of identical nucleons and for odd-odd nuclei involving only two nucleons or a nucleon and a hole in a shell. The latter category contains the interesting nuclei, $_{17}$Cl^{36} and $_{19}$K^{40}, whose spins have been measured.

The second part of the paper treats the ground states of the odd mass number and even Z-even mass number nuclei in the $1f_{7/2}$ shell. Approximate wave functions are constructed for those configurations not consisting of identical nucleons in the shell. Binding energies, magnetic moments, and ft values are computed. In particular, the binding energy differences of isobars are related to the symmetry properties of the wave functions. The symmetry properties depend on the strength of spin-orbit coupling, and the experimental evidence is shown to favor strong coupling.

II. METHOD OF CALCULATION

The calculations are carried out with Hartree-Fock wave functions. While there are objections[4] to the use of such functions, wherein the total wave function is represented by the product of individual nucleon functions, this procedure offers the only feasible means of calculation, and presents reasonable values for spins and magnetic moments.

Two fundamental assumptions are employed in constructing the wave functions:

(1) The Hamiltonian is symmetric to the exchange of any two nucleons, which allows the separation of the wave function into the product of a space spin function and an isotopic spin function. This assumption of the equality of $p-p$, $n-p$, and $n-n$ forces makes the isotopic spin T a good quantum number. The effect of Coulomb forces for determining the wave functions is also assumed to be negligible, which should be a valid approximation since the nuclei treated are rather light.

(2) The intrinsic spin and orbital angular momentum of each individual nucleon are strongly coupled so that the individual particle energy for the state $j=l+\frac{1}{2}$ lies considerably below that for $j=l-\frac{1}{2}$.

The single particle wave functions are taken to be those resulting from a central harmonic oscillator potential. The state of orbital angular momentum l is then represented by the product of a spherical harmonic and a radial wave function. The radial dependence for states encountered in this calculation is given by the functions with no nodes:

$$R_l(r)=N_l r^l \exp[-(r/r_l)^2], \quad (1)$$

where N_l is a normalization factor. The parameter r_l in the oscillator function can be chosen in such a way as to make the function quite similar to that function resulting from a square-well potential with constants

* Much of this material is contained in a thesis submitted to the University of Chicago in partial fulfillment of the requirements for the degree of Doctor of Philosophy (August 1951).

† Present address: Argonne National Laboratory, Lemont, Illinois.

[1] M. G. Mayer, Phys. Rev. **75**, 1969 (1949); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949).

[2] D. Kurath, Phys. Rev. **88**, 804 (1952).

[3] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937); E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937).

[4] H. A. Bethe and R. F. Bacher, Revs. Modern Phys. **8**, 149 (1936).

TABLE I. Coefficients in the potential energy expressions, $(\alpha_0+\alpha_1x+\alpha_2x^2+\alpha_3x^3)K$ and $(\beta_0+\beta_1x+\beta_2x^2+\beta_3x^3)K$, where $x=\rho^4/4(1+\rho^2)$, $\rho=r_f/r_0$ and $K=(A_0/735)(1+\rho^2)^{-9/2}$.

	α_0	α_1	α_2	α_3	β_0	β_1	β_2	β_3
$(1f_{7/2})^2$								
$I=6$	−525	−2520	−630	2100	735	5040	6930	2100
4	−195	−2124	−1422	4212	735	5040	10 494	4212
2	15	−360	−4950	8580	735	8820	18 810	8580
0	105	2268	12 474	36 036	735	11 340	37 422	36 036
$(1f_{7/2})^3$								
$I=15/2$	−1350	−7290	−2430	7740	2205	15 120	23 220	7740
11/2	−930	−5526	−5958	12 948	2205	18 270	32 796	12 948
9/2	−765	−6120	−4770	12 420	2205	16 290	31 410	12 420
7/2	−630	−2772	5544	36 036	2205	21 420	51 282	36 036
5/2	−525	−3528	−9954	18 564	2205	22 050	43 218	18 564
3/2	−450	−5238	−6534	15 444	2205	17 550	36 828	15 444
$(1f_{7/2})^4$								
$I=8$	−2340	−13 428	−7164	19 224	4410	32 040	53 208	19 224
6	−1890	−10 332	−2016	38 136	4410	36 540	72 072	38 136
5	−1710	−11 664	−10 692	25 272	4410	34 560	64 044	25 272
4	−1560	−9684	−14 652	29 832	4410	39 060	72 864	29 832
4	−1560	−9936	−2808	40 248	4410	36 540	75 636	40 248
2	−1350	−11 448	−11 124	27 144	4410	34 020	67 068	27 144
2	−1350	−8172	−6336	44 616	4410	40 320	83 952	44 616
0	−1260	−5544	11 088	72 072	4410	42 840	102 564	72 072

fitting experimental observations. In this way the differences resulting from the use of these two central potential wells can be minimized.

These orbital functions are then coupled with intrinsic spin functions in linear combinations that have individual particle eigenvalues $j=l\pm\frac{1}{2}$. The individual particle eigenfunctions are multiplied by isotopic spin functions and combined into many-particle Hartree-Fock wave functions having total angular momentum I and isotopic spin T. The procedure is illustrated in Appendix III.

With these wave functions as zero-order functions, the potential energy of two-body interactions can be calculated as a first-order perturbation. If the nuclear interactions are restricted to be charge independent static central forces, the general interaction of two nucleons is[5]

$$[C_0+C_1(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+C_2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)+C_3(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)]J(r_{12}), \quad (2)$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are the ordinary spin and isotopic spin vectors, respectively, and $J(r_{12})$ is the spatial dependence. The interaction can be alternatively expressed in the form of exchange interactions

$$[W+MP_{12}+BQ_{12}+HP_{12}Q_{12}]J(r_{12}), \quad (3)$$

where P_{12} and Q_{12} are the operators that exchange the space coordinates and spin coordinates respectively for particles 1 and 2. The coefficients identify the usual name for the interaction, Wigner, Majorana, Bartlett, and Heisenberg. The wave functions are antisymmetric to the complete exchange of two nucleons, so that if one introduces the operator R_{12} which exchanges the charge of two nucleons, all wave functions obey the relationship

$$P_{12}Q_{12}R_{12}\psi=-\psi. \quad (4)$$

The coefficients in (2) and (3) are connected by the relationships for antisymmetric wave functions:

$$\begin{aligned} Q_{12}&=\tfrac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),\\ P_{12}Q_{12}&=-R_{12}=-\tfrac{1}{2}(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2),\\ P_{12}&=-\tfrac{1}{4}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2). \end{aligned} \quad (5)$$

Therefore, by calculating the energies for the four possible interactions (1, P_{12}, Q_{12}, and $P_{12}Q_{12}$), one can obtain the result for any central-force interaction. The central-force solution of the deuteron, where $P_{12}=1$, requires that $W+M\approx0.8$ and $H+B\approx0.2$ in order to account for the difference of singlet and triplet potentials for low energy neutron-proton scattering. Aside from this restriction, the coefficients are rather uncertain.

For the spatial dependence of the interaction, the negative Gaussian is used:

$$J(r_{12})=A_0\exp[-(r_{12}/r_0)^2]. \quad (6)$$

The parameters A_0 and r_0 are related from the determination of the deuteron binding energy.[6] This function provides easy variation of the range of interaction, r_0, and in the limit as r_0 goes to zero it behaves like a delta function. In this limit the ambiguity in exchange mixture is removed since P_{12} approaches unity, and one can also check results of previous delta function calculations. The method of calculating the interaction integrals plus a tabulation of these integrals is presented in Appendix I.

[5] E. Wigner and L. Eisenbud, Proc. Nat. Acad. Sci. U. S. **27**, 281 (1941).

[6] H. A. Bethe, Revs. Modern Phys. **8**, 111 (1936).

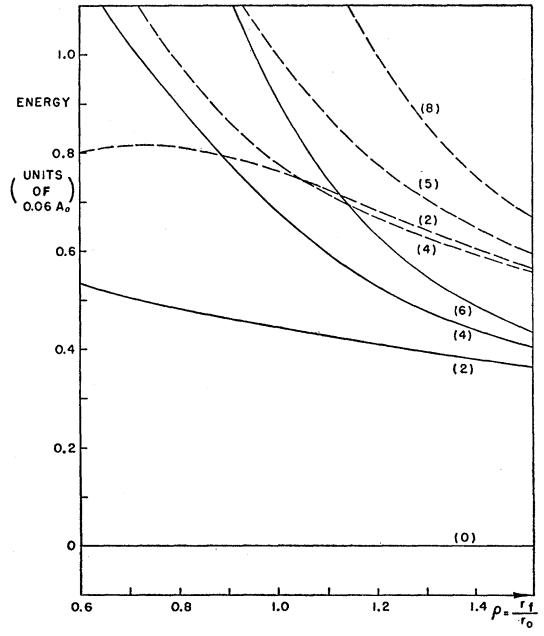

FIG. 1. Order of energy levels of the $(1f_{7/2})^2$ and $(1f_{7/2})^4$ configurations for the interaction (8) as a function of ρ, the ratio of nuclear size parameter to range of nuclear forces. Solid lines are levels appearing in both configurations; broken lines are additional levels for $(1f_{7/2})^4$.

III. DETERMINATION OF GROUND STATES

A. Identical Nucleons

The ordering of energy levels for a given nucleus depends only on the interactions among the nucleons outside closed shells. In configurations where the nucleons in unfilled shells are identical, the predictions of ground state spins have been given for the cases of delta function[7] and infinite range[8] forces. The two approximations give a spin of zero for an even number of identical nucleons or holes, but for an odd number they differ in all but the trivial cases of a single particle or hole in a shell. Since the $1d_{3/2}$ shell contains no configurations where the predictions differ, only the $1f_{7/2}$ shell is treated.

Since the $1f_{7/2}$ shell is filled above $_{20}Ca^{40}$, the nuclei having an unfilled shell of identical $1f_{7/2}$ nucleons are the Ca isotopes, and those nuclei with filled $1f_{7/2}$ neutron shells, $_{21}Sc^{49}$, $_{22}Ti^{50}$, $_{23}V^{51}$, $_{24}Cr^{52}$, $_{25}Mn^{53}$, $_{26}Fe^{54}$, $_{27}Co^{55}$, and $_{28}Ni^{56}$. The potential energy of interaction among $1f_{7/2}$ nucleons is all that splits the levels, so the theory of holes says that the level order for n identical particles is the same as that for n identical holes. Therefore the only configurations one needs in order to study the level splitting are $(1f_{7/2})^2$, $(1f_{7/2})^3$, and $(1f_{7/2})^4$.

For identical nucleons, the charge-exchange operator R_{12} has the eigenvalue one, so from (4) we see that $P_{12}Q_{12}=-1$ or $P_{12}=-Q_{12}$. The level order for the configurations is obtained using the finite range interaction integrals of Appendix I. The general expression of the potential energy of a level is

$$E=(M-B)(\alpha_0+\alpha_1x+\alpha_2x^2+\alpha_3x^3)K + (W-H)(\beta_0+\beta_1x+\beta_2x^2+\beta_3x^3)K, \quad (7)$$

where W, M, B, and H are the strengths of 1, P_{12}, Q_{12}, and $P_{12}Q_{12}$ exchange, respectively. As shown in the appendix, K and x involve the parameters A_0, r_0, of the Gaussian interaction and the oscillator wave function parameter, r_f. The coefficients α_i and β_i are tabulated for the various levels in Table I.

The level order depends on the quantity $\rho=r_f/r_0$, the ratio of the range of a $1f_{7/2}$ nucleon to the range of nuclear forces. Some possible exchange mixtures are:

$$[0.8P_{12}+0.2Q_{12}]J(r_{12}),\ \text{Majorana plus Bartlett}; \quad (8)$$

$$P_{12}[0.8+0.2Q_{12}]J(r_{12}),\ \text{Majorana plus Heisenberg}; \quad (9)$$

$$\tfrac{1}{2}[1+P_{12}][0.8+0.2Q_{12}]J(r_{12}),\ \text{Serber mixture}. \quad (10)$$

The space exchange part of (10) comes from the present interpretation[9] of high-energy neutron-proton scattering, although such a potential violates the saturation requirements for nuclear forces.

The level order of the various configurations is presented as a function of ρ in Figs. 1 and 2 for the interaction (8). For an even number of nucleons, the ground state has $I=0$ and is well separated from the first excited state 2 for all ranges. This is true for any exchange mixture that is not predominantly spin dependent. For three nucleons (or holes) there is a crossing

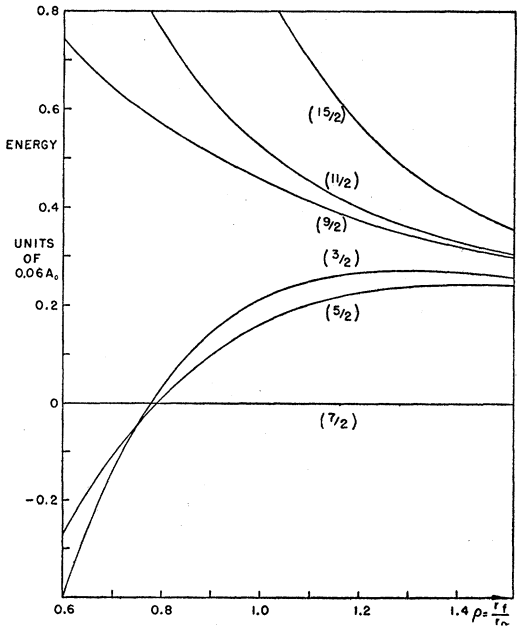

FIG. 2. Order of energy levels of the $(1f_{7/2})^3$ configuration as a function of ρ for the interaction (8).

[7] M. G. Mayer, Phys. Rev. **78**, 22 (1950).
[8] G. Racah, Phys. Rev. **78**, 622 (1950).
[9] R. S. Christian and E. W. Hart, Phys. Rev. **77**, 441 (1950).

of the 7/2 level with those of lower I. The position of crossover depends on the exchange mixture, and the ground state of $(1f_{7/2})^3$ is listed as a function of ρ for several mixtures in Table II.

In order to estimate a value for ρ corresponding to existing nuclei, the parameter r_f is determined by comparing the oscillator wave function with a square-well wave function whose constants are determined from the experimental evidence in this region of masses:

Atomic mass number, $A=43$.
Nuclear radius $=1.48\times10^{-13}A^{\frac{1}{3}}$ cm $=5.19\times10^{-13}$ cm.
Binding energy of last nucleon $=8$ Mev.

For $r_f\approx2.9\times10^{-13}$ cm, the two wave functions are quite similar. If the range of nuclear forces r_0 for a Gaussian potential is assumed to be about 2.8×10^{-13} cm, the ratio $\rho\approx1$. This is near the region of crossing, but would tend to favor $I=7/2$ as ground state. The use of a Yukawa spatial dependence shifts the crossover region so that an I of 7/2 is heavily favored; for the Serber exchange mixture (10), 7/2 would be low for all ranges. The singularity of the Yukawa potential at the origin, which makes it similar to a delta function, accounts for this behavior, as has been pointed out by Talmi,[10] who used a Yukawa dependence with Majorana exchange. Similar level order calculations have recently been published by Flowers and Edmonds,[11] using an elegant group theoretical approach and covering other configurations as well.

The behavior of the levels as given by the Gaussian calculation has some similarity to experimental facts. It presents a possible explanation[12] of the spin of 3/2 observed in the $(1d_{5/2})^3$ configuration of Na^{23}. The fact that the $(1f_{7/2})^3$ configuration of V^{51} has a measured spin of 7/2 while the $(1f_{7/2})^{-3}$ configuration of Mn^{55} is measured as 5/2 can be interpreted as evidence that these nuclei are near the region of level crossing. In the $1g_{9/2}$ shell there is strong experimental evidence that there is a state, 7/2 (even parity), that competes with the 9/2 level in configurations of 3, 5, or 7 nucleons. These are the configurations where a generalization of the calculation results would lead one to expect crossovers with the 7/2+ state being one which drops below 9/2 as one goes from large to small values of r_g/r_0. However, one would also expect the levels 5/2+ and 3/2+ to be low-lying states in configurations $(1g_{9/2})^{3\text{ or }7}$, and one would expect 1/2+ in addition to these for $(1g_{9/2})^5$. These have not been observed in any of the numerous nuclei where such configurations occur. Recent explicit calculations of the $(1g_{9/2})^3$ configuration[13] show that 7/2+ is not expected as ground state with the usual central forces, although the level spacing is quite small. Therefore, while the model does contain the possibility of level crossings in those nuclei where such behavior is observed, it does not explain all the observed facts, and further clarification is needed.

TABLE II. Ground state of $(1f_{7/2})^3$ as function of $\rho=r_f/r_0$.

Potential	$I=7/2$	$I=5/2$	$I=3/2$
(8)	$\rho>0.79$	$0.74<\rho<0.79$	$\rho<0.74$
(9)	$\rho>0.90$	...	$\rho<0.90$
(10)	$\rho>0.75$	$0.58<\rho<0.75$	$\rho<0.58$

B. Odd-Odd Nuclei

The question of ground state spins for odd-odd nuclei has been investigated by Nordheim,[14] who found quite good agreement with experimental evidence by using the following hypotheses.

1. The individual configurations of neutrons and protons in odd-odd nuclei are the same as in odd-A nuclei with the same number of nucleons in the odd-particle group.

2. If the odd neutron and proton groups belong to different Schmidt groups, the resultant spin is the difference. $j_1=l_1+\frac{1}{2}$, $j_2=l_2-\frac{1}{2}$ gives $I=|j_1-j_2|$.

3. If the odd neutron and proton groups belong to the same Schmidt group, the resultant spin is high, often the maximum possible.

$j_1=l_1\pm\frac{1}{2}$, $j_2=l_2\pm\frac{1}{2}$ gives I large, often j_1+j_2.

When the odd-odd nucleus has a configuration of a single proton (or proton-hole) in a shell together with a single neutron (or neutron-hole) in a shell, the wave function is determined uniquely by stipulating I. Calculations for such simple configurations in the $1d_{3/2}$ and $1f_{7/2}$ shell were carried out to compare results with the Nordheim rules and with experimental evidence. The energy levels are listed in the tables of Appendix II for the static central force interactions.

The results can be summarized in the following manner. The space-dependent Majorana and Wigner interactions give different predictions for the coupling of two particles (or holes) and the coupling of a particle and a hole. For two particles, the Majorana interaction shows competition between parallel and antiparallel alignment of the j's. It favors antiparallel alignment for vanishing range of nuclear forces but parallel alignment for any reasonable ratio of nuclear size to range of forces. This is shown in Table III where the crossover value of $\rho=r_f/r_0=r_d/r_0$, the ratio of oscillator parameter to nuclear force range, is given. For the nuclei in question it is expected that this ratio is close to unity. Wigner forces would give antiparallel alignment of j's.

However, for the coupling of a particle and a hole, Majorana forces give competition between parallel j alignment and a value of I one unit less. For vanish-

[10] I. Talmi, Phys. Rev. **82**, 101 (1951).
[11] B. H. Flowers, Proc. Roy. Soc. (London) **A212**, 248 (1952); A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A214**, 515; **A215**, 120 (1952).
[12] D. Kurath, Phys. Rev. **80**, 98 (1950).
[13] I. Talmi, Helv. Phys. Acta **25**, 185 (1952); B. H. Flowers, Proc. Roy. Soc. (London) **A215**, 398 (1952).
[14] L. W. Nordheim, Revs. Modern Phys. **23**, 322 (1951).

TABLE III. Coupling of a single neutron with a single proton. The ratio ρ = (nuclear size parameter)/(range of nuclear forces), is expected to be near one in the region of physical interest. $\rho = r_f/r_0 = r_d/r_0$.

Configuration	Nucleus	I_{exp}	Majorana potential I for $\rho=\infty$	Majorana potential I for $\rho=0.1$	Crossover region	Serber potential I
$(1d_{3/2})(1d_{3/2})$	$_{17}Cl^{34}$	···	0	3	$\rho=2.8$	Always 3
$(1d_{3/2})^{-1}(1d_{3/2})^{-1}$	$_{19}K^{38}$	(2 or 3)[a]	0	3	$\rho=2.8$	Always 3
$(1d_{3/2})(1f_{7/2})$	$_{17}Cl^{38}$	(2)[a]	2	5	$\rho=4.3$	5 for $0.4<\rho<1.85$ 2 for $\rho<0.4$, $\rho>1.85$
$(1f_{7/2})(1f_{7/2})$	$_{21}Sc^{42}$	···	0	7	$\rho=2.0$	7 for $\rho<2.5$

[a] Inferred from beta decay.

ing range of nuclear forces ($\rho=\infty$) an I of one unit less than the maximum is favored, while for long-range forces the result is parallel alignment. This behavior is presented in Table IV, and the crossover values of ρ are seen to be considerably smaller than for the particle-particle cases. Wigner forces favor the smaller I, namely the $\rho=\infty$ result. The results for space-dependent interactions do not show any of the dependence on Schmidt line groups postulated by the Nordheim rules.

On the other hand, the spin-exchange Bartlett and Heisenberg interactions do not differentiate between particle-particle and particle-hole couplings. The Bartlett interaction gives results in agreement with the Nordheim rules, and the Heisenberg interaction also gives this agreement but less unequivocally. From the deuteron one expects that space-dependent interaction has a weight of about 0.8 while spin dependence has a weight of about 0.2. So the separation of particle-particle cases from particle-hole cases should be real. The I's expected for an exchange mixture like the Serber potential (10) are given in the last column of Tables III and IV.

In comparing the particle-particle predictions with experimental results Sc^{42} is unknown, and the decay scheme of Cl^{34} cannot be easily interpreted. The allowed decay of K^{38} to the $I=2$ level of A^{38} is consistent with an assignment of 3 for the spin of K^{38}, in agreement with calculation. The spin of Cl^{38} seems to be 2. Here Majorana forces would give 5, but spin-dependent forces favor 2 in agreement with the Nordheim rule. The Serber mixture bring 2 close to 5, but still above it for the region of physical interest $\rho\approx 1$.

For particle-hole coupling there is more certain experimental evidence. The measured I's of Cl^{36} and K^{40} are both one unit less than the maximum for the shell model assignments. These agree with the results of using a Serber potential, although other combinations would give these results also. The spin assignment of 6 for Sc^{48} has been found to be compatible with recent experimental results.[15] A further experimental result of interest[16] is the measured spin of 6 for V^{50}, $[(1f_{7/2})^3 \times (1f_{7/2})^{-1}]$. An extensive calculation[17] of V^{50} also found a spin of 6 for the case of vanishing range of nuclear forces.

The results of this calculation agree with the Nordheim rule for coupling of two $l+\frac{1}{2}$ or two $l-\frac{1}{2}$ particles. For the coupling of an $l+\frac{1}{2}$ particle with an $l-\frac{1}{2}$ particle the Majorana interaction gives parallel and antiparallel j alignment as the low states but favors parallel alignment for expected values of the range parameter ρ. Spin-dependent interactions favor antiparallel alignment, in agreement with Nordheim's rule, but interactions with 20 percent spin dependence still favor parallel alignment. This disagrees with the experimental result for Cl^{38}, which fits the Nordheim rule.

For the coupling of a particle and a hole, the present calculation gives strong evidence for an I of one unit less than that of parallel j alignment regardless of the Schmidt line groups involved. This seems to be in good agreement with experiment and explains K^{40}, an outstanding exception to the Nordheim rule.

The over-all picture for odd-odd nuclei is still obscure. While the calculation of particle-hole coupling seems to fit in well with experimental evidence, the particle-particle results are not well supported. Furthermore, there must be a region of transition from the particle-particle picture to the particle-hole picture as one considers nuclei other than the simple cases treated here, and the calculation becomes formidable. For this reason, odd-odd nuclei are not treated further in this paper.

IV. POTENTIAL ENERGY CONTRIBUTIONS WITHIN THE $1f_{7/2}$ SHELL

When one deals with configurations having both neutrons and protons in unfilled shells, there are generally several states with the same total angular momentum and isotopic spin. The problem of selecting the state presumed to have lowest energy without solving secular determinants is decided as follows.

The ground-state I's are presumed to be 7/2 for odd-A nuclei and 0 for even-even nuclei together with $T=T_z$. Those magnetic moments that have been measured are found to lie near the Schmidt line. The configurations of identical nucleons have calculated magnetic moments on the Schmidt lines, and also have the property that for an $I=7/2$, $M=7/2$ state the configurations consist of an individual particle with $m=7/2$

[15] Hamermesh, Hummel, Goodman, and Engelkemeir, Phys. Rev. 87, 528 (1952); D. Kurath, Phys. Rev. 87, 528 (1952).
[16] Kikuchi, Sirvetz, and Cohen, Phys. Rev. 88, 142 (1952).
[17] A. Hitchcock, Phys. Rev. 87, 664 (1952).

while the m's of the remaining particles cancel in pairs, (i.e., for every particle in a $+m$ state there is one in a $-m$ state). Similarly the $I=0, M=0$ states are made up of configurations wherein the individual particle m's cancel in pairs. The property that only such configurations occur is assumed to carry over to the ground states of nuclei with both neutrons and protons in the unfilled $1f_{7/2}$ shell. As a result of restriction to such configurations,[18] the quantum numbers I and T serve to determine a wave function uniquely. The validity of this assumption is tested in Appendix III for the case of three $1f_{7/2}$ nucleons where three states with $I=7/2$, $T=1/2$ are possible. Solution of the secular determinant shows that the lowest energy eigenvalue is quite close to that found by calculating the diagonal matrix element with the approximate wave function formed using the procedure stated above.

By constructing the approximate wave functions for all odd-A and even-even nuclei in the $1f_{7/2}$ shell in accordance with these assumptions, as illustrated in Appendix III, one can then calculate diagonal energy matrix elements for the various static central force interactions. The resulting potential energy contributions from within the $1f_{7/2}$ shell can be summarized in the expression:

$$V_n=\left[\frac{n(n+2)-4T(T+1)}{4}\right]b(\rho)+\left[\frac{n-I/(7/2)}{2}\right]D(\rho)+\left[\frac{n(n-2)+I/(7/2)}{4}\right]d(\rho). \quad (11)$$

Here T is the isotopic spin, I the total angular momentum, and n is the number of nucleons in the $1f_{7/2}$ shell. In this expression $b(\rho)$, $D(\rho)$, and $d(\rho)$ are quantities which depend on the interaction potential used and the nuclear size through the ratio of oscillator parameter to range of nuclear forces, $\rho=r_f/r_0$ introduced in Sec. I. Explicit expressions for these quantities are given in Eq. (A8) of Appendix I, but considerable information can be obtained without them. In order to obtain an idea of the physical meaning of the quantities b, d, and D, we notice that for identical particles $T=n/2$, so that b is not present. For two neutrons, $V_2=D$, and for three neutrons, $V_3=D+d$ so that D is the binding of two identical nucleons and d is the amount added when another identical one is present. On the other hand, b arises when both neutrons and protons are present; its coefficient depends on the isotopic spin T, that is, the symmetry properties of the wave functions. It is the only term of V_n that is involved when the difference in binding energy for isobars with the same I is evaluated.

ISOBAR DIFFERENCES

A. Comparison of Calculation and Experiment

The experimentally measured odd-A beta decays in the $1f_{7/2}$ shell consist of one mirror decay, and ten others which have logft values around five and are thus allowed transitions. These provide the most accurate experimental data for comparing energy differences with the calculated values. In addition, there are three isobar pairs of even-even nuclei whose mass differences have been measured.

The theoretical binding energy difference consists of two parts:

1. the Coulomb energy difference;
2. the difference in that part of the potential energy term due to interactions of the $1f_{7/2}$ nucleons with themselves.

The Coulomb difference has two contributions, one arising from the repulsion of a $1f_{7/2}$ proton by the protons in the closed shells, and the other coming from the Coulomb interaction of the $1f_{7/2}$ protons with each other. The interaction with the core can be estimated by the change in electrostatic energy of a uniformly charged sphere of nuclear dimensions upon adding a proton

$$\epsilon=(6/5)Ze^2/R. \quad (12)$$

At $_{20}Ca^{41}$, $Z=20$, $R=(1.48\times10^{-13})(41)^{\frac{1}{3}}=5.10\times10^{-13}$ cm; these give a value of $\epsilon=6.75$ Mev. This value is in good agreement with the experimental difference obtained from the mirror decay, $Sc^{41}(\beta^+)Ca^{41}$, where the binding energy difference is the sum of the kinetic energy of the beta particle, the neutron-proton mass difference, and two electron masses. Numerically this is $4.94\pm0.07+0.78+1.02=6.74\pm0.07$ Mev. Therefore, this part of the Coulomb difference is included in the calculation as $\epsilon=6.75(41/A)^{\frac{1}{3}}$ Mev to provide for the dependence on nuclear radius. The electrostatic interaction of the $1f_{7/2}$ protons is calculated in the same way

TABLE IV. Coupling of a single nucleon with a single hole for odd-odd nuclei. The ratio $\rho=$ (nuclear size parameter)/(range of nuclear forces) is expected to be near one in the region of physical interest. $\rho=r_f/r_0=r_d/r_0$.

Configuration	Nucleus	I_{exp}	Majorana potential I for $\rho=\infty$	Majorana potential I for $\rho=0.1$	Majorana potential Crossover region	Serber potential I
$(1d_{3/2})(1d_{3/2})^{-1}$	$_{17}Cl^{36}$	(2)	2	3	$\rho=1.5$	2 for $\rho>0.85$
$(1d_{3/2})^{-1}(1f_{7/2})$	$_{19}K^{40}$	(4)	4	5	$\rho=1.3$	4 for $\rho>0.6$
$(1f_{7/2})(1f_{7/2})^{-1}$	$_{21}Sc^{48}$	...	6	6	6, 7 degenerate at $\rho=0$	6 for $\rho>0.5$ 7 for $\rho<0.5$

[18] These ground states are believed to be identical with those found by B. H. Flowers using the symplectic group. See reference 11.

TABLE V. Energy resulting from electrostatic interaction of the $1f_{7/2}$ protons. The unit of energy is $c=0.76e^2/r_f=[1.10\times10^{-13}/r_f]$ Mev$\approx$0.35 Mev.

Element	Energy
$_{22}$Ti	$1.0c$
$_{23}$V	$2.5c$
$_{24}$Cr	$5.0c$
$_{25}$Mn	$8.1c$
$_{26}$Fe	$12.1c$
$_{27}$Co	$16.6c$

as the interaction due to nuclear forces. However, since the term is small and variations of the term for a group of isotopes are at most 5 percent of the total term, the contribution is listed only for each element. This is done in Table V in units of c, where $c=0.76e^2/r_f$; r_f is the oscillator range parameter. In Sec. III it was estimated that $r_f\approx3\times10^{-13}$ cm for $A=43$; so an average value of c over the shell is $c=0.35$ Mev.

The nuclear part of the interaction of $1f_{7/2}$ particles is given by (11), and it is clear that for isobar differences, $V_n(T-1)-V_n(T)=2Tb(\rho)$. The calculated difference of binding energy is therefore a very simple expression, consisting of a multiple of $b(\rho)$ plus a numerical Coulomb term obtained by using $\epsilon=6.75\times(41/A)^{\frac{1}{3}}$ Mev and $c=0.35$ Mev. The calculated differences are listed in the last column of Table VI.

The experimental binding energy differences are gotten chiefly from beta-decay observations. They are obtained by the equations:

$$\begin{aligned}(\text{B.E.})_{\text{final}}-(\text{B.E.})_{\text{initial}}&=(\text{K.E.})_{\max}\\&\quad+2m_0c^2+({}_0n^1-{}_1\text{H}^1)\text{ for }\beta^+\text{ decay},\\(\text{B.E.})_{\text{final}}-(\text{B.E.})_{\text{initial}}&=(\text{K.E.})_{\max}\\&\quad-({}_0n^1-{}_1\text{H}^1)\text{ for }\beta^-\text{ decay},\end{aligned}\tag{13}$$

where $(\text{K.E.})_{\max}$ is the observed maximum kinetic energy of the beta particle. Some additional binding energy differences are found in the mass spectroscopic data at $A=48$ and 50, the successive beta decays from $_{26}\text{Fe}^{52}\rightarrow{}_{25}\text{Mn}^{52}\rightarrow{}_{24}\text{Cr}^{52}$, and the (p, n) threshold on V^{51}. Unless otherwise noted, the data are taken from references given in the National Bureau of Standards Circular 499. They are listed in the first two columns of Table VI, and the experimental binding energy differences obtained by using $m_0c^2=0.51$ Mev and $({}_0n^1-{}_1\text{H}^1)=0.78$ Mev are given in column three.

TABLE VI. Binding energy differences for isobars in the $1f_{7/2}$ shell. Column two lists the experimental information as well as method and year of observation. Unless noted, data are from National Bureau of Standards Circular 499. Column three lists the experimental binding energy difference, and column four the calculated difference.

Isobars	Maximum K.E. (Mev) method and year of observation[a]	Exp. B.E. diff. (Mev)	Calc. B.E. diff. (Mev)
$\text{Sc}^{43}(\beta^+)\text{Ca}^{43}$	1.12 ± 0.05 *a, s* (45)	2.92	$6.65-3b(\rho)$
$\text{Ti}^{45}(\beta^+)\text{Sc}^{45}$	1.00 ± 0.02 *s* (50)	2.80	$6.90-3b(\rho)$
$\text{Ca}^{45}(\beta^-)\text{Sc}^{45}$	0.255 ± 0.005 *s* (50)	−0.525	$-6.55+5b(\rho)$
$\text{V}^{47}(\beta^+)\text{Ti}^{47}$	1.65 *a* (49)	3.45	$7.00-3b(\rho)$
$\text{Sc}^{47}(\beta^-)\text{Ti}^{47}$	0.61 *a* (49)	−0.17	$-6.80+5b(\rho)$
$\text{Cr}^{49}(\beta^+)\text{V}^{49}$	1.45 ± 0.05 *a, cc* (42)	3.25	$7.25-3b(\rho)$
$\text{Sc}^{49}(\beta^-)\text{Ti}^{49}$	2.4 *a* (49)	1.6	$-6.70+7b(\rho)$
$\text{Mn}^{51}(K)\text{Cr}^{51}$	2.0 *a* (38)	3.8	$7.35-3b(\rho)$
$\text{Cr}^{51}(K)\text{V}^{51}$	Q for $\text{V}^{51}(p, n)\text{Cr}^{51}$ (48)	1.534	$7.15-5b(\rho)$
$\text{Fe}^{53}(\beta^+)\text{Mn}^{53}$	2.5 ± 0.1 *s* (51)	4.3	$7.60-3b(\rho)$
Ca^{48}—Ti^{48}	Mass spec.[b] (52)	2.7	$-13.15+14b(\rho)$
Ti^{50}—Cr^{50}	Mass spec.[b] (52)	−2.8	$-14.05+10b(\rho)$
Cr^{52}—Fe^{52}	Sum of β^+ and γ energies is 4.76 *s* (47) *a* (51)	−8.36	$-14.95+6b(\rho)$

[a] *a*—absorption; *s*—spectrometer; *cc*—cloud chamber.
[b] See reference 25.

By equating the experimental values to the calculated binding energy difference expressions, one obtains points for a curve of b as a function of A. As can be seen from Fig. 3, the points give b a magnitude of about 1.2 Mev, and show a tendency for b to decrease with increasing A. The transitions from $T=1/2$ to $T=3/2$ all have the term $3b$ in the calculated binding energy difference; similarly the $T=0$ to $T=2$ transition contains $6b$. These are the lowest nonvanishing multiples of b that occur in odd-A and even-A transitions respectively, and as will be shown later, the proportionality coefficient would be the same for weak spin-orbit coupling. The points arising from these transitions are the crosses of Fig. 3. The extent to which they scatter about a smooth curve, which is well beyond the average experimental error of about 1 percent, is an indication of the degree of approximation of the model used. The points arising from differences between states of greater T are encircled in Fig. 3. The fact that these points make the scattering no worse is an indication that the proportionality constants of b are given correctly by (11). The significance of these proportionality constants for strong spin-orbit coupling as against weak spin-orbit coupling will be discussed in Part B of this section.

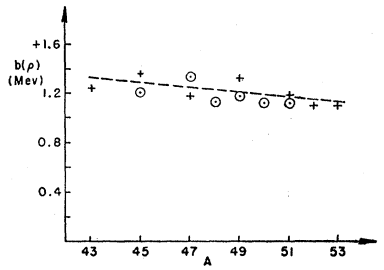

FIG. 3. Plot of $b(\rho)$ as a function of A from the last two columns of Table VI. Broken line is a fitted theoretical curve.

The experimentally determined results for $b(\rho)$ can be used to estimate some nuclear parameters in the theoretical expression. The expressions for $b(\rho)$ in the four central force cases are given in Appendix I; they depend on the parameters in the spatial dependence of the interaction potential A_0, r_0 and on the oscillator parameter r_f in the wave functions. A_0 and r_0 are related by the solution of the deuteron problem with inverse Gaussian potential, while from comparison of oscillator and square-well functions it was found that $r_f\approx3.0\times10^{-13}$ cm. The value of r_0 is expected to lie between 2.0×10^{-13} and 3.0×10^{-13} cm; numerical values of $b(\rho)$ for the four exchange potentials are given in Table VII using these parameters. It is evident that as long as there is a preponderance of potentials involving

space exchange (P_{12}), it is possible to fit the experimental magnitude of $b(\rho)$ with reasonable values of the nuclear parameters. Since all the expressions show a decrease of $b(\rho)$ for increasing r_f and constant r_0, this feature of the experimentally obtained points of Fig. 3 will be present.

For example, the potentials (9) and (10) with $J(r_{12})=A_0\exp[-(r_{12}/r_0)^2]$, $A_0=35.6$ Mev, $r_0=2.25\times10^{-13}$ cm will give curves for $b(\rho)$ as a function of mass number that present a reasonable fit to the points of Fig. 3 for:

$$\begin{aligned} &r_f=3.16(A/53)^{\frac{1}{3}}\times10^{-13}\text{ cm for (9)},\\ &\qquad\text{where } A \text{ is the mass number;}\\ &r_f=2.65(A/53)^{\frac{1}{3}}\times10^{-13}\text{ cm for (10)},\\ &\qquad\text{where } A \text{ is the mass number.}\end{aligned}\tag{14}$$

To observe the effect of a spatial dependence having a singularity at the origin, the delta function, $J(r_{12})=A_\delta\delta(r_{12})$ is treated. In this case, P_{12} is unity, so the two exchange mixtures (9) and (10) coincide. All energies of $1f_{7/2}$ interactions in the delta-function case are proportional to the integral

$$G=2A_\delta\int_0^\infty R_f{}^4(r)r^2dr=0.865A_\delta/r_f{}^3.\tag{15}$$

For the delta-function case the expression $b(\rho)=0.4G$, which is seen to be proportional to r_f^{-3} and hence inversely proportional to the mass number. The best fit to the experimental points is the curve $b=(57.3/A)$ Mev. The three cases evaluated give substantially the same curve for b as a function of A as is drawn in Fig. 3.

The theoretical binding energy difference resulting from use of the curve of Fig. 3 is generally a value that is small compared to the quantities which contribute to it, since the Coulomb term and the potential energy term have about the same magnitude but opposite sign. These values are listed together with the experimental binding energy differences in Table VIII; the agreement is generally quite good, as would be expected from the agreement in Fig. 3.

B. Symmetry Properties of the Wave Functions

The expression $b(\rho)$, which occurs in the binding energy differences of isobars, has coefficients which can be shown to result from the symmetry properties of the wave functions. The following discussion is based on articles by Hund[19] and Wigner.[20]

The total wave function is antisymmetric to exchange of any two nucleons. Since the isotopic spin is assumed to be a good quantum number, the wave function is separable into the product of an isotopic spin function with a space-spin function. Under the assumption of strong spin-orbit coupling, the space dependence and the ordinary spin dependence are not separable. The symmetry properties of the wave function are

[19] F. Hund, Z. Physik **43**, 788 (1927); **105**, 202 (1937).
[20] E. Wigner, Phys. Rev. **51**, 105 (1937); **51**, 937 (1937).

TABLE VII. Numerical values of the expression $b(\rho)$ for the central force potentials. In both cases $r_f=3\times10^{-13}$ cm, $\rho=r_f/r_0$; in case 1, $A_0=42.8$ Mev, $r_0=2.0\times10^{-13}$ cm; in case 2, $A_0=23.1$ Mev, $r_0=3.0\times10^{-13}$ cm.

Exchange potential	b for $\rho=1.5$	b for $\rho=1.0$
1	0.59 Mev	0.41 Mev
P_{12}	1.06 Mev	1.44 Mev
$P_{12}Q_{12}$	1.18 Mev	2.24 Mev
Q_{12}	0.30 Mev	0.20 Mev

designated by the notation:

$\varphi(\{1\ 2\}3)$ is a function antisymmetric to exchange of 1 and 2 and is a function with the symmetry property $A(2+1)$.

$\varphi([1\ 2]3)$ is a function symmetric to the exchange of 1 and 2 and is a function with the symmetry property $S(2+1)$.

The theory of the permutation group says that in order to be able to obtain a completely antisymmetric product wave function, the two components of the product must have adjoint symmetry character. The functions with symmetry properties $A(\lambda_1+\lambda_2)$ and $S(\lambda_1+\lambda_2)$ have adjoint character.

The component wave functions are formed so that they give irreducible representations of the permutation group on n particles, where $\lambda_1+\lambda_2=n$ represents a given partition of n. This means that if one tries to change a wave function of symmetry $A(2+1)$ by permutation and combination to form $A(3)$,

$$\varphi(\{1\ 2\}3)-\varphi(\{3\ 2\}1)-\varphi(\{1\ 3\}2)=\psi(\{1\ 2\ 3\}),$$

one will find that $\psi=0$. The general statement is that if the λ's of a partition are arranged in order of nonincreasing magnitude, then if the wave function belongs to an irreducible representation, one cannot increase any of the λ's by permutation of particles and combination of terms.

The isotopic spin variable is two-valued; hence a many-particle isotopic spin function cannot be made

TABLE VIII. Comparison of theoretical and experimental binding energy differences.

Isobars	Theor. B.E. difference[a] (Mev)	Exp. B.E. difference (Mev)
$Sc^{41}(\beta^+)Ca^{41}$	+ 6.75	+6.74
$Sc^{43}(\beta^+)Ca^{43}$	+ 6.65− 3.90=+2.75	+2.92
$Ti^{45}(\beta^+)Sc^{45}$	+ 6.90− 3.80=+3.10	+2.80
$Ca^{45}(\beta^-)Sc^{45}$	− 6.55+ 6.30=−0.25	−0.525
$V^{47}(\beta^+)Ti^{47}$	+ 7.00− 3.65=+3.35	+3.45
$Sc^{47}(\beta^-)Ti^{47}$	− 6.80+ 6.10=−0.70	−0.17
$Cr^{49}(\beta^+)V^{49}$	+ 7.25− 3.55=+3.70	+3.25
$Sc^{49}(\beta^-)Ti^{49}$	− 6.70+ 8.25=+1.55	+1.6
$Mn^{51}(\beta^+)Cr^{51}$	+ 7.35− 3.40=+3.95	+3.8
$Cr^{51}(K)V^{51}$	+ 7.15− 5.70=+1.45	+1.534
$Fe^{53}(\beta^+)Mn^{53}$	+ 7.60− 3.30=+4.30	+4.3
Ca^{48}—Ti^{48}	−13.15+16.80=+3.65	+2.7
Ti^{50}—Cr^{50}	−14.05+11.60=−2.45	−2.8
Fe^{52}—Cr^{52}	−14.95+ 6.70=−8.25	−8.36

[a] Theoretical values are given to the nearest 0.05 Mev.

antisymmetric in more than two nucleons. This is equivalent to saying that when the isotopic spin function is written in its $S(\lambda_1+\lambda_2+\cdots+\lambda_k)$ form, there can be only two parts, λ_1 and λ_2, to the partition of n. The difference $\lambda_1-\lambda_2$ is twice the isotopic spin quantum number T. Therefore, since the space-spin part of the wave function must have adjoint character if the total wave function is to be antisymmetrized, its symmetry properties are determined by T. This is illustrated in Table IX for the case of seven $1f_{7/2}$ nucleons, that is, mass number 47.

The central interaction potentials involve only the space and spin coordinates, so the symmetry properties of the space-spin functions are the relevant features. With respect to simultaneous permutation of the space and spin coordinates, a given space-spin wave function of n particles ψ_k belongs to an irreducible representation of the permutation group $D(R)$,

$$P_R\psi_k[(X,\sigma)_1,\cdots(X,\sigma)_n]=\sum_{j=1}^{f}D_{jk}(R)\psi_j[(X,\sigma)_1,\cdots(X,\sigma)]. \quad (16)$$

Here P_R is a definite simultaneous permutation of the space (X) and spin (σ) coordinates of the n nucleons; $D(R)$ is the matrix representing this permutation; ψ_j is one of the wave functions representing the group (the ψ_j are an orthonormal set); f is the dimension of the representation (it equals the number of independent wave functions needed to represent the group); $1\leq k$, $j\leq f$. The potential energy of an interaction using this wave function is

$$V_{kk}=\sum_{\alpha<\beta}^{n}\int\psi_k^*V_{\alpha\beta}\psi_k. \quad (17)$$

By using the group properties as defined in (16) in the same way that Wigner[21] proceeds, one obtains

$$V_{kk}=\frac{n(n-1)}{2f}\sum_{j=1}^{f}\int\psi_j^*V_{12}\psi_j. \quad (18)$$

A representation for a particular group can be chosen so that all of its wave functions are either symmetric or antisymmetric to exchange of the particular coordinates 1 and 2. In this way (18) may be split into

$$V_{kk}=\frac{n(n-1)}{2f}\left[\sum_{j=1}^{s}\int\psi_j^*V_{12}\psi_j+\sum_{j=s+1}^{f}\int\psi_j^*V_{12}\psi_j\right], \quad (19)$$

TABLE IX. Symmetry properties of the component wave functions for seven $1f_{7/2}$ nucleons.

Element	T	Isotopic spin component	Space-spin component
Ca^{47}	7/2	$S(7)$	$A(7)$
Sc^{47}	5/2	$S(6+1)$	$A(6+1)$
Ti^{47}	3/2	$S(5+2)$	$A(5+2)$
V^{47}	1/2	$S(4+3)$	$A(4+3)$

[21] E. Wigner, Phys. Rev. **51**, 947 (1937).

TABLE X. Coefficients of b in the potential energy expression for isobars of mass number 47.

Element	Space-spin symmetry	$\frac{n(n-1)}{2}\frac{2s}{f}b$	Difference
Ca^{47}	$A(7)$	0	
			$7b$
Sc^{47}	$A(6+1)$	$7b$	
			$5b$
Ti^{47}	$A(5+2)$	$12b$	
			$3b$
V^{47}	$A(4+3)$	$15b$	

where the first sum contains all the functions symmetric to exchange of the coordinates of 1 and 2, there being s such functions. Therefore V_{kk} contains two terms, the first of which remains unchanged if the interaction V_{12} is replaced by $V_{12}P_{12}Q_{12}$, whereas the second term will change sign under such replacement.

The result of calculating the $1f_{7/2}$ potential energy contributions is given in Sec. IV, where it was found that all the V_{kk} depend on three expressions D, d and b. For configurations of nucleons of like charge, where the operator $P_{12}Q_{12}$ has the eigenvalue -1, only D and d occur; so these both change sign when V_{12} is replaced by $V_{12}P_{12}Q_{12}$. Since b has no such simple symmetry properties, the second sum must be a linear combination of D and d. There is only one linear combination of D, d, and b which does not change sign under replacement of V_{12} by $V_{12}P_{12}Q_{12}$:

$$S=(-D+5d+18b). \quad (20)$$

Therefore, all the integrals in the first sum of (19) are proportional to S. From the fact that $P_{12}Q_{12}\psi_j=\psi_j$ for all wave functions on the first sum, plus the fact that in the limit of delta-function range dependence $\int\psi_j^*V_{12}P_{12}\psi_j=\int\psi_j^*V_{12}\psi_j$ the proportionality constants can be shown to be independent of the type of exchange. In the limit of infinite range interaction and no exchange, the integrals are all alike,

$$\int\psi_j^*V_\infty\psi_j=V_\infty=c_jS,$$

so the proportionality constants are all alike. From the value of S in this limit, one finds $c=1/9$. Thus we can write (19) as

$$V_{kk}=\frac{n(n-1)}{2f}\left[\frac{sS}{9}+\sum_{j=s+1}^{f}\int\psi_j^*V_{12}\psi_j\right]. \quad (19a)$$

Section IV shows that only b occurs in the differences of V_{kk}'s for isobars; so what we seek is the coefficient of b in V_{kk}. Since b occurs only in S, this coefficient is:

$$\frac{n(n-1)}{2f}(2s).$$

The coefficient of b depends on the ratio $2s/f$, which is simply connected to the characters of the repre-

TABLE XI. Coefficients of b' in the potential energy expression under the assumption of separability of space and spin dependence for isobars of mass number 47.

Element	Spatial symmetry	$\frac{n(n-1)}{2}\frac{2sb'}{f}$	Difference
Ca^{47}	$S(2+2+2+1)$	$15b'$	
			$5b'$
Sc^{47}	$S(3+2+2)$	$20b'$	
			$4b'$
Ti^{47}	$S(4+2+1)$	$24b'$	
			$3b'$
V^{47}	$S(4+3)$	$27b'$	

sentation:

$2s-f=X(t)$, the character of a transposition

$f=X(E)$, the dimension of the representation.

The ratio $X(t)/X(E)$ can be obtained[22] from the symmetry properties of the space-spin wave functions, $A(\lambda_1+\lambda_2)$. Murnaghan[23] gives a general formula for the result when the wave functions are in the form $S(\lambda_1+\lambda_2+\cdots+\lambda_k)$:

$$n(n-1)\frac{X(t)}{X(E)}=\sum_{j=1}^{k}\lambda_j(\lambda_j-2j+1). \quad (21)$$

For wave functions in the $A(\lambda_1+\lambda_2+\cdots+\lambda_k)$ form, the sign of the left-hand side of (21) is changed, and with only two λ's the term of V_{kk} which involves b becomes

$$\tfrac{1}{2}n(n-1)(2s/f)b = \tfrac{1}{2}[n(n-1)-\lambda_1(\lambda_1-1)-\lambda_2(\lambda_2-3)]b. \quad (22)$$

Substitution of $\lambda_1+\lambda_2=n$ and $\lambda_1-\lambda_2=2T$ in (22) shows that this term agrees with the term containing b in (11); however, the form (22) shows its origin more clearly. The resulting isobar differences for mass number 47 are listed in Table X.

If instead of assuming strong spin-orbit coupling, one assumes the opposite extreme of negligible spin-orbit coupling, with space-dependent interactions dominant, one has the first approximation[20] in Wigner's supermultiplet theory. Here the wave function is separable into the product of space dependence, spin dependence, and isotopic spin dependence. The symmetry property of the space-dependent component of the wave function is the determining feature in this approximation. The ground states have been classified,[24] and are ^2F for odd-A nuclei, and $^1S_0$ for even-even nuclei. The potential energy differences will again depend on the coefficient $\frac{1}{2}n(n-1)(2s/f)$, where $(2s-f)/f=X(t)/X(E)$. The space wave function can have at most four λ's in the partition and is written in the form $S(\lambda_1+\lambda_2+\lambda_3+\lambda_4)$. From Eq. (21) one obtains

$$\tfrac{1}{2}n(n-1)(2s/f)=\tfrac{1}{2}[n(n-1)+\lambda_1(\lambda_1-1)+\lambda_2(\lambda_2-3)+\lambda_3(\lambda_3-5)+\lambda_4(\lambda_4-7)]. \quad (23)$$

For isobar differences, the coefficient (23) times a function $b'(\rho)$ will be all that is involved[21] in the potential energy differences. The case of mass number 47 is listed in Table XI.

In a Hartree model, the potential energy difference is the only difference in the binding energy of isobars aside from the Coulomb difference. The isobar differences for odd-A and even-even nuclei as given by the two models are summarized in Table XII.

The strong spin-orbit coupling model was compared with experiment in Fig. 3. If the coefficients predicted by the weak spin-orbit coupling model were used instead to plot b' as a function of A, the points for $T=3/2$ and $T=2$ would be unchanged since the coefficients are alike in the two models. However, for greater T's the weak coupling coefficients are smaller; so the encircled points of Fig. 3 would all be raised to values between 1.4 and 1.68 Mev. The points would then be badly scattered, and no smooth dependence of b' on A would give a good fit to the data.

The dependence on the coefficients can be tested without involving b and b' for mass numbers where more than one nonmirror decay is present. By taking ratios of non-Coulomb part of the binding energy differences, the b (or b') cancels. The comparison of the models with experiment is given for the four experimentally measured cases in Table XIII.

The experimental values provide a measure of the degree to which space and spin dependence are separable in the wave function for a Hartree model. The fact that the experimental values lie closer to the values obtained from the approximate functions of the jj model provides a good argument for strong spin-orbit coupling as an approximation preferable to that of weak spin-orbit coupling.

VI. TOTAL BINDING ENERGY OF STABLE NUCLEI

Mass spectroscopic results for nuclei as large as those of the $1f_{7/2}$ shell are generally not so reliable as the results for light nuclei. When comparing the results of various laboratories, the measurements of differences between nuclei are often in much better agreement than the absolute masses. For this reason, it is very fortunate that all the masses in this shell have

TABLE XII. Binding energy differences for isobars, aside from the Coulomb difference, as given by the strong spin-orbit coupling and weak spin-orbit coupling models.

Change of isotopic spin	Strong coupling	Weak coupling
T to $T-1$	$2Tb$	$(T+\frac{3}{2})b'$
T to $T-2$	$(4T-2)b$	$(2T+2)b'$

[22] See reference 3, footnote on p. 98.

[23] F. D. Murnaghan, *The Theory of Group Representation* (Johns Hopkins Press, Baltimore, 1938), p. 140, Eq. (5.6).

[24] B. H. Flowers, Proc. Roy. Soc. (London) **A210**, 497 (1951).

TABLE XIII. Ratios of binding energy differences in the $1f_{7/2}$ shell (Coulomb contribution removed).

Ratio	Exp. value	Strong coupling	Weak coupling
$[Ti^{45}(\beta^+)Sc^{45}/Ca^{45}(\beta^-)Sc^{45}$	0.68	0.60	0.75
$[V^{47}(\beta^+)Ti^{47}/Sc^{47}(\beta^-)Ti^{47}$	0.54	0.60	0.75
$[Cr^{49}(\beta^+)V^{49}/Sc^{49}(\beta^-)Ti^{49}$	0.48	0.43	0.60
$[Mn^{51}(\beta^+)Cr^{51}/Cr^{51}(K)V^{51}$	0.63	0.60	0.75

been measured recently at one laboratory.[25] The calculations had been completed before these results were available, and it was gratifying to discover that agreement with the calculation, particularly in regard to the even-Z isobar differences at mass number 48 and 50, is better for the new data than for that which was previously available.

The experimental binding energies are obtained from the mass spectroscopic data with the following conversion formula:

$$\text{B.E.(Mev)}=0.931[1.008146A+0.000840(A-Z) -\text{mass of neutral atom}], \quad (24)$$

where B.E. is binding energy, A is the mass number, and Z is the charge. The values are listed in Table XIV, and the points are plotted in Fig. 4. The only deviation from a smooth function of mass number A for the binding energy curve is the variation for even and odd isotopes found in the isotopes of Ca, Ti, and Cr. The magnitude of this variation is found to be about 1.7 Mev, in good agreement with the term from the semi-empirical Bohr-Wheeler mass formula, which gives this variation as $33.5A^{-\frac{3}{4}}$ Mev, or 1.8 Mev at $A=49$.

In calculating the total binding energy with Hartree wave functions constructed according to Sec. II, one can separate the contributions as follows:

1. Coulomb interaction of $1f_{7/2}$ protons with themselves and with the other closed shells of protons.
2. Potential energy of interaction of $1f_{7/2}$ nucleons with themselves, exclusive of the Coulomb interaction.
3. Contributions directly proportional to the number of $1f_{7/2}$ nucleons. These include spin-orbit-coupling energy, kinetic energy, and potential energy of interaction with the core nucleons.
4. The total binding energy of the $_{20}Ca^{40}$ core.

TABLE XIV. Experimental binding energy in Mev.

Nucleus	Binding energy	Nucleus	Binding energy
Ca^{40}	341.9	Ti^{48}	418.7
Ca^{41}	350.2[a]	Ti^{49}	426.6
Ca^{42}	361.7	Ti^{50}	437.6
Ca^{43}	369.7	Cr^{50}	434.8
Ca^{44}	381.1	V^{51}	445.4
Sc^{45}	387.9	Cr^{52}	456.2
Ti^{46}	398.4	Mn^{53}	462.6[b]
Ti^{47}	407.0	Fe^{54}	471.5
Ca^{48}	415.9	Co^{55}	476.5[c]

[a] From $Ca^{41}-Ca^{40}$ mass difference. V. L. Sailor, Phys. Rev. **75**, 1836 (1949).
[b] Related to $Cr^{53}(p,n)Mn^{53}$ threshold. Lovington, McCue, and Preston, Phys. Rev. **85**, 585 (1952).
[c] Related to Mn^{55} by radioactive decay data. All other values taken from reference 25.

[25] Collins, Nier, and Johnson, Phys. Rev. **84**, 717 (1951); **86**, 408 (1952).

The Coulomb contribution is that obtained in Sec. V,

$$\text{Coulomb term (Mev)}=[Z-20][23.28A^{-\frac{1}{3}}]+0.35k, \quad (25)$$

where Z is charge, A mass number, and k is the coefficient of c in Table V. The contribution from nuclear forces among $1f_{7/2}$ nucleons is given by (11). This is based on the assumption that odd-A and even-even nuclei have ground states of $I=7/2$ and $I=0$, respectively.

The remaining binding energy contributions, 3 and 4, while contributing the bulk of the absolute value, are expected to give a smoothly varying function of the mass number A. Calculations of absolute values with product wave functions, done chiefly in the p shell, have been quite unsuccessful, but they have been able to account for the structure of the binding energy curve that is superimposed on a smoothly varying function of A. The procedure of this investigation is to see whether the first two terms can account for the fluctuations of the experimental curve and then determine the uniform function of A needed to give the absolute values of the experimental curve.

The even-odd variation in the binding energy for a given Z is the only irregular dependence on A that is experimentally evident. Among the terms of the calculation for a fixed Z, only the nuclear $1f_{7/2}$ contribution (11) exhibits such behavior. We can rewrite (11) in terms of Z by using the fact that it was assumed that $T=T_Z$ for ground states and that $2T_Z=A-2Z$. Since $A=n+40$, where n is the number of $1f_{7/2}$ nucleons, (11) becomes

$$V_n=[(Z-20)(n+21-Z)]b + \left[\frac{n-I/(7/2)}{2}\right]D+\left[\frac{n(n-2)+I/(7/2)}{4}\right]d. \quad (11a)$$

The even-odd variation is given by the second difference of Eq. (11a) with Z constant. The small changes in b, D, and d arise from changing the mass number by one are neglected, and in the second difference only the terms involving I or those quadratic in n can contribute.

$$[V_{n+1}-V_n]-[V_n-V_{n-1}] = \frac{d}{2}+\left[\frac{I_{n+1}+I_{n-1}-2I_n}{7/2}\right]\left[\frac{d-2D}{4}\right] = \Delta^2V_n=\left\{\begin{matrix}+D \text{ for odd } n \\ -D+d \text{ for even } n\end{matrix}\right\}. \quad (26)$$

The experimental points within the shell give a nu-

merical value for this second difference:

$$\Delta^2V_n=\left\{\begin{matrix}+3.3 \text{ Mev for odd } n\\ -3.3 \text{ Mev for even } n\end{matrix}\ (\text{error } \pm 0.3)\right\}. \quad (27)$$

This indicates that in order to obtain agreement with experiment, one must have $|d|\ll|D|$, and $D\approx 3.3$ Mev.

So far no particular central force interaction has been used. Before investigating particular potentials, some further properties of d and D will be used to show what properties a potential must have to provide agreement with experiment.

The expressions for D and d are given in Appendix I for the four types of exchange with central forces. These expressions occur in calculating interactions for configurations of identical nucleons; wave functions of such configurations are antisymmetric to simultaneous exchange of space and spin. Therefore, D and d have the property that if the interaction $P_{12}Q_{12}V_{12}$ is used instead of V_{12}, the result is to merely change the sign of D and d. If one uses the coefficients W, M, H, and B to represent the amount of exchange of type 1, P_{12}, $P_{12}Q_{12}$, and Q_{12}, respectively, the general expression for d or D is a linear combination of the expressions listed in the Appendix with these coefficients. Using the further result that from the deuteron problem one expects $W+M=0.8$ and $B+H=0.2$, the general expressions for D and d become

$$\begin{aligned} d&=0.6d_W+(M-B)(d_M-d_W), \quad (28)\\ D&=0.6D_W+(M-B)(D_M-D_W). \end{aligned}$$

Furthermore, the Appendix shows that $d_M=(-1/2)d_W$, so that

$$d=[0.6-1.5(M-B)]d_W. \quad (28a)$$

With the nuclear parameters that were determined from the isobar differences in Sec. V, the magnitudes of D_W, D_M, and d_W are the following:

Finite range	Delta-function range
$D_W\approx 3.8$ Mev	$D_W\approx 6.0$ Mev
$D_M\approx 1.3$ Mev	$D_M\approx 6.0$ Mev
$d_W\approx 2.6$ Mev	$d_W=0$

From these magnitudes it is clear that in order to satisfy the requirement $|d|\ll|D|$, the coefficient of d_W must be small in the finite range case. This would give $M-B\approx 0.4$ and consequently $D\approx 1.3$ Mev, too small a value for the other requirement, $D\approx 3.3$ Mev, resulting from comparison with experiment. On the other hand, the case of delta-function range gives $d=0$ and $D\approx 3.6$ Mev and is independent of $M-B$. These values will give an even-odd variation term quite similar to observation.

The values resulting from the particular finite range potentials (8), (9), and (10), are the following:

(8) ($M=0.8$, $B=0$) gives $D=0.3$ Mev, $d=-1.5$ Mev;

(9) ($M=0.8$, $B=0.2$) gives $D=0.8$ Mev, $d=-0.8$ Mev;

(10) ($M=0.4$, $B=0.1$) gives $D=1.5$ Mev, $d=+0.4$ Mev.

From these sample cases the indication is that in order to have simultaneous quantitative agreement with isobar differences and with the even-odd variation for isotopes in the binding energy curve, the delta-function limit is the best approximation. This means that the use of a range dependence like the Yukawa potential would improve the agreement of the finite range cases considerably since its results are expected to lie closer

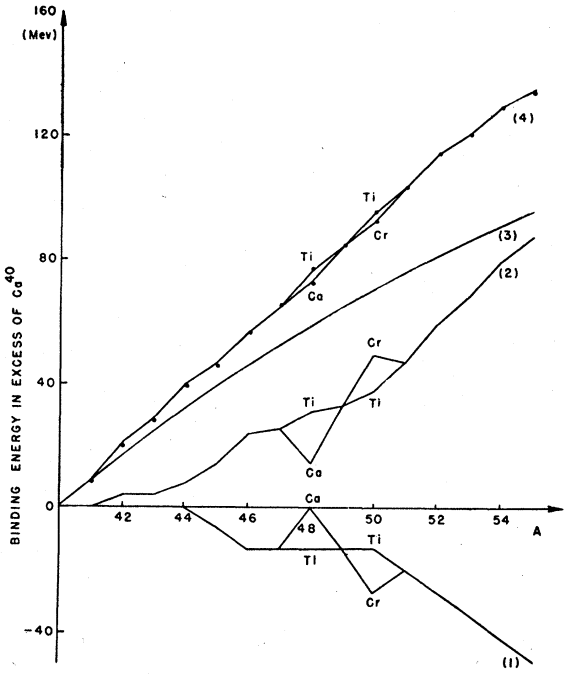

FIG. 4. Binding energy of stable nuclei in the $(1f_{7/2})$ shell: (1) Coulomb contribution; (2) $(1f_{7/2})$ interactions from Eq. (11); (3) $(A-40)$ times the smooth function $W(A)=353A^{-1}$ Mev; (4) Sum of these contributions. The circles are the experimental values.

to those of the delta-function dependence than the inverted Gaussian range dependence.

The even-odd variation discussed above is the only irregular contribution to the binding energy for a given Z. In general, for the nuclei in the shell the Coulomb contribution and the contribution due to the $1f_{7/2}$ interactions cancel each other's irregularities. The contribution 3 directly proportional to the number of $1f_{7/2}$ nucleons can, therefore, be represented by a smooth function of A, namely $(A-40)W(A)$, where $W(A)$ is obtained by fitting the experimental points. The separate contributions and their sum which represents the binding energy in excess of that of Ca^{40} are presented in Fig. 4, for the delta-function case where $W(A)=8.6(41/A)$ Mev gives a good fit. This dependence on

A is reasonable since one expects the terms which it represents to depend on A to some negative power around unity. For two-thirds of the points agreement is within 0.4 Mev and all are within 1.0 Mev.

VII. *FT* VALUES AND MAGNETIC MOMENTS

A. *ft* Values

The allowed *ft* values in the shell can be calculated for a Gamow-Teller interaction by evaluating the nuclear matrix elements of the operator $\sum_i \sigma^i \tau_x^i$. Here τ_x is the isotopic spin operator which changes the charge variable of a nucleon, and σ is the Pauli spin vector. The transitions are assumed to be between states of $I=7/2$, and the matrix element is averaged over initial states and summed over the final states with different magnetic quantum numbers. The square of the matrix element is listed in the second column of Table XV; the decays are listed by pairing each decay with that of its hole image since their theoretical matrix elements are the same.

The decay $Sc^{41}(\beta^+)Ca^{41}$ is a mirror decay, namely from $T_z=-1/2$ to $T_z=+1/2$. The square of its GT matrix element is seen to be from two to ten times as great as those of the nonmirror decays. Experimentally, mirror decays have very low *ft* values and are called super-allowed; if only the GT matrix element were involved, the ratio of squared matrix elements should be about 30:1 to account for the difference between observed *ft* values in allowed and super-allowed transitions. However, the mirror transition can have Fermi interaction as well as the Gamow-Teller interaction contributing to its matrix element, which is not true for the allowed transitions. The matrix element for the Fermi interaction $\sum_i \tau_x^i$ is unity for the mirror transition and zero for all the rest. Therefore, the lack of a difference between calculated allowed and super-allowed transitions comparable to that observed experimentally is the result of the omission of the Fermi interaction and possibly also to lack of cancellation in the matrix element by the many-particle wave functions.

If the calculated GT matrix elements are assumed to be correct, the relative strength of Fermi interaction to GT interaction must be about 20:1 to account for the low *ft* of the super-allowed transition. This is not compatible with recent interpretations[26] which indicate an upper limit of about 1:1 for this ratio.

Therefore, the discrepancy must lie in the calculated GT matrix elements. That this is likely is indicated by the results of Appendix III, where the secular determinant is solved with delta-function interaction in the case of Sc^{43}. While the energy of the resulting ground state differs little from that obtained with the approximate wave function constructed according to Sec. IV, $|M|_{GT}^2$ for the $Sc^{43}(\beta^+)Ca^{43}$ transition drops by 60 percent. The value of the matrix element is thus quite sensitive to the wave function and hence to the type of interaction used when solving the secular determinants; so calculation throughout the shell is not feasible. The fact that the magnitude drops when wave functions resulting from solution of secular determinants are used is expected since these should not have so much similarity as wave functions constructed according to Sec. IV.

A reduction of the calculated $|M|_{GT}^2$ by a factor of about ten is needed to give the magnitude of the observed log *ft*'s for allowed transitions. If a constant reduction factor is assumed and incorporated by fitting the calculated log *ft* for $Sc^{43}(\beta^+)Ca^{43}$, the resulting relationship is

$$\log|M|_{GT}^2 ft = 4.66. \tag{29}$$

Values obtained by using (29) on column 2 of Table XV give column 3. There is still little semblance of any correlation with column 4, so the wave functions of Sec. IV are inadequate for calculating *ft* values. The fact that these wave functions are nevertheless suitable for energy calculations is not surprising since such calculations are an order of magnitude less sensitive to changes in the wave function.

TABLE XV. Comparison of calculated and experimental *ft* values.

Decay	M_{GT}^2	$(\log ft)_{theor}$	$(\log ft)_{exp}$
$Sc^{41}(\beta^+)Ca^{41}$	1.28		3.45
$Sc^{43}(\beta^+)Ca^{43}$	0.57	4.9[a]	4.9
$Fe^{53}(\beta^+)Mn^{53}$	0.57	4.9	4.9
$Ca^{45}(\beta^-)Sc^{45}$	0.46	5.0	5.9
$Cr^{51}(K)V^{51}$	0.46	5.0	5.2
$V^{47}(\beta^+)Ti^{47}$	0.30	5.2	≥ 4.6[b]
$Cr^{49}(\beta^+)V^{49}$	0.30	5.2	≥ 4.5[b]
$Sc^{49}(\beta^-)Ti^{49}$	0.25	5.3	6.0
$Sc^{47}(\beta^-)Ti^{47}$	0.18	5.4	5.6
$Ti^{45}(\beta^+)Sc^{45}$	0.11	5.6	≥ 4.4[b]
$Mn^{51}(\beta^+)Cr^{51}$	0.11	5.6	≥ 5.1[b]

[a] Fitted to experimental value of $Sc^{43}(\beta^+)Ca^{43}$.
[b] Lower limit for log *ft* since branching ratio for *K* capture is unknown.

B. Magnetic Moments

The magnetic moments obtained for the stable nuclei with the many-particle wave functions are listed in the second column of Table XVI. Comparison with the experimental determinations given in the third column shows that the theoretical values are considerably higher. Still, the experimentally observed tendency for the moment to be reduced when going from a nucleus like V^{51}, which has only protons outside closed shells, to Sc^{45}, which has both neutrons and protons outside shells, is given by the model, as has been pointed out by Flowers.[27] As in the case of the beta-decay matrix elements, the magnetic moment is considerably more sensitive to the nuclear wave function than is the energy, so that nuclei having approximate wave functions are affected as is shown in Appendix III.

However, there is no ambiguity in the construction of the wave function for V^{51} whose moment is predicted

[26] G. L. Trigg, Phys. Rev. **86**, 506 (1952); J. M. Blatt, Phys. Rev. **89**, 83 (1953).
[27] B. H. Flowers, Phil. Mag. **43**, 1330 (1952).

to lie on the Schmidt line. There have been proposals[28] that the spin contribution to the magnetic moment is suppressed from its free particle value when a nucleon is in the field of other nucleons. If one follows this suggestion and uses the V^{51} experimental moment to estimate the amount by which the anomalous spin moment of both neutron and proton is suppressed, the results are $\mu_p=2.15\mu_0$ and $\mu_n=-1.27\mu_0$. Magnetic moments calculated with these values are listed in the last column of Table XVI.

TABLE XVI. Magnetic moments for odd-A nuclei in the $1f_{7/2}$ shell.

Nucleus	Calculated[a]	Experimental	Calculated[b]
Ca^{43}	−1.91	···	−1.27
Sc^{45}	+5.11	+4.75[c]	+4.58
Ti^{47}	−1.57		−0.99
		−1.10[d]	
Ti^{49}	−1.67		−1.01
V^{51}	+5.79	+5.15[e]	+5.15

[a] Calculated with $\mu_p=+2.79$, $\mu_n=-1.91$.
[b] Calculated with suppressed moments to fit V^{51}; $\mu_p=+2.15$, $\mu_n=-1.27$.
[c] W. G. Proctor and F. C. Yu, Phys. Rev. **78**, 471 (1950).
[d] Jeffries, Loliger, and Staub, Helv. Phys. Acta **24**, 643 (1951) (isotope uncertain).
[e] W. D. Knight and V. W. Cohen, Phys. Rev. **76**, 1421 (1959).

VIII. CONCLUSIONS

In summary, the following conclusions can be drawn from comparison of the jj-coupling model with experiment in the $1d_{3/2}$ and $1f_{7/2}$ regions.

For identical nucleons, even configurations give the observed order of $I=0, 2$. Odd configurations give the I of the single odd nucleon, although the $(1f_{7/2})^3$ configuration has levels of smaller I that are calculated to lie close to the ground state in energy.

For odd-odd nuclei, the coupling of a single particle with a single hole is given in agreement with experiment. For coupling of two nucleons or two holes, calculation does not always agree with experiment.

For other $1f_{7/2}$ shell nuclei, the wave functions constructed according to the assumptions of Sec. IV seem suitable for obtaining binding energy differences. They can account for the kinks in the binding energy curve of the stable odd-A and even-even nuclei if the range dependence of interaction is closer to the delta function than the Gaussian. They also account for the binding energy differences of isobars within the Hartree model, which the wave functions of the weak spin-orbit coupling model do not.[29] The $1f_{7/2}$ shell is peculiarly suited for such a comparison, since it is the only section of the periodic table where isobars of sufficiently great differences in neutron excess have both neutrons and protons within the same shell. This makes the binding energy difference of neighboring isobars a single, simple quantity, directly related to the symmetry properties of the wave functions.

Although the approximate wave functions are good for energy calculations and give reasonable values for the few measured magnetic moments, they are not adequate for calculation of allowed ft values. The results indicate that while the wave functions formed according to Sec. IV are good first approximations, they are not sufficiently exact for the sensitive test of ft values. Furthermore, if the configuration $(1f_{7/2})_{5/2}^{-3}$, which is the ground state of $_{25}Mn^{55}$, is involved in any of these decays, it would seriously affect the ft calculations. On the other hand, the energy differences would not be seriously affected since the $(1f_{7/2})_{7/2}^{-3}$ configuration which was assumed to be the ground state in the calculations should lie very close to the $(1f_{7/2})_{5/2}^{-3}$ in energy.[29a]

The author wishes to express his gratitude to Professor M. G. Mayer for suggesting the investigation and providing liberal amounts of discussion and guidance during the course of the work.

APPENDIX I. EVALUATION OF INTERACTION INTEGRALS

The potential energy of interaction caused by nuclear forces is obtained by calculating the matrix element for two-body interactions using the many-particle spin-orbit coupled wave functions. This is a sum of integrals involving the individual particle functions with coefficients determined by the wave function of the particular nucleus in question. The individual particle integrals can be integrated over ordinary spin, and are then linear combinations of the fundamental integrals

$$(ab|V|cd)=\int\int u_a^*(1)u_b^*(2)V_{12}u_c(1)u_d(2)d\tau_1 d\tau_2, \quad \text{(A1)}$$

where $u=R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\varphi)$, the subscripts on the u's referring to the set of quantum numbers n, l, and m. The functions are normalized so that

$$\int_0^\infty R^2r^2dr=1; \quad \int_0^\pi \Theta^2\sin\theta d\theta=1; \quad \int_0^{2\pi}\Phi^2d\varphi=1.$$

The fundamental integrals all involve the spatial dependence of the interaction between pairs of nucleons. This is taken to be the negative Gaussian:

$$J(r_{12})=A_0\exp[-(r_{12}/r_0)^2]$$
$$=A_0\exp[-(r_1^2+r_2^2-2\mu r_1r_2)/r_0^2]. \quad \text{(A2)}$$

[28] F. Bloch, Phys. Rev. **83**, 839 (1951); H. Miyazawa, Progr. Theoret. Phys. (Japan) **6**, 263 (1951); A. de-Shalit, Helv. Phys. Acta **24**, 296 (1951).

[29] E. Wigner (reference 21) accounts for binding energy differences by using a model with potential energy from the Hartree model with negligible spin-orbit coupling and kinetic energy calculated with the statistical model. For the nuclei considered here, the use of two models is not necessary if jj coupling is assumed.

[29a] *Note added in proof:*—Recent experimental data indicate that odd-A nuclei having the configuration $(1f_{7/2})^{-3}$ have a ground state of 5/2 with the state 7/2 a few hundred kilovolts higher, whereas this situation is reversed for the configuration $(1f_{7/2})^3$. The new data also provide closer agreement with the binding energy differences for 7/2 states calculated in Sec. IVA). See L. S. Cheng and M. L. Pool, Phys. Rev. **90**, 886 (1953); B. Crasemann and H. T. Easterday, Phys. Rev. **90**, 1124 (1953); C. D. Jeffries, Phys. Rev. **90**, 1130 (1953) and Bull. Am. Phys. Soc. **5**, 24 (1953).

TABLE XVII. Direct and exchange interaction integrals for $1f_{7/2}$ nucleons. The general expression is $[\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3]K$, where $x=\rho^4/4(1+\rho^2)$, $K=(A_0/735)(1+\rho^2)^{-9/2}$ and $\rho=r_f/r_0$.

	Wigner interaction				Majorana interaction			
	α_0	α_1	α_2	α_3	α_0	α_1	α_2	α_3
$77D$	735	8820	22 050	14 700	735	8820	22 050	14 700
$75D$	735	6930	14 490	8400	105	3150	10 710	8400
E	0	1890	7560	6300	630	5670	11 340	6300
$73D$	735	5670	11 340	6720	0	1260	6930	6720
E	0	630	4410	4620	525	3780	7560	4620
$71D$	735	5040	10 332	6216	0	378	4914	6216
E	0	0	2268	3444	420	2772	5796	3444
$7\bar{1}D$	735	5040	10 332	6216	0	0	3402	6216
E	0	0	1134	2772	315	2268	4536	2772
$7\bar{3}D$	735	5670	11 340	6720	0	0	1890	6720
E	0	0	0	2100	210	1890	3780	2100
$7\bar{5}D$	735	6930	14 490	8400	0	0	0	8400
E	0	0	0	2100	105	1260	3150	2100
$7\bar{7}D$	735	8820	22 050	14 700	0	0	0	14 700
E	0	0	0	0	0	0	0	0
$55D$	735	6660	16 650	11 820	555	5580	15 570	11 820
$53D$	735	6480	14 580	8520	180	2610	9090	8520
E	0	1440	5220	4980	480	4860	10 260	4980
$51D$	735	6390	12 494	7296	0	1404	5778	7296
E	0	1350	4266	4428	405	3780	6696	4428
$5\bar{1}D$	735	6390	12 492	7296	0	864	3618	7296
E	0	0	1296	2868	330	2610	5796	2868
$5\bar{3}D$	735	6480	14 580	8520	0	0	1890	8520
E	0	0	2430	3540	255	1620	4320	3540
$5\bar{5}D$	735	6660	16 650	11 820	0	0	3240	11 820
E	0	0	0	0	180	1080	1080	0
$33D$	735	7020	15 930	11 340	435	4860	12 690	11 340
$31D$	735	7290	16 362	9456	225	2790	7542	9456
E	0	450	2664	3612	390	4158	10 476	3612
$3\bar{1}D$	735	7290	16 362	9456	0	1008	4464	9456
E	0	1800	6768	5844	345	3132	5886	5844
$3\bar{3}D$	735	7020	15 930	11 340	0	1080	5400	11 340
E	0	0	0	0	300	2160	3240	0
$11D$	735	7740	19 026	13 068	375	4068	10 818	13 068
$1\bar{1}D$	735	7740	19 026	13 068	240	3096	9360	13 068
E	0	0	0	0	360	3672	8208	0

Following the method applied to the case of Coulomb interaction in atomic physics,[30] the interaction is expanded in a series of Legendre polynomials of argument $\mu=\cos\theta$:

$$J(r_{12})=A_0\exp[-(r_1^2+r_2^2)/r_0^2]\sum_k a_k(r_1,r_2)P_k(\mu). \quad \text{(A3)}$$

The fundamental integrals are then linear combinations of Slater integrals with coefficients obtained from the angular part of the integration. These coefficients are the same as for the atomic case and are tabulated in reference 30. For the $1f$ shell, the Slater integrals are the following:

$$F^{(k)}=\int_0^\infty\int_0^\infty R_{1f}^2(r_1)R_{1f}^2(r_2)\exp[-(r_1^2+r_2^2)/r_0^2]\times a_k(r_1,r_2)r_1^2r_2^2dr_1dr_2.$$

The values of k are 0, 2, 4, and 6 with the following a_k:

$$\begin{aligned}
a_0&=(1/y)\sinh y,\\
a_2&=(5/y^3)[(3+y^2)\sinh y-3y\cosh y],\\
a_4&=(9/y^5)[(105+45y^2+y^4)\sinh y-(105y+10y^3)\cosh y],\\
a_6&=(13/y^7)[(10395+4725y^2+210y^4+y^6)\sinh y-(10395y+1260y^3+21y^5)\cosh y],
\end{aligned} \quad \text{(A4)}$$

where $y=2r_1r_2/r_0^2$.

The radial functions for the $1f$ shell are the harmonic oscillator functions

$$R_{1f}=N_{1f}r^3\exp[-(r/r_f)^2]. \quad \text{(A5)}$$

The Slater integrals are evaluated with these functions, giving the expressions,

$$\begin{aligned}
F^{(0)}&=[105+9(105)x+9(11)(21)x^2+9(11)(13)x^3]7K,\\
F^{(2)}&=[0+9(63)x+9(11)(18)x^2+9(11)(13)x^3]35K,\\
F^{(4)}&=[0+0+9(11)(11)x^2+9(11)(13)x^3]63K,\\
F^{(6)}&=[0+0+0+9(11)(13)x^3]91K,
\end{aligned} \quad \text{(A6)}$$

where $x=\rho^4/4(1+\rho^2)$, $K=(A_0/735)(1+\rho^2)^{-9/2}$, $\rho=r_f/r_0$.

The integrals desired involve the spin-orbit coupled individual particle functions, and since the integrals are linear combinations of the Slater integrals, they can be expressed as polynomials in x. The formulation in terms of x is convenient since the limiting cases of zero range and infinite range of interaction are immediately evident as x goes respectively to infinity or zero. The integrals for nucleons in the $1f_{7/2}$ shell are given in the form,

$$\begin{aligned}
&\int[7(1)\bar{5}(2)]^*V_{12}[7(1)\bar{5}(2)]d\tau_1d\tau_2=7\bar{5}\text{ direct}=7\bar{5}D.\\
&\int[7(1)\bar{5}(2)]^*V_{12}[\bar{5}(1)7(2)]d\tau_1d\tau_2=7\bar{5}\text{ exchange}=7\bar{5}E.
\end{aligned} \quad \text{(A7)}$$

Here V_{12} is the two-particle interaction potential, and $7(1)\bar{5}(2)$ means that particle 1 is in the state with z component of angular momentum $m=7/2$ and that particle 2 is in the state $m=-5/2$. The integrals are given for the interactions $V_{12}=J(r_{12})$, the Wigner

[30] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1951), Chap. VI.

interaction, and for $V_{12}=J(r_{12})P_{12}$, the Majorana or space exchange interaction. The integrals for Heisenberg interaction can be obtained from the Wigner integrals since the Heisenberg operator $P_{12}Q_{12}$ merely exchanges particles 1 and 2 in the space-spin function. Thus the direct integral with Heisenberg interaction is the exchange integral of Wigner interaction and vice versa. The same relationship holds between Bartlett (Q_{12}) and Majorana integrals.

The resulting integrals which allow calculation with any type of static central-force interaction are given in Table XVII. There the coefficients of the powers of x in the polynomial $(\alpha_0+\alpha_1x+\alpha_2x^2+\alpha_3x^3)K$ are listed.

The quantities arising in the expression for potential energy within the $1f_{7/2}$ shell (11) can naturally also be expressed as polynomials in x. For the four interactions these are:

Wigner Interaction

$$\begin{aligned}D_W&=(105+1620x+5346x^2+5148x^3)7K,\\ d_W&=(210+1440x+1980x^2+0)7K,\\ b_W&=(0+200x+968x^2+1144x^3)7K.\end{aligned}$$

Majorana Interaction

$$\begin{aligned}D_M&=(+15+324x+1782x^2+5148x^3)7K,\\ d_M&=(-105-720x-990x^2+0)7K,\\ b_M&=(+60+536x+1232x^2+1144x^3)7K.\end{aligned}\tag{A8}$$

Heisenberg Interaction

$$\begin{aligned}&D_H=-D_W,\ d_H=-d_W,\\ &b_H=(+105+820x+1474x^2+572x^3)7K.\end{aligned}$$

Bartlett Interaction

$$\begin{aligned}&D_B=-D_M,\ d_B=-d_M,\\ &b_B=(0+100x+484x^2+572x^3)7K.\end{aligned}$$

APPENDIX II. ENERGY EXPRESSIONS FOR ODD-ODD NUCLEI

The Slater integrals for configurations of $1d_{3/2}$ nucleons and for mixed $1d_{3/2}$ and $1f_{7/2}$ nucleons involve the parameter r_d from the radial $1d$ wave function

$$R_{1d}=N_{1d}r^2\exp[-(r/r_d)^2]\tag{A9}$$

as well as the parameters r_f, A_0, and r_0 defined in the evaluations of Appendix I.

For $1d_{3/2}$ configurations the Slater integrals are:

$$\begin{aligned}F^{(0)}&=[15+70x+63x^2]K_d,\\ F^{(2)}&=[0+49x+63x^2]5K_d,\\ F^{(4)}&=[0+0+63x^2]9K_d,\end{aligned}\tag{A10}$$

where

$$x=\rho_d^4/4(1+\rho_d^2);\quad \rho_d=r_d/r_0;\quad K_d=(A_0/15)(1+\rho_d^2)^{-7/2}.$$

For mixed $1f_{7/2}$ and $1d_{3/2}$ configurations the integrals are:

$$\begin{aligned}F^{(0)}&=[5+30w+33w^2]Q,\\ F^{(2)}&=[0+21w+33w^2]5Q,\\ F^{(4)}&=[0+0+33w^2]9Q,\end{aligned}\tag{A11}$$

where

$$w=\frac{1}{4}\left(\frac{2\rho_f^2\rho_d^2}{2+\rho_f^2+\rho_d^2}\right);\quad Q=(256A_0/5)(2+\rho_d^2)(w/\rho_f^2\rho_d^2)^{9/2}.$$

$$\begin{aligned}F^{(1)}&=[35/3+42y+33y^2]3C,\\ F^{(3)}&=[0+27y+33y^2]7C,\\ F^{(5)}&=[0+0+33y^2]11C,\end{aligned}\tag{A12}$$

where

$$y=\frac{\rho_f^4\rho_d^4}{(\rho_f^2+\rho_d^2)(\rho_f^2+\rho_d^2+2\rho_f^2\rho_d^2)};$$

$$C=(256A_0/5)\rho_d^2(y/\rho_f^2\rho_d^2)^{9/2}.$$

In the limit $\rho_f=\rho_d=\rho$, one obtains the simplification $y=w=x$. The level crossings given in the text are obtained for this particular case.

The expressions for the level order of the various odd-odd nuclei are given in Tables XVIII, XIX, and XX. In this polynomial form it is easy to see the order in the limit of delta-function interaction since in this limit $y=w=x$ and $x\gg1$. The infinite range interaction case is also obvious since here $x, y, w\ll1$.

TABLE XVIII. Energy level order for $1d_{3/2}$ configurations. The expressions are in the form $[\alpha_0+\alpha_1x+\alpha_2x^2](K_d/25)$, where $x=\rho_d^4/4(1+\rho_d^2)$, $\rho_d=r_d/r_0$, and $K_d=(A_0/15)(1+\rho_d^2)^{-7/2}$. The level $I=0$ is set at zero as a reference level.

	$Cl^{34}[(1d_{3/2})(1d_{3/2})]$ and $K^{38}[(1d_{3/2})^{-1}(1d_{3/2})^{-1}]$											
	Majorana			Heisenberg			Wigner			Bartlett		
I	α_0	α_1	α_2	α_0	α_1	α_2	α_0	α_1	α_2	α_0	α_1	α_2
3	330	1930	−1260	750	4970	5040	0	−980	−1260	180	1580	5040
2	−90	−70	−2520	0	1960	2520	0	−1960	−2520	90	70	2520
1	180	280	−1260	750	4970	5040	0	−980	−1260	30	−70	5040
0		0			0			0			0	
	$Cl^{36}[(1d_{3/2})(1d_{3/2})^{-1}]$											
	Majorana			Heisenberg			Wigner			Bartlett		
I	α_0	α_1	α_2	α_0	α_1	α_2	α_0	α_1	α_2	α_0	α_1	α_2
3	750	3470	1260	750	3010	2520	0	980	1260	30	850	2520
2	750	3010	2520	750	2030	1260	0	1960	2520	−60	320	1260
1	600	1820	1260	750	3010	2520	0	980	1260	−120	−800	2520
0		0			0			0			0	

TABLE XIX. Energy level order for $(1d_{3/2}, 1f_{7/2})$ configurations. The level $I=2$ is set equal to zero as a reference level. The expressions for Majorana and Heisenberg interaction are in the form $[\alpha_0+\alpha_1 y+\alpha_2 y^2](C/35)$, where $y=(\rho_f{}^4\rho_d{}^4/\rho_f{}^2+\rho_d{}^2)(\rho_f{}^2+\rho_d{}^2+2\rho_f{}^2\rho_d{}^2)^{-1}$ and $C=(256A_0/5)\rho_d{}^2(y/\rho_f{}^2\rho_d{}^2)^{9/2}$. The expressions for Wigner and Bartlett interaction are in the form $[\beta_0+\beta_1 w+\beta_2 w^2](Q/35)$, where $w=\rho_f{}^2\rho_d{}^2/4+2\rho_f{}^2+2\rho_d{}^2$ and $Q=(256A_0/5)(\rho_d{}^2+2)(w/\rho_f{}^2\rho_d{}^2)^{9/2}$.

$Cl^{38}[(1d_{3/2})(1f_{7/2})]$

	Majorana			Heisenberg			Wigner			Bartlett		
I	α_0	α_1	α_2	α_0	α_1	α_2	β_0	β_1	β_2	β_0	β_1	β_2
5	432	1884	−440	0	60	−1840	0	−280	−440	−120	−1256	−1840
4	−168	84	−1540	0	210	−1540	0	−980	−1540	−70	−1106	−1540
3	72	264	−1100	0	−210	−1540	0	−700	−1100	−30	−746	−1540
2		0			0			0			0	

$K^{40}[(1d_{3/2})^{-1}(1f_{7/2})]$

	Majorana			Heisenberg			Wigner			Bartlett		
I	α_0	α_1	α_2	α_0	α_1	α_2	β_0	β_1	β_2	β_0	β_1	β_2
5	672	2484	440	0	0	−1400	0	280	440	−120	−976	−1400
4	672	2184	1540	0	0	0	0	980	1540	−70	−126	0
3	672	1764	1100	0	−360	−440	0	700	1100	−30	−46	−440
2		0			0			0			0	

APPENDIX III

The nucleus Sc^{43} is treated as an example of the construction of wave functions. This nucleus has one proton and two neutrons in the $1f_{7/2}$ shell so that there are several states with the same total angular momentum I and isotopic spin T. Therefore, the approximate wave function is constructed in accordance with the assumptions discussed in Sec. IV, and an exact wave function is also found by solving a secular determinant. The expectation values of various observables are calculated with the approximate function, and the exact function and the results are compared to check the validity of the approximation method.

The individual particle states are represented by symbols like $7p$ and $\bar{3}n$. The first of these symbols represents the state with z component of angular momentum $m=7/2$ and isotopic spin projection $t_z=-1/2$ (proton); similarly, the second state has $m=-3/2$ and $t_z=1/2$ (neutron). The general wave function for Sc^{43} with $M=\sum m^i=7/2$ and $T_z=\sum t_z{}^i=1/2$ is the linear combination:

$$\begin{aligned}\psi=&A\sum 7p(1)7n(2)\bar{7}n(3)+B_1\sum 7p(1)5n(2)\bar{5}n(3)\\&+B_2\sum 7n(1)5p(2)\bar{5}n(3)+B_3\sum 7n(1)5n(2)\bar{5}p(3)\\&+C_1\sum 7p(1)3n(2)\bar{3}n(3)+C_2\sum 7n(1)3p(2)\bar{3}n(3)\\&+C_3\sum 7n(1)3n(2)\bar{3}p(3)+D_1\sum 7p(1)1n(2)\bar{1}n(3)\\&+D_2\sum 7n(1)1p(2)\bar{1}n(3)+D_3\sum 7n(1)1n(2)\bar{1}p(3)\\&+F_1\sum 5p(1)3n(2)\bar{1}n(3)+F_2\sum 5n(1)3p(2)\bar{1}n(3)\\&+F_3\sum 5n(1)3n(2)\bar{1}p(3)+G\sum 3p(1)3n(2)1n(3)\\&+H\sum 5p(1)5n(2)\bar{3}n(3)+J\sum 5n(1)1p(2)1n(3).\end{aligned}$$

A component $\sum 7p(1)7n(2)\bar{7}n(3)$ represents the sum over all particle permutations in the three individual particle states $7p$, $7n$, and $\bar{7}n$ with appropriate signs to make the function antisymmetric to exchange of any two nucleons.

The arbitrary coefficients are then determined by requiring the desired values $I=7/2$ and $T=1/2$. This means that operating on ψ with $I=I_x+iI_y$ should give zero because of the general equation for angular momenta

$$I^+\psi_{I,M}=[I(I+1)-M(M+1)]^{\frac{1}{2}}\psi_{I,M+1}. \quad \text{(A13)}$$

Similarly, operating with $T^+=T_x+iT_y$ should give zero. The procedure for obtaining the approximate wave function according to Sec. IV is to set the coefficients F_1, F_2, F_3, G, H, and J equal to zero. Then applying the conditions $I^+\psi=0$ and $T^+\psi=0$ will determine the other coefficients. These are $A=-3$, $B_1=-C_1=D_1$

TABLE XX. Energy level order for $1f_{7/2}$ configurations. The expressions are in the form $[\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3]K_f$, where $x=\rho_f{}^4/4(1+\rho_f{}^2)$, $\rho_f=r_f/r_0$, $K_f=(A_0/735)(1+\rho_f{}^2)^{-9/2}$. The level $I=0$ is set equal to zero as a reference level.

$Sc^{42}[(1f_{7/2})(1f_{7/2})]$

	Majorana				Heisenberg			
I	α_0	α_1	α_2	α_3	α_0	α_1	α_2	α_3
7	630	6552	9576	21 336	1470	20 160	59 472	50 736
6	−630	−4788	−13 104	−33 936	0	6300	30 492	33 936
5	240	−468	−4464	−27 576	1470	15 480	47 772	44 496
4	−300	−4392	−13 896	−31 824	0	6300	26 928	31 824
3	−30	−576	−9432	−27 144	1470	18 180	49 176	44 928
2	−90	−2628	−17 424	−27 456	0	2520	18 612	27 456
1	−180	−3492	−15 048	−17 160	1470	21 780	66 924	54 912
0		0				0		

	Wigner				Bartlett			
7	0	−2520	−15 372	−21 336	840	11 088	34 524	50 736
6	0	−6300	−30 492	−33 936	630	4788	13 104	33 936
5	0	−7200	−27 072	−27 576	450	4068	20 484	44 496
4	0	−6300	−26 928	−31 824	300	4392	13 896	31 824
3	0	−4500	−25 668	−27 144	180	3960	15 516	44 928
2	0	−2520	−18 612	−27 456	90	2628	17 424	27 456
1	0	−900	−7920	−17 160	30	1044	9900	54 912
0		0				0		

$Sc^{48}[(1f_{7/2})(1f_{7/2})^{-1}]$

	Majorana				Heisenberg			
I	α_0	α_1	α_2	α_3	α_0	α_1	α_2	α_3
7	2940	26 460	58 012	21 336	5880	52 920	116 424	72 072
6	2940	26 460	58 012	33 936	5880	52 920	116 424	67 872
5	2940	26 460	50 992	27 576	5880	52 920	116 424	72 072
4	2940	26 460	54 448	31 824	5880	52 920	109 296	63 648
3	2940	22 896	46 132	27 144	5880	52 920	116 424	72 072
2	2940	22 680	46 132	27 456	5880	45 360	92 664	54 912
1	1680	13 608	28 112	17 160	5880	52 920	116 424	72 072
0		0				0		

	Wigner				Bartlett			
7	0	2520	15 372	21 336	840	13 608	49 896	72 072
6	0	6300	30 492	33 936	630	11 088	43 596	67 872
5	0	7200	27 072	27 576	450	11 268	47 556	72 072
4	0	6300	26 928	31 824	300	10 692	40 824	63 648
3	0	4500	25 668	27 144	180	8460	41 184	72 072
2	0	2520	18 612	27 456	90	5148	36 036	54 912
1	0	900	7920	17 160	30	1944	17 820	72 072
0		0				0		

$=+2$, $B_2=B_3=-C_2=-C_3=D_2=D_3=-1$ with normalizing factor $(27)^{-\frac{1}{2}}$.

If one does not set coefficients equal to zero as in the approximate case, the I^+ and T^+ operators will lead to equations reducing the number of arbitrary constants from 16 to 3. This indicates that there are three states with $I=7/2$, $T=1/2$, which can only be resolved by solving a secular determinant of the matrix elements for nuclear interactions. Three arbitrary wave functions are constructed, consistent with the $I=7/2$, $T=1/2$ requirements and mutually orthogonal. The secular determinant is solved for the case of a two-particle delta-function interaction

$$V_{12}=A_\delta(0.8+0.2Q_{12})\delta(r_{12}), \qquad \text{(A14)}$$

where Q_{12} is spin exchange. The various physical quantities calculated with the resulting wave functions are given in Table XXI and compared with values obtained by using the approximate wave function.

TABLE XXI. Expectation values with the approximate and exact wave functions for Sc^{43}.

Wave function	$1f_{7/2}$ shell energy	Mag. mom. nm	$\lvert M\rvert_{GT}^2$ for $Ti^{43}(\beta^+)Sc^{43}$	$\lvert M\rvert_{GT}^2$ for $Sc^{43}(\beta^+)Ca^{43}$
ψ (approximate)	2.400 G	+4.08	0.40	0.57
ψ_1	2.516 G	+4.82	0.72	0.24
ψ_2	1.313 G	+1.03		
ψ_3	0.981 G	+0.02		

From Table XXI it is evident that the lowest energy solution of the secular determinant ψ_1 represents a state with about the same energy as the approximate state. However, while the energies agree within 5 percent, the magnetic moments and Gamow-Teller matrix elements for beta decay are considerably different. This indicates the degree of validity in using the approximate wave functions and is satisfactory for energies.

PHYSICAL REVIEW VOLUME 91, NUMBER 6 SEPTEMBER 15, 1953

The Linear Combination in β Decay*

D. C. PEASLEE
Columbia University, New York, New York

(Received April 29, 1953)

Empirical evidence on the shapes of first- and second-forbidden spectra is surveyed to determine the linear combination of invariants in the β-decay interaction. Absence of $1/W$ terms in allowed shape first-forbidden spectra excludes combinations SA and VT. Spectrum shapes for $\Delta I=2$, no, transitions exclude VA and indicate that for negatron emission the relative algebraic sign of the terms is $(S-T)$. These spectra also indicate the need for considerable correction of nuclear matrix element estimates because of the presence of pseudoscalar-coupled forces in the nucleus. Extrapolation of these empirical corrections to the case of RaE shows that the full linear combination should be $[S-T+(1/\delta)P]$, where δ is positive and of order unity. Arguments based on symmetry principles indicate that the correct interaction should in fact be $(S\pm T+P)$, where $\pm$ refers to $\beta^\pm$ emission. The effects of the difference in sign of the T term should in principle be observable. This law for β decay is compared with the decay of the μ meson, and it is found that if the linear combinations are the same, the coupling constants also have identical absolute values. The mean coupling constant is $|f|=1.44\times0.04\times10^{-49}$ erg cm^3. The chief experimental doubt concerns the μ-meson spectrum, which is expected to go through zero at its end point if the β and μ interactions are identical. It appears possible to deduce the combination $(S-T+P)$ uniquely from the postulates that the neutrino field is (1) unique; (2) massless; (3) part of a universal four-particle interaction.

I. INTRODUCTION AND SUMMARY

THE shapes of forbidden β spectra are analyzed to provide information about the linear combination of invariants, $SVTAP$, in the β-decay interaction. It is assumed as a basis throughout that (1) the interaction contains equal parts of Fermi (S, V) and Gamow-Teller (T, A) invariants; and (2) that there is no appreciable ($\lesssim$10 percent) mixture of S and V or of T and A in the interaction. These assumptions are indicated by the most recent analyses of allowed spectra.[1,2]

In forbidden transitions only those spectra involving a mixture of different nuclear matrix elements can give information about the linear combination; by assumption (2) this implies that the only useful spectra are those with a spin change $\Delta I<n+1$, where n is the order of forbiddenness. To extract the parameters λ of the linear combination, it is necessary to have some independent estimate of the ratios of nuclear matrix elements, which always occur in combination with λ. Section II attempts to form such estimates for the general matrix element, extending a procedure previously given[3] for first-forbidden matrix elements and

* Work performed under the research program of the U. S. Atomic Energy Commission.

[1] O. Kofoed-Hansen and A. Winther, Phys. Rev. **86**, 428 (1952).

[2] J. P. Davidson and D. C. Peaslee, Phys. Rev. **91**, 1232 (1953).

[3] T. Ahrens and E. Feenberg, Phys. Rev. **86**, 64 (1952).