Interaction between Classical Electron and Quantized Electromagnetic Field*

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(Received May 14, 1953)

The equations of motion for a quantized electromagnetic field subject to the influence of a classical electron of arbitrarily prescribed velocity are solved by a simple method involving the corresponding difference equations. The solution yields some results which are contrary to those of an earlier theory. An expression for the transition probabilities between a high-energy state and neighboring states of the field inside a resonant cavity is obtained.

THE behavior of a quantized field which is subjected to the influence of a classical electron with prescribed motion has been discussed by several authors.[1–4] One of the treatments[2] is not correct, and contradicting results are obtained below. Two of the other treatments[1,3] are mainly concerned with the case in which the electron velocity is independent of the time and also are not very suitable for calculating transition probabilities between quantum states of the field. Another treatment[4] makes use of methods developed by Schwinger[5] to obtain an expression for the state of the field in terms of the initial state. This expression is then used to derive transition probabilities between the lowest-energy state and higher states of an unconfined free space field. In the present note, the above-mentioned expression is derived briefly by elementary methods and is then used to obtain transition probabilities between a high-energy state and neighboring states of the field inside a resonant cavity.

The system under consideration is the radiation field inside a cavity resonator described by the vector potential operator $\mathbf{A}$ and its conjugate $\mathbf{P}$. In the usual way,[6] we expand

$$\mathbf{A}=\sum_k q_k(t)\mathbf{u}_k(\mathbf{r}),\quad \mathbf{P}=\sum_k p_k(t)\mathbf{u}_k(\mathbf{r}),$$

where the subscript k refers to the kth normal mode of the cavity, $\mathbf{u}_k(\mathbf{r})$ is a normalized function describing the spatial dependence of the field, and q and p are the coordinate and momentum operators of the radiation oscillators satisfying the commutation relationship $[q_k, p_k]=i\hbar$, all other commutator pairs vanishing.

Considering the effect of the electron as an external influence, the nonrelativistic Hamiltonian for the field is

$$H=H_0+(e/c)\mathbf{v}\cdot\mathbf{A}(\mathbf{r}_0),$$

where

$$H_0=\sum_k[2\pi c^2 p_k^2+(\omega_k^2/8\pi c^2)q_k^2],$$

ω_k is the angular frequency of the kth mode, $\mathbf{r}_0$ is the position of the electron, and the components of $\mathbf{v}$ are ordinary numbers, since the electron is treated classically. In a representation in which H_0 is diagonal, the vector with components $a(t)_{n_1, n_2}\cdots$, which describes the state of the radiation field at time t (with the ith index referring to the ith mode), is given by the equation of motion,

$$\dot{a}(t)=S(t)a(0), \tag{1}$$

where $S(t)$ is a matrix with elements

$$S(t)_{n_1, n_2\cdots;\, n_1', n_2'\cdots}.$$

The only nonvanishing elements of S are

$$S_{n_1\cdots n_j\cdots;\, n_1\cdots n_j\pm1\cdots}=(e/i\hbar c)[q_j]_{n, n\pm1}\mathbf{v}\cdot\mathbf{u}_j(\mathbf{r}_0)e^{\mp i\omega_j t}$$

and

$$[q_j]_{n, n+1}=c[2\pi\hbar(n+1)/\omega_j]^{\frac{1}{2}}=[q_j]_{n+1, n}.$$

It can be seen by a straightforward calculation that the commutator $[S(t_1), S(t_2)]$ is a pure imaginary multiple of the unit matrix.

In order to solve Eq. (1), we consider the following difference equation:

$$[a(t_{i+1})-a(t_i)]/\Delta t_i=S(t_i)a(t_i), \tag{2}$$

where

$$\Delta t_i=t_{i+1}-t_i.$$

A solution of Eq. (2), to the first order in Δt, is

$$a(t_i)=e^{S(t_{i-1})\Delta t_{i-1}}e^{S(t_{i-2})\Delta t_{i-2}}\cdots e^{S(t_0)\Delta t_0}a(0). \tag{3}$$

Using the fact that if the commutator of two matrices, A_1 and A_2, is a multiple of the unit matrix, then

$$e^{A_1}e^{A_2}=e^{A_1+A_2}e^{\frac{1}{2}[A_1,A_2]}, \tag{4}$$

and applying it successively to the right side of Eq. (3) (and bearing in mind that all commutators may be treated as numbers) we obtain

$$a(t_i)=\{\exp\sum_{r=0}^{i-1}S(t_r)\Delta t_r\}$$
$$\times\{\exp\sum_{p=1}^{i-1}\sum_{q=0}^{p-1}[S(t_p), S(t_q)]\Delta t_p\Delta t_q\}a(0).$$

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* Parts of this paper were presented before the 1952 Annual Meeting of the American Physical Society.

[1] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).
[2] L. P. Smith, Phys. Rev. **69**, 195 (1946).
[3] W. Thirring and B. Touschek, Phil. Mag. **42**, 244 (1951).
[4] R. R. Glauber, Phys. Rev. **84**, 395 (1951).
[5] J. Schwinger, Phys. Rev. **75**, 651 (1949).
[6] See, for instance, Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), Sec. 50.

Taking the limit as $\Delta t \to 0$, we have

$$a(t)=\left\{\exp\int_0^t S(t_1)dt_1\right\}\times\left\{\exp\int_0^t dt_1\int_0^{t_1} dt_2[S(t_1),S(t_2)]\right\}a(0). \quad (5)$$

This is the solution of Eq. (1). A simple check of this solution may be obtained by evaluating the expression $[a(t+\Delta t)-a(t)]/\Delta t$, using Eq. (4). The limit of this expression, as $\Delta t \to 0$, reduces to the right side of Eq. (1).

We apply the above theory to the problem examined by Smith.[2] He considers the effect of only one mode of oscillation, of angular frequency ω, and takes $\mathbf{u}(\mathbf{r}_0)$ as constant inside the cavity along the path of the electron. We ask for the probability of the emission or absorption of a given number of photons by the field when the transit time of the electron is an integral number of cycles of the frequency of oscillation, and the electron velocity is constant. For $t=2\pi m/\omega$, m being any integer, all the matrix elements of $\int_0^t S(t_1)dt_1$ vanish, and $\exp\int_0^t S(t_1)dt_1$ is the unit matrix. Since $[S(t_1), S(t_2)]$ is a pure imaginary number, the absolute values of the elements of $a(t)$ are the same as the absolute values of the corresponding elements of $a(0)$. Thus the probability that the state of the field will be changed is zero. This contradicts the results of Smith, and necessitates a different interpretation of the experimental work of Shulman.[7]

We consider, finally, the situation where a mode of the field is initially in a high-energy state, that is, the cavity contains a large number of photons, and ask for the probability of absorption or emission of a number of photons, small compared to the initial value. We neglect all but the oscillating mode, assuming that all the other modes are in their lowest-energy state, and that the probability of their absorption of quanta (virtual or real) is negligibly small.[8] We write only the index referring to the oscillating mode, considering all the others to be zero. We take $a(0)_m=\delta_{mp}$. Then, from Eq. (5) we obtain

$$|a_k(t)|=\left|\sum_{n=0}^{\infty}(1/n!)(B^n)_{kp}\right|, \quad (6)$$

where

$$B=\int_0^t S(t_1)dt_1.$$

Now,

$$(B^n)_{kp}=\sum_{i_1,i_2\cdots} B_{ki_1}B_{i_1i_2}\cdots B_{i_{n-1}p}, \quad (7)$$

and since the only nonvanishing elements of B are $B_{m,m-1}$ and $B_{m-1,m}$, m being any positive integer, we have, setting $r=|k-p|$, $(B^n)_{kp}=0$ for $n<r$, or for $n-r$ being odd. For $n\geqslant r$ and $n-r$ even, the number of nonvanishing terms on the right side of Eq. (7) is equal to the number of possibilities of going from k to p in n steps, each step being ± 1, namely,

$$n![(\tfrac{1}{2}n+\tfrac{1}{2}r)!(\tfrac{1}{2}n-\tfrac{1}{2}r)!]^{-1}.$$

Each of these terms has a common factor, corresponding to the creation (if $k>p$) or destruction (if $k<p$) of r real photons. The remaining factor in each term corresponds to the creation and destruction of $\frac{1}{2}(n-r)$ virtual photons. Because of the convergence of the right side of Eq. (6), we can neglect all terms in it with $n>N(r)$, say.

We consider the case $p\gg r$ and $p\gg N(r)$. This means that the initial number of photons is large, and that the change in the number of photons is small compared to the initial number. Then, the $B_{i,i\pm1}$'s occurring in the sum of Eq. (7) do not differ much from $B_{p,p\pm1}$, since $p-r<i<p+r$ in the common factor (corresponding to emission or absorption of real quanta), and $p-\frac{1}{2}(N-r)<i<p+\frac{1}{2}(N+r)$ in the remaining factors (corresponding to emission and absorption of virtual quanta). The common factor is, therefore, approximately $(B_{p,p\pm1})^r$, the upper sign holding for $p>k$ and the lower sign for $p<k$. The remaining $B_{i,i\pm1}$'s in each term of the sum in Eq. (11) can be considered in pairs. For each $B_{i,i+1}$ there is a $B_{i+1,i}$ in the same term, and their product is $-|B_{i,i+1}|^2$. We therefore have

$$(B^n)_{kp}\cong(-1)^{\frac{1}{2}(n-r)}n!\times\left[\left(\frac{n+r}{2}\right)!\left(\frac{n-r}{2}\right)!\right]^{-1}(B_{p,p\pm1})^r|B_{p,p+1}|^{n-r}$$

for $n-r$ even and positive, and zero otherwise. Substituting in Eq. (6) we obtain as the probability that the field absorb or emit r quanta,

$$|a(t)_{p\pm r}|^2\cong|J_r(2|B_{p,p+1}|)|^2, \quad (12)$$

where J_r is the Bessel function of the rth order. If we take $\mathbf{u}(\mathbf{r}_0)=\mathbf{u}_0$, a constant, along the electron path inside the cavity and zero outside the cavity, and if we

[7] C. Shulman, Phys. Rev. **82**, 116 (1951). Smith's results imply that transitions are produced when the electron transit time is an integral number of cycles, and Shulman measured this supposed effect, interpreting his results by Smith's theory. The main error in Smith's paper is the use of ordinary probability theory instead of quantum theory to obtain the probability of a transition involving more than one quantum. Another aspect of Smith's treatment has been criticized by D. Gabor, Phil. Mag. **41**, 1180 (1950). However, Gabor's own quantum-mechanical analysis does not seem to be correct.

[8] This is the case if the coupling between the electron and the nonoscillating modes is negligible. However, irrespective of the coupling, our interest is in the transition probabilities of the oscillating mode, regardless of the transitions made by the other modes. We can therefore sum over all the quantum states of the other modes. Setting

$$S_{n_1\cdots n_i\cdots;\,m_1\cdots m_i\cdots}=\sum_i S_{n_im_i}{}^{(i)}\prod_{j\neq i}\delta_{n_jm_j},$$

we have S expressed as a sum of commuting matrices. Then, $\exp(\int_0^t S(t_1)dt_1)$ becomes a product of terms, each one of which refers essentially to one mode. Summing over all the quantum states of the initially nonoscillating modes makes all the factors referring to these modes equal to unity. The subsequent treatment is applicable, therefore, even when the absorption probabilities for the nonoscillating modes are not negligible, provided the notation is understood to mean that a summation is taken over all the quantum states of these modes.

take the electron velocity to be constant, then

$$|B_{p,\,p+1}|=edu_0\left(\frac{2\pi p}{\hbar\omega}\right)^{\frac{1}{2}}\frac{\sin(\omega t/2)}{(\omega t/2)},$$

where d is the distance of electron path in the cavity. If we set

$$\mathbf{E}_0=2\mathbf{u}_0(2\pi p\hbar\omega)^{\frac{1}{2}}$$

(this may be considered the amplitude of the corresponding classical field), Eq. (8) becomes

$$|a(t)_{p\pm r}|^2=\left|J_r\left(\frac{edE_0}{\hbar\omega}\frac{\sin(\omega t/2)}{(\omega t/2)}\right)\right|^2. \tag{9}$$

Equation (9) is identical with one obtained by Ward,[9] as the approximate probability that a quantum-mechanical electron will absorb or emit r quanta in passing through a cavity containing a classical field of amplitude E_0. The similarity of the two results is due to the fact that the approximations made in going from Eq. (5) to Eq. (9), as well as Ward's approximations, reduce the problem to one of the absorption or emission of energy according to essentially classical rules except that the change takes place in discrete quanta.

The author is indebted to Professor Julian Schwinger for enlightening discussions related to the foregoing subject matter.

[9] J. C. Ward, Phys. Rev. 80, 119 (1950).

PHYSICAL REVIEW VOLUME 91, NUMBER 6 SEPTEMBER 15, 1953

Optical Studies of Injected Carriers. I. Infrared Absorption in Germanium

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(Received June 3, 1953)

The infrared absorption of injected carriers in germanium has been observed by a modulation technique. For injection into low-resistivity material, the data indicate a linear relationship between injected carrier density and injection current. For injection into high-resistivity material, departure from a linear relationship is indicated. The absorption spectrum of injected carriers resembles that of extrinsic carriers.

THIS note describes some preliminary observations on the infrared absorption of injected carriers in germanium.[1] The process was studied for two cases: injection into almost intrinsic material (40 ohm cm, n type) and injection into doped material (0.5 ohm cm, p type).

The experimental technique was as follows: A dc light source was focused on a small region of a grown p-n junction diode cut with plane parallel faces. The diode was masked by a slit so that only a 15-mil region immediately adjacent to the barrier on the low-conductivity side was exposed. A 50 percent on-off square-wave generator pulsed the diode at about 360 cps and injected carriers. The latter modulated the transmission of the light through the germanium. After dispersal by a monochromator (Perkin-Elmer) the modulated portion of the dispersed light was detected by a dry-ice-cooled PbS cell used in conjunction with an L-C tuned regenerative feedback amplifier and recording potentiometer.

Where extreme sensitivity was not important, an ac thermocouple was employed as a detector. The diode current was then modulated at 13 cps by a mechanical switch operating in conjunction with the synchronous rectifier of the thermocouple amplifier.

If the dimensions of the region of the diode that is studied are small, compared to characteristic decay lengths for injection, then the modulation in light level at some particular wavelength is

$$\Delta I=I_0T_0(1-e^{-k(i)d}).$$

Here I_0 is the incident light intensity, T_0 is the trans-

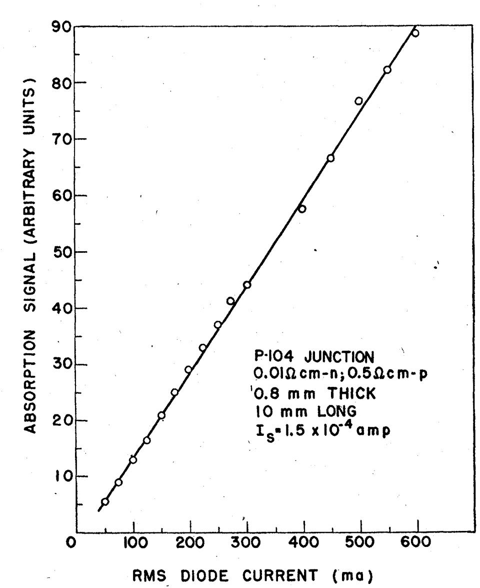

FIG. 1. Absorption signal at 2.3μ as a function of diode current at 25°C for injection into 0.5-ohm cm material.

[1] K. Lehovec, Proc. Inst. Radio Engrs. 40, 1407 (1952).