S Matrix and Causality Condition. II. Nonrelativistic Particles

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The application of the "causality condition" to the S matrix for nonrelativistic particles encounters several difficulties: (a) there is no maximum velocity; (b) the interference of ingoing and outgoing waves has to be taken into account; (c) wave packets with a sharp front do not exist. The condition is therefore reformulated as follows: At any time the total probability of finding the particle outside the scattering center shall not be greater than 1, for every form of the incident wave packet. From this follows for spherical waves that S, as a function of the momentum p, is analytic and holomorphic in the first quadrant and that $e^{2iap}S(p)$ (where a is the radius of the scattering center) has an imaginary part $\leqslant 1$. That suffices to give an explicit integral representation and a product expansion for S, but these permit a more general form for S than is usually envisaged. If, however, the usual symmetry relation $S(-p)=S(p)^*$ is assumed in addition to the causality condition, more specific equations can be derived, which are direct generalizations of those in Part I. In particular, integral relations between the real and imaginary parts of S, and the properties that Wigner found for the R matrix can be deduced.

I. FORMULATION OF THE CAUSALITY CONDITION

IN part I[1] the consequences of the causality condition for the scattering of a Maxwell field by a fixed scattering center or "core" of finite size were investigated. It was assumed that outside a sphere with radius a the free-field equations hold but that nothing is known about the interior of the sphere. The causality condition was formulated as follows: If at a large distance r_1 from the center of the sphere the ingoing wave packet is zero for all $t<t_1$, then the outgoing wave packet shall be zero at r_1 for all $t<t_1+2(r_1-a)/c$.

Obviously, for nonrelativistic particles a modification is necessary, since no maximum velocity exists, and one is inclined to postulate: If at any distance r_1 the ingoing wave packet is zero for all $t<t_1$, then the outgoing wave packet must also be zero for $t<t_1$. This would have to be true for all $r_1\geqslant a$, but it is sufficient to take the strongest form, namely $r_1=a$. This formulation of the causality condition was indeed used by Schützer and Tiomno,[2] who treated the same problem. However, as mentioned in I, there is a serious objection.

The difficulty is that *there are no ingoing or outgoing wave packets that are rigorously zero up to a certain time*, as can be seen as follows. All (spherically symmetrical) superpositions of ingoing waves have the form

$$r\psi_{\text{in}}(r,t)=\int_0^\infty A(p)e^{-ipr-iEt}dp, \tag{1}$$

where $E=p^2/2m$, $\hbar=1$, and $A(p)$ is an arbitrary square integrable function. At a given point $r=r_1$ this is a Fourier expansion of $r\psi_{\text{in}}(r_1,t)$ with respect to t, but the frequency E only runs from 0 to ∞. This puts a severe restriction on ψ_{in} as a function of t, and in particular it can be shown that $\psi_{\text{in}}(r_1,t)$ cannot be zero over any length of time.[3] This difficulty can be overcome by formulating the causality condition in the following way: The probability of finding an outgoing particle at r_1 prior to t_1 cannot be greater than the probability of finding an ingoing particle at r_1 prior to t_1.

A second difficulty, however, arises. At any finite distance r_1 the total probability cannot be uniquely decomposed into an ingoing and an outgoing part, because there is an interference term corresponding to a rapidly oscillating probability current. In the case of relativistic particles this second difficulty can be overcome by letting r_1 go to infinity, but that is not possible for nonrelativistic particles. For, owing to the absence of a maximum velocity, the above condition becomes weaker as r_1 increases; and if it is only postulated for $r_1=\infty$, it leads to practically no restrictions on the S matrix. One is thus forced to the following formulation: The outgoing probability current, integrated from $t=-\infty$ to $t=t_1$, cannot exceed the integrated ingoing current by more than the absolute value of the integral of the interference term. The presence of the interference term makes the condition considerably weaker than in the electromagnetic case and will turn out to be the reason why the S matrix can now have singularities on the positive imaginary axis.

The condition can be given a more familiar form by considering the behavior of ψ as a function of r rather than of t. It is then readily seen to be equivalent to postulating that *if the ingoing wave packet is so normalized as to represent at $t=-\infty$ one incident particle, the total probability at $t=t_1$ of finding a particle outside of any sphere of radius $r_1\geqslant a$ cannot be greater than 1.*

[1] N. G. van Kampen, Phys. Rev. 89, 1072 (1953). This paper will be referred to as I; the formulas contained in it are denoted by (I, $\cdot\cdot$).

[2] W. Schützer and J. Tiomno, Phys. Rev. 83, 249 (1951). The difficulty becomes apparent in their Eq. (7), in which the lower limit of integration should be 0 instead of $-\infty$.

[3] According to (1) ψ is analytic in the lower half of the complex t plane; if it were zero in an interval on the real axis, it could be continued by Schwarz's reflection principle [E. C. Titchmarsh, *The Theory of Functions* (Clarendon Press, Oxford, 1939), second edition], which would give an analytic function vanishing in an interval. For this simple proof I am indebted to Professor N. Levinson; a more general result is mentioned in L. Bieberbach, *Lehrbuch der Funktionentheorie*, Vol. 2 (B. G. Teubner, Leipzig, 1931), second edition, p. 159.

Clearly we get the strongest condition by choosing $r_1=a$ and do not lose anything by taking $t_1=0$. Our final form of the causality condition thus becomes

$$\int_a^\infty 4\pi r^2|\psi_{\rm in}(r,0)+\psi_{\rm out}(r,0)|^2dr\leqslant 1, \qquad (2)$$

ψ being normalized by

$$\int_a^\infty 4\pi r^2|\psi_{\rm in}(r,-\infty)|^2dr=1. \qquad (3)$$

Obviously this condition is satisfied whenever it is possible to define a probability density (obeying a conservation law) in the interior of the core.

The scattering process is described by an S matrix in the usual way. We shall assume that the core is spherically symmetric and we consider only s waves. Then, similarly to Eq. (1),

$$r\psi_{\rm out}(r,t)=-\int_0^\infty B(p)e^{ipr-iEt}dp, \quad (a<r<\infty), \qquad (4)$$

and $S(p)$ is defined by

$$B(p)=S(p)A(p) \quad (0<p<\infty). \qquad (5)$$

The ingoing probability current, integrated with respect to t from $-\infty$ to $+\infty$, is

$$8\pi^2\int_0^\infty |A(p)|^2dp, \qquad (6)$$

and, of course, the same value is found for (3). This is the physical reason for imposing on $A(p)$ the restriction of square-integrability. The integrated outgoing current is given by the same expression with $B(p)$ instead of $A(p)$; as we shall assume that no particles can be absorbed by the core, both expressions must be equal, so that S must be unitary:

$$|S(p)|=1 \quad (0<p<\infty). \qquad (7)$$

It should be emphasized that in physically meaningful scattering states only positive values of p occur, so that any extension of the definition of $S(p)$ to negative p is purely conventional. One may postulate, for example, the familiar equation

$$S(-p)=S^*(p), \qquad (8)$$

or, alternatively, one may define $S(-p)$ as the analytic continuation of $S(p)$ (if it exists and is unique). It is known[4] that if the interaction inside the core can be described by a potential field, both these definitions lead to the same values for $S(-p)$ but there is no reason to assume that they will also coincide in the general case treated here. We shall therefore consider $S(p)$ as a function defined *a priori* for real positive p only.

[4] R. Jost, Helv. Phys. Acta 20, 256 (1947).

We now use (6) for the right-hand side of (2) and substitute (1) and (4) on the left. On inverting the order of integration such terms occur as

$$\int_a^\infty e^{i(p-p')r}dr=2\pi e^{i(p-p')a}\delta_+(p-p') = e^{i(p-p')a}\left\{\frac{1}{-i(p-p')}+\pi\delta(p-p')\right\}.$$

The terms with the δ functions cancel owing to (7) and only the principal-value terms remain. The exponentials can be absorbed in A and B by putting

$$\begin{aligned} e^{-ipa}A(p)&=A_a(p), & e^{+ipa}B(p)&=B_a(p),\\ B_a(p)&=S_a(p)A_a(p), & S_a(p)&=e^{2ipa}S(p). \end{aligned} \qquad (9)$$

One is then left with

$$\int_0^\infty\int_0^\infty \frac{B_a(p)B_a^*(p')-A_a(p)A_a^*(p')}{i(p-p')}dpdp' \geqslant \int_0^\infty\int_0^\infty \frac{B_a(p)A_a^*(p')-A_a(p)B_a^*(p')}{i(p+p')}dpdp'. \qquad (10)$$

The causality condition requires that this inequality be satisfied for all square integrable functions $A_a(p)$, the function $B_a(p)$ being related to $A_a(p)$ by (9).

From this condition we shall deduce in Sec. II that S has an analytic continuation without singularities in the first quadrant of the complex p plane and in Sec. III that the imaginary part of this analytic function is bounded from above. These results are used in Sec. IV to derive the integral representation (24) and the product expansion (28), which, however, still contain the undetermined functions $\tilde{\beta}$ and α. In Sec. V the consequences of the additional assumption (8) are investigated, and the representations (30) and (35) are derived. In that case it is possible to find integral equations and sum rules similar to those in Part I and to derive the properties of Wigner's R matrix. Some of these equations are not new and others might have been conjectured as generalizations of the results in I. The purpose of the present work, however, is to show that they actually follow rigorously from general assumptions. Unfortunately, the use of some rather unfamiliar theorems about complex functions is inevitable. It has been attempted to present the mathematical arguments in no more detail than is necessary to make a reconstruction of the rigorous proofs possible. In Sec. VI the physical significance of the resulting equations is discussed, and it is emphasized that the use of the analytic continuation of S should be regarded merely as a mathematical procedure, which happens to be the adequate tool for treating certain properties of the physically significant function $S(p)$ on the positive real axis, and not for instance as a means of computing bound states.

II. ANALYTIC CONTINUATION OF S

An essential difference with the electromagnetic case shows up in the range of integration in (10), which is now $(0, \infty)$ rather than $(-\infty, +\infty)$. This makes it impossible to cope with the integral kernel $i(p-p')^{-1}$ by using the theory of Hilbert transforms, but fortunately it is still possible to find its eigenfunctions. Indeed, one easily verifies

$$\int_0^\infty \frac{(p')^{i\tau-\frac{1}{2}}}{i(p'-p)}dp' = (\pi \tanh\pi\tau)p^{i\tau-\frac{1}{2}},$$

whence it follows that $p^{i\tau-\frac{1}{2}}$ $(-\infty<\tau<+\infty)$ are improper eigenfunctions with eigenvalues $\pi \tanh\pi\tau$. The expansion in eigenfunctions amounts to a Mellin transformation[5]:

$$A_a(p)=\int_{-\infty}^{+\infty} \mathfrak{A}(\tau)p^{i\tau-\frac{1}{2}}d\tau,$$
$$\mathfrak{A}(\tau)=(1/2\pi)\int_0^\infty A_a(p)p^{-i\tau-\frac{1}{2}}dp. \tag{11}$$

This establishes a one-to-one correspondence of all functions $A_a(p)$ of the class[6] $L_2(0, \infty)$ with all functions $\mathfrak{A}(\tau)$ of the class $L_2(-\infty, +\infty)$, and

$$\int_0^\infty |A_a(p)|^2dp=2\pi\int_{-\infty}^{+\infty} |\mathfrak{A}(\tau)|^2d\tau. \tag{12}$$

When $B_a(p)$ is similarly transformed,

$$B_a(p)=\int_{-\infty}^{+\infty} \mathfrak{B}(\tau)p^{i\tau-\frac{1}{2}}d\tau,$$

the relation (5) between the ingoing and outgoing waves takes the form

$$\mathfrak{B}(\tau)=\int_{-\infty}^{+\infty} \mathfrak{S}(\tau-\tau')\mathfrak{A}(\tau')d\tau'. \tag{13}$$

The inequality (10) can now be transformed; using the identity

$$\int_0^\infty \frac{(p')^{i\tau-\frac{1}{2}}}{i(p+p')}dp'=\frac{-\pi i}{\cosh\pi\tau}p^{i\tau-\frac{1}{2}},$$

one finds

$$\int_{-\infty}^{+\infty} \tanh\pi\tau\{\mathfrak{B}(\tau)\mathfrak{B}^*(\tau)-\mathfrak{A}(\tau)\mathfrak{A}^*(\tau)\}d\tau$$
$$\geqslant\int_{-\infty}^{+\infty} \frac{i}{\cosh\pi\tau}\{\mathfrak{A}(\tau)\mathfrak{B}^*(\tau)-\mathfrak{A}^*(\tau)\mathfrak{B}(\tau)\}d\tau. \tag{14}$$

If $\mathfrak{A}(\tau)$ and $\mathfrak{B}(\tau)$ are L_2 functions connected by (13), the same is true for $\mathfrak{A}(\tau-\mu)$ and $\mathfrak{B}(\tau-\mu)$, for any real constant μ. Hence one may write $\tanh\pi(\tau+\mu)$ and $\cosh\pi(\tau+\mu)$ in (14). Also one may subtract from the left-hand side,

$$\int_{-\infty}^{+\infty} \{|\mathfrak{B}(\tau)|^2-|\mathfrak{A}(\tau)|^2\}d\tau,$$

which vanishes according to (12) and (7). The result is[7]

$$\int_{-\infty}^{+\infty} \frac{e^{-2\pi(\tau+\mu)}}{1+e^{-2\pi(\tau+\mu)}}\{|\mathfrak{A}(\tau)|^2-|\mathfrak{B}(\tau)|^2\}d\tau$$
$$\geqslant 2\int_{-\infty}^{+\infty} \frac{e^{-\pi(\tau+\mu)}}{1+e^{-2\pi(\tau+\mu)}}\mathcal{I}[\mathfrak{A}^*(\tau)\mathfrak{B}(\tau)]d\tau. \tag{15}$$

This inequality must be satisfied by any square integrable $\mathfrak{A}(\tau)$ and for all real μ. We choose for $\mathfrak{A}(\tau)$ a function that vanishes so fast for $\tau\to-\infty$ that $e^{-\pi\tau}\mathfrak{A}(\tau)$ is also square integrable, and let μ tend to $+\infty$. The first term on the left can be written

$$e^{-2\pi\mu}\int_{-\infty}^{+\infty} \frac{|e^{-\pi\tau}\mathfrak{A}(\tau)|^2}{1+e^{-2\pi(\tau+\mu)}}d\tau,$$

and is of order $e^{-2\pi\mu}$ because the integral is clearly bounded. Similarly, the right-hand side of (15) becomes

$$2\mathcal{I}e^{-\pi\mu}\int_{-\infty}^{+\infty} \frac{e^{-\pi\tau}\mathfrak{A}^*(\tau)\mathfrak{B}(\tau)}{1+e^{-2\pi(\tau+\mu)}}d\tau$$

and is therefore of order $e^{-\pi\mu}$. The remaining term must also be of order $e^{-\pi\mu}$, i.e.,

$$\int_{-\infty}^{+\infty} \frac{|e^{-\pi\tau/2}\mathfrak{B}(\tau)|^2}{\cosh\pi(\tau+\mu)}d\tau$$

must be bounded. Consequently, for any positive T, the expression

$$\frac{1}{\cosh\pi T}\int_{-\mu}^{-\mu+T} |e^{-\pi\tau/2}\mathfrak{B}(\tau)|^2d\tau$$

must be bounded, which implies that there is a constant K depending on T such that

$$\int_{\tau_1}^{\tau_2} |e^{-\pi\tau/2}\mathfrak{B}(\tau)|^2d\tau\leqslant K|\tau_2-\tau_1| \quad \text{for} \quad |\tau_2-\tau_1|\geqslant T.$$

[5] See, for instance, E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford, 1937). The formulas in the text can also be found by putting $p=e^x$.

[6] That is the class of functions for which the square integral in the interval $(0, \infty)$ exists (in the Lebesgue sense); see Titchmarsh, reference 3.

[7] $\mathcal{I}$ denotes the imaginary part, $\mathcal{R}$ the real part.

From this it follows that

$$\int_{-\infty}^{+\infty} |e^{-\theta\tau}\mathfrak{B}(\tau)|^2 d\tau < \infty \quad \text{for} \quad 0 \leqslant \theta < \tfrac{1}{2}\pi.$$

This result has to be translated in terms of $A_a(p)$ and $B_a(p)$ by means of the transformation (11). If both $\mathfrak{A}(\tau)$ and $e^{-\pi\tau}\mathfrak{A}(\tau)$ are L_2,

$$A_a(pe^{i\theta}) = e^{-i\theta/2} \int_{-\infty}^{+\infty} \mathfrak{A}(\tau) e^{-\theta\tau} p^{i\tau-\frac{1}{2}} d\tau$$

exists and is square integrable over p, for every value of θ between 0 and π. Moreover, the square integral has a uniform bound for $0 \leqslant \theta \leqslant \pi$:

$$\int_0^\infty |A_a(pe^{i\theta})|^2 dp = 2\pi \int_{-\infty}^{+\infty} |\mathfrak{A}(\tau)e^{-\theta\tau}|^2 d\tau$$
$$\leqslant 2\pi \int_0^\infty |\mathfrak{A}(\tau)|^2 d\tau + 2\pi \int_{-\infty}^0 |\mathfrak{A}(\tau)e^{-\pi\tau}|^2 d\tau,$$

so that $A_a(\lambda)$ is a regular analytic function of $\lambda = pe^{i\theta}$ for $0 < \theta < \pi$ and tends to the boundary function $A_a(p)$ for almost all p as θ goes to zero.[8] Conversely, when $A_a(\lambda)$ has these properties $e^{-\theta\tau}\mathfrak{A}(\tau)$ is $L_2(-\infty, +\infty)$ for $0 \leqslant \theta \leqslant \pi$. Thus the above theorem becomes: *When $A_a(p)$ is the boundary function of a function $A_a(\lambda)$ analytic in the angle $0 < \theta < \pi$ and satisfying*

$$\int_0^\infty |A_a(pe^{i\theta})|^2 dp \leqslant M \quad \text{for} \quad 0 \leqslant \theta \leqslant \pi,$$

then $B_a(p)$ is the boundary function of a function $B_a(\lambda)$ analytic in the angle $0 \leqslant \theta < \frac{1}{2}\pi$ and satisfying

$$\int_0^\infty |B_a(pe^{i\theta})|^2 dp \leqslant N_\delta \quad \text{for} \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi - \delta,$$

with arbitrary positive δ. By a slightly more sophisticated argument one can show that it is sufficient to assume for $A_a(\lambda)$ the same properties as are concluded for $B_a(\lambda)$, *viz.*, regularity for $0 < \theta < \frac{1}{2}\pi$ and uniform square integrability for $0 \leqslant \theta < \frac{1}{2}\pi - \delta$.

It is now possible to define the analytic continuation $S(\lambda)$ of $S(p)$ in the first quadrant by means of

$$B_a(\lambda) = S_a(\lambda)A_a(\lambda) = e^{2ia\lambda}S(\lambda)A_a(\lambda).$$

It has no singularities and the behavior at infinity can be specified as follows. One may choose

$$A_a(\lambda) = (\lambda + i\beta)^{-1};$$

since the corresponding outgoing wave packet,

$$B_a(\lambda) = S_a(\lambda)/(\lambda + i\beta),$$

must be square integrable, it follows that for large $|\lambda|$,

$$S_a(\lambda) = e^{2ia\lambda}S(\lambda) = o(|\lambda|) \tag{16}$$

in the angle $0 < \arg\lambda < \frac{1}{2}\pi$. The analytic continuation can be extended to the fourth quadrant by putting

$$S(\lambda^*) = [S(\lambda)]^{*-1},$$

which reduces to (7) for real positive λ. The only possible singularities are poles corresponding to the zeros above the real axis. *In this way $S(\lambda)$ is defined as a meromorphic function in the right half of the complex λ plane, excluding the imaginary axis.*

III. BEHAVIOR OF S ON THE IMAGINARY AXIS

The fact that a part of the boundary of the domain in which S could be continued analytically had to be excluded, makes it impossible to arrive at a product expansion and integral relations by a method similar to that in I. Our next task is therefore to find more information about the behavior of $S(\lambda)$ on the imaginary axis. It may be expected that the interference term on the right of (15) is important for this purpose, because that was the term which prevented us from extending the analytic continuation of S beyond the i axis. The result will be that $\mathcal{I}S_a$ is bounded from above, which in Sec. IV will be seen to contain implicit information about the behavior on the boundary and to be as useful as a more explicit specification.

Let us choose for $\mathfrak{A}(\tau)$ a function such that, for some $\Delta > 0$,

$$e^{-\theta\tau}\mathfrak{A}(\tau) \quad \text{is} \quad L_2 \quad \text{for} \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi + \Delta. \tag{17}$$

According to the result in the previous section one then has certainly

$$e^{-\theta\tau}\mathfrak{B}(\tau) \quad \text{is} \quad L_2 \quad \text{for} \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi - \Delta.$$

Because of (17) the first term on the left in (15) is of order $e^{-(\pi+2\Delta)\mu}$, whereas the second term may be omitted since it is negative. After multiplying through by $e^{\pi\mu}$ and putting $\mu = +\infty$, one finds for the right-hand side of (15) the condition

$$\mathcal{I} \int_{-\infty}^{+\infty} e^{-(\pi/2+\Delta)\tau}\mathfrak{A}^*(\tau) \cdot e^{-(\pi/2-\Delta)\tau}\mathfrak{B}(\tau) d\tau \leqslant 0,$$

which by means of (11) and (5) can be written

$$\mathcal{I}e^{-i\Delta} \int_0^\infty [A_a(ipe^{i\Delta})]^* A_a(ipe^{-i\Delta}) S_a(ipe^{-i\Delta}) dp \leqslant 0. \tag{18}$$

It is clear that the special choice,

$$A(\lambda) = (\lambda + z^*)^{-1}$$

(z^* complex conjugate of a point $z = x + iy$ in the first quadrant), satisfies the restriction (17), provided Δ is taken less the $\frac{1}{2}\pi - \arg z$. With this choice, (18) can be

[8] On substituting $\lambda = e^z$ these assertions reduce to well-known theorems about analytic functions in a strip, see, e.g., R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain* (American Mathematical Society, New York, 1934).

written

$$\Re \int_0^{\infty \exp[i(\pi/2-\Delta)]} \frac{S_a(\lambda)d\lambda}{(\lambda+z^*)(-\lambda+z)} \geqslant 0.$$

On shifting the integration path to the real axis, the integral becomes[9]

$$\int_0^{\infty} \frac{S_a(p)dp}{(p+z^*)(-p+z)} + 2\pi i \frac{S_a(z)}{z+z^*}.$$

Hence we get

$$\frac{\pi}{x}\mathcal{I}S_a(z) \leqslant \Re \int_0^{\infty} \frac{S_a(p)dp}{(p+z^*)(-p+z)}$$

$$\leqslant \left\{ \int_0^{\infty} \left| \frac{S_a(p)}{p+z^*} \right|^2 dp \cdot \int_0^{\infty} \frac{dp}{|p-z|^2} \right\}^{\frac{1}{2}}$$

$$\leqslant \int_{-\infty}^{+\infty} \frac{dp}{p^2+y^2} = \frac{\pi}{y},$$

so that

$$\mathcal{I}S_a(z) \leqslant x/y. \qquad (19)$$

This shows that the imaginary part of $S_a(z)$ is bounded from above in any angle $0<\delta\leqslant \arg z<\frac{1}{2}\pi$. Now consider the function,

$$F(z)=\exp[-iS_a(z)],$$

in an angular region $0\leqslant \arg z\leqslant\theta$, where $0<\theta<\frac{1}{2}\pi$. It has an upper bound e on the real axis, and on the radius $\arg z=\theta$ it is bounded by $\exp(\cot\theta)$. Also

$$\int_0^{\theta} \log^+|F(pe^{i\theta'})| \cdot \sin(\pi\theta'/\theta)d\theta'$$

is bounded as $p\rightarrow\infty$, so that according to Phragmén-Lindelöf's theorem[10] $|F(z)|$ attains its maximum value on the boundary of the region. By letting θ go to $\frac{1}{2}\pi$ one thus finds $|F|\leqslant e$ in the whole quadrant, or

$$\mathcal{I}S_a(z)\leqslant 1 \quad \text{for} \quad 0\leqslant \arg z<\tfrac{1}{2}\pi. \qquad (20)$$

This result will be used in the next section to derive an integral representation and a product expansion, but we remark here that it implies that $|S_a|$ is bounded, except in the neighborhood of the i axis. Indeed, since (20) followed from the causality condition for a core with radius a, it must also be fulfilled for any $a'>a$, i.e.,

$$\mathcal{I}e^{2ia'z}S_a(z)\leqslant 1 \quad \text{for} \quad \alpha'\geqslant 0.$$

Or, if $\Phi=\arg S_a$,

$$e^{-2\alpha' y}|S_a(z)|\sin(2\alpha' x+\Phi)\leqslant 1.$$

For each value of x, y it is possible to choose an α' between 0 and π/x for which the sine takes the value 1, so that

$$|S_a(z)|\leqslant e^{2\pi y/x}. \qquad (21)$$

We summarize the results: $S_a(\lambda)$ *is meromorphic in the right half-plane* $\Re\lambda>0$. *In the first quadrant it is regular and its imaginary part has the upper bound 1* (which is attained only on the positive real axis). *The absolute value* $|S_a(\lambda)|$ *is bounded in every angle* $0\leqslant\arg\lambda\leqslant\frac{1}{2}\pi-\delta$.

IV. EXPLICIT REPRESENTATIONS OF S

For the derivation of an integral representation it is convenient to use the variable $w=\frac{1}{2}\lambda^2$, which on the real axis reduces to the energy $E=\frac{1}{2}p^2$ (we put from now on $m=1$). The function $S_a(w)-i$ is regular in the upper half-plane, and its imaginary part is $\leqslant 0$. According to a remarkable theorem of Herglotz,[11] this entails the validity of the Poisson integral

$$S_a(w)-i=\Theta-\int_{-\infty}^{+\infty}\frac{1+Ew}{E-w}d\beta(E), \qquad (22)$$

where Θ is a real constant and $\beta(E)$ is some *bounded nondecreasing* real function. β is connected with the boundary values of the imaginary part of S_a by[12]

$$\mathcal{I}S_a(E)-1=-\pi(1+E^2)\beta'(E). \qquad (23)$$

This equation is valid for all (real) E for which S_a is analytic and therefore certainly for $E>0$. For $E<0$, however, $\beta(E)$ may have discontinuities with a positive jump:

$$\beta_n=\beta(E_n+0)-\beta(E_n-0)>0.$$

As β is bounded, the E_n form a denumerable set and $\Sigma\beta_n<\infty$. Denoting by $-\tilde{\beta}(-E)$ the continuous function that remains after subtracting the jumps, one obtains

$$S_a(w)-i=\Theta+\frac{1}{\pi}\int_0^{\infty}\frac{1+Ew}{E-w}\frac{\mathcal{I}S_a(E)-1}{1+E^2}dE$$

$$+\Sigma\beta_n\frac{1+E_nw}{w-E_n}+\int_0^{\infty}\frac{Ew-1}{E+w}d\tilde{\beta}(E). \qquad (24)$$

[9] That there is no contribution from the circular arc at infinity can be shown in much the same way as for functions in a strip (Paley and Wiener, reference 8). The origin too has to be cut off by a small arc, but there is no difficulty in showing that this contribution also vanishes.

[10] This formulation of the theorem is due to F. and R. Nevanlinna, see, e.g., L. Bieberbach, reference 3, p. 129. ($\log^+x$ is defined as $\log x$ for $x>1$, and 0 for $x<1$.) In order to apply it to the present case the right half-plane had to be mapped onto the angle $0\leqslant\arg z\leqslant\theta$, which is the reason why $\sin(\pi\theta'/\theta)$ appears rather than $\cos\theta'$.

[11] A. Herglotz [Ber. Verhandl. K. sächs. Ges. Wiss. Leipzig, Math.-phys. Kl. **63**, 501 (1911)] proved this theorem for functions in the unit circle; see also M. H. Stone, *Linear Transformations in Hilbert Space* (American Mathematical Society, New York, 1932), p. 570. W. Cauer [Bull. Am. Math. Soc. **38**, 713 (1932)] and J. A. Shohat and J. D. Tamarkin [*The Problem of Moments*, American Mathematical Society, New York, 1943] applied it to functions in a half-plane; J. S. Toll [thesis, Princeton, 1952] used it in connection with the causality condition for the propagation of light in a medium. In the actual theorem an additional linear term $\Theta'w$ appears on the right, but since $\Theta'=\lim S_a(w)/w$ for $|w|\rightarrow\infty$, $\delta\leqslant\arg w\leqslant\pi-\delta$ (see Shohat and Tamarkin), it follows from (16) or (21) that in our case $\Theta'=0$.

[12] This can be found directly from (22) by letting w approach a point E on the real axis (see J. S. Toll, reference 11).

The third term on the right exhibits the poles of $S_a(w)$ on the negative real axis with the positive residues[13] $b_n=\beta_n(1+E_n^2)$. There can be no E_n equal to zero, because $S_a(E)$ is bounded for $E>0$. Hence this term can also be written

$$\Sigma b_n\left\{\frac{1}{w-E_n}+\frac{1}{E_n}\right\}-\Sigma\frac{\beta_n}{E_n}, \tag{25}$$

which is a Mittag-Leffler expansion with respect to the poles E_n. The finite constant $\Sigma\beta_n/E_n$ can be absorbed in Θ.

As will be shown in Sec. V, the fourth term on the right stems from the fact that we have not assumed the symmetry relation (8). It may contain additional singularities in $S_a(w)$ on the negative real axis, but they give rise to a slower increase than the poles.[14] It thus follows from (24) that $S(\lambda)$ can have no poles of higher order than the first on the $+i$ axis, which can also be concluded directly from (20).

Although the integral representation (22) or (24) lends itself for the study of the behavior of S on the boundary, it has two serious disadvantages. It contains the arbitrary constant a implicitly in the Θ and β, and it disregards the information that $|S(E)|=1$ for $E>0$. We proceed to derive a product expansion, which does not suffer from these shortcomings, and therefore seems to be a more natural representation for S.

In order to apply more readily some theorems in function theory, it is convenient to map the first quadrant of the λ plane onto the unit circle by putting

$$\zeta=(i\lambda^2+1)/(i\lambda^2-1).$$

The integral (22) then becomes

$$S_a(\zeta)=\Theta_1-2i\zeta\int_0^{2\pi}\frac{d\beta(\theta)}{e^{i\theta}-\zeta},$$

where Θ_1 is now a complex constant. It has been shown[15] that one may conclude from this that $S_a(\zeta)$ belongs to the so-called Hardy class[16] H_p for every $p<1$, which means that, for each such p,

$$\int_0^{2\pi}|S_a(\rho e^{i\varphi})|^p d\varphi$$

is bounded as $\rho\to 1$.

The importance of this result lies in the fact that for functions of that type a canonical representation is known.[17] They can be decomposed in two factors, $S_a(\zeta)=b(\zeta)f_a(\zeta)$, where

$$b(\zeta)=\prod_n|\zeta_n|\frac{1-\zeta/\zeta_n}{1-\zeta\zeta_n^*}$$

is the "Blaschke-product" containing all zeros ζ_n inside the unit circle, and $f_a(\zeta)$ can be represented as follows

$$f_a(\zeta)=\exp\int_0^{2\pi}\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}d\alpha_a(\varphi), \tag{26}$$

$\alpha_a(\varphi)$ being some function of bounded variation. Since $|S_a(\lambda)|=1$ on the positive real axis, $|f_a(\zeta)|=1$ on the arc$\pi<\varphi<2\pi$ of the unit circle and hence $d\alpha_a(\varphi)=0$ for $\pi<\varphi<2\pi$. Translating (26) back to the variable λ one thus finds, putting $\cot(\varphi/2)=s^2$,

$$f_a(\lambda)=\exp\left[i\Theta'\lambda^2-i\int_0^\infty\frac{s^2\lambda^2-1}{s^2+\lambda^2}d\alpha_a(s)\right]. \tag{27}$$

The constant Θ' has to be added, as mentioned in footnote 11, to allow for a possible discontinuity of $\alpha_a(\varphi)$ at $\varphi=0$ (or $\varphi=2\pi$).

The factor $e^{2ia\lambda}$ contained in $S_a(\lambda)$ can be written in the same form (27) and may therefore be combined with $f_a(\lambda)$. *Hence we obtain for* $S(\lambda)$ *the canonical representation*

$$S(\lambda)=\prod\frac{1-\lambda^2/\lambda_n^2}{1-\lambda^2/\lambda_n^{*2}}\times\exp\left[i\Theta'\lambda^2-i\int_0^\infty\frac{s^2\lambda^2-1}{s^2+\lambda^2}d\alpha(s)\right]. \tag{28}$$

This is the counterpart of (I, 16). The right-hand side does not involve a but is uniquely determined by $S(\lambda)$. It has been derived only for the first quadrant but it is clearly valid for all λ in the half-plane $\Re\lambda>0$.

The Blaschke product in (28) contains only the zeros and poles on the right of the imaginary axis. The poles on the imaginary axis itself arise from the (logarithmic) discontinuities of $\alpha(s)$; that they do not appear explicitly is reasonable, because they have no special role among all the singularities that may occur in the half-plane $\Re\lambda\leqslant 0$. From the fact that the product converges one finds a sum rule for the λ_n; in terms of the corresponding energies $w_n=u_n+iv_n=\frac{1}{2}\lambda_n^2$ (u_n resonance energy, $2v_n$ level width) it takes the same form as in I:

$$\Sigma v_n/|w_n|^2<\infty.$$

The exponential factor in (28) corresponds to some sort of potential scattering. Instead of the single con-

[13] That the residue must be positive is well known for potential fields, see C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 22, No. 19 (1946).

[14] Let $w=E+iv$ ($E<0$, $v>0$) and $v\to 0$; then it can be shown that the last term in (24) is $o(1/v)$, owing to the continuity of β. This term may also be singular at E_n, in which case E_n is not just a pole of $S_a(w)$.

[15] V. J. Smirnoff, J. Soc. Phys.-Math. Léningrade 2[2], 22 (1929).

[16] F. Riesz, Math. Z. 18, 87 (1923).

[17] See references 15 and 16. I am indebted to Professor A. Beurling for the remark that this is a special case of Nevanlinna's theorem for "beschränktartige" functions [R. Nevanlinna, *Eindeutige Analytische Funktionen* (J. Springer, Berlin, 1936)], because $\{S_a(\zeta)-2i\}^{-1}$ is bounded in the unit circle.

stant α in (I, 16) it contains an undetermined function $\alpha(s)$ and in addition the constant Θ', owing to our ignorance about the analytic behavior in the left half-plane. They are not completely arbitrary but have to be such that (20) is satisfied. We shall not derive the explicit form of this restriction, but mention only that Θ' must not be negative, as can also be seen from (21). The physical meaning of Θ' is clear from (4): Combined with the time factor e^{-iEt} it gives $\exp[-\frac{1}{2}p^2(t-2\Theta')]$, so that it simply delays the outgoing wave packet—which does not violate the causality condition.

V. THE SYMMETRY PROPERTY

In this section we shall show how the above results can be simplified if the symmetry relation (8) is assumed. More precisely, it is assumed that $S(\lambda)$ has a unique analytic continuation across the positive imaginary axis, which on the negative real axis assumes the values defined by (8). This analytic continuation must satisfy

$$S(-\lambda^*)=[S(\lambda)]^*, \tag{29}$$

and it is therefore a regular function in the second quadrant and meromorphic for $\Re\lambda<0$. On the imaginary axis $S(\lambda)$ is real at all points except the singularities.

We shall now make the additional assumption that there are no other than isolated singularities on the imaginary axis. Then there can only be simple poles. For poles of higher order were already shown to violate (20); if a branch point occurred the analytic continuation would not be unique; and essential singularities can also be shown[18] to be incompatible with (20). Because of (7) there can be no pole at $\lambda=0$, so that (29) yields $S(0)=\pm 1$. For brevity we shall confine ourselves to the case $S(0)=1$. In the literature only S functions of this restricted class have been envisaged (for pure scattering), and, indeed, Wigner[19] showed that S must be of this form if the interaction can be described by some self-adjoint Hamiltonian. On the other hand, these additional properties cannot be derived from our causality condition, as is shown by the following example. Let α, β, γ be positive constants and $\beta>\alpha$. Then

$$S(\lambda)=\frac{(\lambda+\gamma+\alpha i)(\lambda+\gamma-\beta i)}{(\lambda+\gamma-\alpha i)(\lambda+\gamma+\beta i)}$$

does not satisfy (29) and even has a pole $-\gamma+\alpha i$ in the second quadrant, but it can be verified by direct calculation that it satisfies our basic inequality (10) (with $a=0$).

With these additional assumptions the integral relation (24) can be simplified considerably. When w tends to a point $E<0$ (which does not happen to be one of the poles E_n), one obtains according to (23)

$$1=\pi(1+E^2)\tilde{\beta}'(-E).$$

On substituting this in (24), the last term cancels the $-i$ on the left and the -1 in the remaining integral on the right. Furthermore this integral can be rewritten in a fashion similar to (25), because $\mathcal{I}S_a(0)=0$. The new constant Θ is then obviously $S_a(0)=1$, so that the final form becomes

$$S_a(w)-1=-\frac{1}{\pi}\int_0^\infty\left(\frac{1}{w-E}+\frac{1}{E}\right)\mathcal{I}S_a(E)\cdot dE +\Sigma b_n\left(\frac{1}{w-E_n}+\frac{1}{E_n}\right). \tag{30}$$

This equation is valid in the whole complex plane, but it gives only one branch of the two-valued function $S_a(w)$. It is therefore preferable to use the variable λ; (30) then takes the form

$$\frac{S_a(\lambda)-1}{\lambda^2}=\frac{2}{\pi}\int_0^\infty\frac{\mathcal{I}S_a(p')}{p'(p'^2-\lambda^2)}dp'-\Sigma\frac{2b_n}{\kappa_n^2(\kappa_n^2+\lambda^2)}, \tag{31}$$

where $\kappa_n^2=-2E_n$. If now λ tends to a real value p, one finds a relation between real and imaginary parts of S:

$$\Re S_a(p)=1+\frac{2p^2}{\pi}\int_0^\infty\frac{\mathcal{I}S_a(p')}{p'(p'^2-p^2)}dp' -2p^2\Sigma\frac{b_n}{\kappa_n^2(\kappa_n^2+p^2)} \tag{32a}$$

(principal value at $p'=p$). By inverting this equation one obtains

$$\mathcal{I}S_a(p)=-\frac{2p}{\pi}\int_0^\infty\frac{\Re S_a(p')}{p'^2-p^2}dp'+2p\Sigma\frac{b_n}{\kappa_n(\kappa_n^2+p^2)}. \tag{32b}$$

These equations are generalizations of Eqs. (I, 30).

A *product expansion* for $S(\lambda)$ in the present case cannot be obtained simply as a specialization of the general representation (28), because the latter is essentially restricted to the right half of the λ plane, e.g., it involves zeros $-\lambda_n$ which are not zeros of $S(\lambda)$. Instead, $S(\lambda)$ should now be considered as a meromorphic function defined in the whole upper half-plane, for which a product expansion with respect to the zeros *and* the poles has to be found.

We shall first construct a product containing all the

[18] If $i\kappa$ is an isolated singularity, $S(\lambda)$ can be split up according to Laurent's theorem (see Titchmarsh, reference 3) into two terms: $S=g_1+g_2$, where g_1 is regular in the neighborhood of $i\kappa$ and g_2 is regular everywhere but in $i\kappa$. Moreover, both are real on the i axis. Hence $\mathcal{I}g_1(x+iy)\leqslant Cx$ in the neighborhood, and the same is true for g_2 because of (19). Put $\lambda=i\kappa-i/\zeta$ and $g_2(\lambda)=g(\zeta)$, so that g is an entire function. Then $\mathcal{I}g(\xi+i\eta)\leqslant C(-\eta)/(\xi^2+\eta^2)$ in the lower half of the ζ plane and therefore $\mathcal{I}g(\zeta)<0$, since it cannot have a maximum. It has been proved in I that from this follows $g(\zeta)=\alpha_1\zeta+\alpha_2$, which means that $S(\lambda)$ has a simple pole at $i\kappa$. Instead of (19) one can also use (20) and the Phragmén-Lindelöf theorem.

[19] E. P. Wigner, Phys. Rev. 70, 15 and 606 (1946).

poles. Since $S(iy)$ is a real function of y, having poles at $y=\kappa_n$ whose residues are all of the same sign, there must be at least one zero ν_n between κ_n and κ_{n+1}. The product

$$\prod_{n=1}^{\infty}\frac{1+\lambda/i\kappa_n}{1-\lambda/i\kappa_n}\frac{1-\lambda/i\nu_n}{1+\lambda/i\nu_n} \tag{33}$$

contains all the poles of S_a in the upper half-plane, has modulus 1 on the real axis, and can readily be shown to be convergent. For[20] the (absolute) convergence of the product depends on the convergence of the sum

$$\Sigma\left|\frac{1}{\kappa_n}-\frac{1}{\nu_n}\right|\leqslant\Sigma\left(\frac{1}{\kappa_n}-\frac{1}{\kappa_{n+1}}\right)\leqslant\frac{1}{\kappa_1}.$$

Now put

$$g_N(\lambda)=S_a(\lambda)\Big/\prod_{n=1}^{N}\frac{1+\lambda/i\kappa_n}{1-\lambda/i\kappa_n}\frac{1-\lambda/i\nu_n}{1+\lambda/i\nu_n}.$$

This function is regular in the first quadrant, and its imaginary part is not greater than 1 on the boundaries. By applying Phragmén-Lindelöf's theorem to $e^{-ig_N(\lambda)}$, one finds $\mathscr{I}g_N(\lambda)\leqslant 1$ in the whole quadrant. Consequently the limiting function $S_a^0(\lambda)=\lim g_N(\lambda)$, which is regular in the upper half plane and of modulus 1 on the real axis, has also an imaginary part $\leqslant 1$ in the first quadrant. Hence it satisfies an equation of the type (30) or (31) without the term containing the poles:

$$\frac{S_a^0(\lambda)-1}{\lambda}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathscr{I}S_a^0(p')}{p'(p'-\lambda)}dp'.$$

From this it follows, as in I, that $S_a^0(\lambda)=O(|\lambda|)$ and therefore necessarily bounded,[21] and subsequently, that $S_a^0(\lambda)$ can be expanded into a product of the form

$$S_a^0(\lambda)=e^{+2i\alpha'\lambda}\prod_n\frac{1-\lambda/\Lambda_n}{1-\lambda/\Lambda_n^*}\frac{1+\lambda/\Lambda_n^*}{1+\lambda/\Lambda_n}\prod_m\frac{1-\lambda/iL_m}{1+\lambda/iL_m}, \tag{34}$$

where again $\alpha'\geqslant 0$.

The iL_m are the zeros of $S(\lambda)$ on the imaginary axis that were not yet included in the ν_n. As the distinction between the zeros $i\nu_m$ and iL_m depends on an arbitrary choice, it is desirable to write the expansion of $S(\lambda)$ in a unique way, e.g., as follows:

$$S(\lambda)=e^{-2i\alpha\lambda}\prod\frac{1-\lambda/\Lambda_n}{1-\lambda/\Lambda_n^*}\frac{1+\lambda/\Lambda_n^*}{1+\lambda/\Lambda_n}\prod\frac{1-\lambda/iP_m}{1+\lambda/iP_m}. \tag{35}$$

Here $\alpha\leqslant a$ and iP_m runs over all zeros on the imaginary axis, taken in the order of increasing absolute value of P_m; a positive P_m corresponds to a zero on the positive imaginary axis, a negative P_m to a pole.

This product expansion is the direct generalization of (I, 16). The same formulas for the cross section in the neighborhood of resonance follow from it. The sum rule (I, 19) carries over:

$$\sigma(0)=4\pi\left\{-\alpha+2\Sigma\frac{\Gamma_n}{|\Lambda_n|^2}+\Sigma\frac{1}{P_m}\right\}^2,$$

provided the signs of the P_m are taken into account. The generalization of the integral relations in I has been given above. The relation (I, 25), for the case $\alpha=0$, goes over into

$$\eta(\infty)-\eta(0)=\pi(2N+M-K),$$

where N is again the number of zeros Λ_n in the first quadrant, and M and K are the numbers of zeros and poles on the $+i$ axis. However, Levinson[22] showed that, if the interaction is caused by a potential field, $\eta(0)-\eta(\infty)=\pi K$, so that necessarily $N=M=0$. But this cannot be true, because there must be a zero between any two successive poles on the imaginary axis. Consequently it is not possible for a potential field of finite range that $\alpha=0$ in (35).[23]

We shall again derive for the R matrix

$$R(w)=\frac{1}{i\lambda}\frac{S_a(\lambda)-1}{S_a(\lambda)+1}\quad(w=\tfrac{1}{2}\lambda^2), \tag{36}$$

the general properties by which Wigner[24] defined the class of "R functions." It is clear that R is meromorphic and real both for $w>0$ and $w<0$. We proceed to show that $R(w)$ is holomorphic in the upper half-plane, i.e., that $S_a(\lambda)+1$ has no zeros in the first quadrant.

First let $\alpha'=0$ in (34) and let the products consist of N and L factors, respectively, and let (33) consist of K factors. Then S_a is a rational function of order $2N+2K+L$ and has the limit $+1$ at infinity (because L is necessarily even). Hence it assumes the value -1 in $2N+2K+L$ points of the λ plane. Now on the imaginary axis S_a is real and has $2K$ poles whose residues have the same sign, so that it takes the value -1 at least $2K$ times. On the real axis S has modulus 1 and the phase changes $2N+L$ times 2π, so that it takes the value -1 at least $2N+L$ times. There can be no other points where $S_a(\lambda)+1=0$. In the second place let $\alpha'>0$ in (34). Take a large rectangle in the λ plane whose vertical sides intersect the real axis at $\pm\pi h/\alpha'$, where h is some large integer. The variation of the phase of $S_a(\lambda)+1$ around the rectangle will be the same as for $e^{2i\alpha'\lambda}+1$, that is $2h\cdot 2\pi$, since the rational factor in $S_a(\lambda)$ tends

[20] This proof seems somewhat shorter than the one given by P. I. Richards, Duke Math. J. **14**, 777 (1947).

[21] This point was treated too summarily in I, because it does not follow from (I, 39) that $(\lambda+i\beta)^{-1}S(\lambda)$ is *uniformly* bounded in the upper half-plane. Rather than the ordinary Phragmén-Lindelöf theorem one should therefore use Nevanlinna's formulation (see reference 10).

[22] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, No. 9 (1949).

[23] A similar argument shows that α must be positive; for examples see C. Møller (reference 13) and D. ter Haar, Physica **12**, 501 (1946).

[24] E. P. Wigner, Ann. Math. **53**, 36 (1951).

to 1. That means that the factor $e^{2i\alpha'\lambda}$ gives rise to $2h$ new zeros of $S_a(\lambda)+1$; but they lie necessarily on the real axis, because the phase between $-\pi h/\alpha'$ and $+\pi h/\alpha'$ now changes $2N+L+2h$ times 2π. Finally let N, K, and L tend to infinity: according to a theorem of Hurwitz (see Titchmarsh[3]) the zeros of the limiting function must also lie on either the real or the imaginary axis. Hence in the w plane $R(w)$ can only have poles on the real axis.

To show that $\mathfrak{I}R(w)>0$ for $\mathfrak{I}w>0$ let us first return to the case N, K, L finite, $\alpha'=0$. In a point on the $+i$ axis of the $\lambda=x+iy$ plane where $S_a(iy)+1$ vanishes, one must have $dS_a(iy)/dy>0$, because otherwise there would be at least three zeros of $S_a(\lambda)+1$ between the neighboring poles, which would result in more than $2N+2K+L$ zeros in total. It then follows from (36) that $R(w)$ has here a simple pole with negative residue. The same is true for the poles on the positive real axis of λ, because there one must have $d\eta_a/dx>0$. Consequently all poles in the w plane may be cut off from the upper half-plane by small semicircles on which $\mathfrak{I}R>0$. Moreover, for large $|w|$ one has $R(w)\sim\lambda^{-2}$ so that $\mathfrak{I}R(w)$ vanishes uniformly at infinity. It thus follows that $\mathfrak{I}R$ is nowhere negative, since it cannot have a minimum. If now $\alpha'>0$ there is an infinity of poles on the real axis, but by using a large rectangle as before, one can prove the same result. Finally, if N, K, and L tend to infinity, $\mathfrak{I}R(w)$ cannot, of course, become negative; neither can it become zero in any point, because that would be a minimum.

This completes the proof that the properties of the S matrix which were found from the causality condition (2) together with the symmetry relation (8), imply the known properties of the R matrix. In another paper (to be published in Rev. Mex. Fis.) we shall show that the converse is also true.

VI. DISCUSSION

The representations (24) and (28) of S, which followed from our causality condition alone, contain an undetermined function and are therefore of little practical use. Only by making the additional assumption (8) was it possible to obtain the more specific representations (31) and (35), which are similar to those in the electromagnetic case, but for the presence of poles in the upper half-plane. The main effect of these poles is that $S(\lambda)$—for complex λ—is no longer expressed in its values on the real axis alone, but one also needs the poles and their residues in (31). This seems a poor result, because an analytic function is already determined by its values in an arbitrarily small interval, but the following has to be borne in mind.

When two analytic functions take the same values in a small interval, they are identical throughout the region where they are defined. However, if one only knows that their difference in the interval is less than ϵ, then, no matter how small ϵ is, their values in any other point of the region may vastly differ. In other words, if the function $S(p)$ is given in a small interval on the real axis, its analytic continuation $S(\lambda)$ does not depend on the given values in a continuous way. It follows from this remark that it is impossible to express $S(\lambda)$ explicitly in terms of the values in a certain interval on the real axis. Nor is it possible to find an algorithm for computing the values in the complex plane. Although $S(\lambda)$ is unique when $S(p)$ is given, the problem to construct the actual values of $S(\lambda)$ from $S(p)$ is not a correctly set problem.[25]

On the other hand, if a function is holomorphic in a region, the problem of finding its values in the interior from its values on the boundary (or even only from its real or its imaginary part on the boundary) is correctly set and an explicit expression is possible, namely the Cauchy integral (respectively, the Poisson integral). If this region is the upper half-plane, the values on the whole real axis are required and in addition some information about the behavior at infinity; this was the basic idea of I. In the present article $S(\lambda)$ may have poles, but if they are known together with their residues, they can be subtracted and the remaining holomorphic function can again be expressed by means of a Poisson integral in terms of its imaginary part on the boundary. This is the underlying idea of (31); the roundabout derivation was necessary because of the infinite region and the infinite number of poles.

An interaction that decreases rapidly with increasing distance is physically not very different from an interaction of finite extent. But our mathematical derivation is no longer valid; in fact even an exponentially decreasing potential field can give an S with other singularities than just simple poles.[4] Let such an interaction be cut off at some large distance; that may alter the values of S in the complex plane considerably, but not the values on the real axis, because they are directly connected with physically measurable quantities. It can therefore be asserted that equations like (32), connecting physical quantities, are approximately correct if the interaction is approximately zero beyond a certain distance. One might even say that they are as correct as the whole idea of an S matrix, because applying S matrix theory to any actual experiment implies that one regards the scattered particles as free when they reach the observing apparatus.

It has been suggested that the bound states can be found from the scattering data by analytic continuation of the S matrix.[26] That turned out to be true if the interaction is caused by a field of finite extent, but not for instance for the potential e^{-r}. This distinction between sharply cut-off and rapidly decreasing potentials

[25] Actually the only way to give the values of $S(p)$ in the interval with infinite precision is by means of some analytic representation. It is a common procedure to use that representation (or a derived one) for an explicit definition of the analytic continuation. However, this has to be done for each case separately and does not solve the problem of finding a generally valid explicit expression of $S(\lambda)$ in terms of $S(p)$.

[26] W. Heisenberg, Z. Naturforsch. 1, 608 (1946).

seems unnatural to a physicist. According to the above considerations it has, indeed, no physical meaning, because the analytic continuation is not determined in a constructive way so that it is *never* possible to compute the energies of the bound states from the phase shift.

Jost and Kohn[27] showed that the potential can be calculated from the scattering phase shift, when, in addition, the energies and the normalization factors of the bound states are given. If it is known *a priori* that the potential has a finite extent, these constants are already determined by the analytic continuation of the scattering phase shift. Nevertheless, there is again no abrupt difference between the case of potential fields of finite extent and rapidly decreasing potentials, because the constants are determined in a nonconstructive way, so that for actual calculations they still have to be given explicitly.

When the interaction at large distances arises from a potential that does not fall off rapidly, Eqs. (32) are still practically valid, as mentioned above, provided that a is chosen very large. However, for large a all terms on the right tend to zero except the contribution of the singularity in the integral, and the equation reduces to a triviality. It is therefore preferable to solve the wave equation explicitly as far as possible and enclose in the sphere with radius a only the interaction which is unknown or cannot be treated explicitly. That amounts to replacing $e^{\pm ipr}$ with the explicit solutions in the outer region, in the manner of Wigner and Eisenbud.[28] It will then be possible to generalize the present treatment, provided the analytic behavior of these solutions (as functions of p) has roughly the same features as that of $e^{\pm ipr}$. Presumably a sufficient condition is that the potential is a regular analytic function of r for $|r|>a$, including the point $r=\infty$. A special case is the centrifugal potential $l(l+1)/r^2$ which comes in when the higher multipole waves are treated.

[27] R. Jost and W. Kohn, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 9 (1952).

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[28] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).