Fourth-Order Radiative Corrections to Atomic Energy Levels*†

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In this paper the fourth-order radiative corrections to the elastic scattering of an electron in the field of a fixed potential are examined, using the Dyson S-matrix formulation of quantum electrodynamics. The result can be represented as an addition to the interaction energy density of the electron with the external potential:

$$-ie\bar{\psi}(x)\gamma_\mu\psi(x)\Box^2/\kappa^2 A_\mu{}^e(x)[(\alpha^2/4\pi^2)(0.52\pm0.21)],$$

plus the anomalous magnetic moment already known. The contributions that arise from the vacuum-polarization currents are omitted. This term calculated contributes to the level shift in a hydrogenic atom an energy of

$$(\alpha^4Z^4/n^3)\ \mathrm{Ry}\delta_{l,0}[(4/\pi^2)(0.52\pm0.21)].$$

For the $2S$ level of hydrogen this is 0.24 ± 0.10 Mc/sec.

INTRODUCTION

EVIDENCE that fourth-order terms may well be of experimental significance comes from the recently completed precise measurements of the fine structure of the $n=2$ levels in hydrogen and deuterium. The application of the second order of quantum electrodynamics to the interpretation of atomic energy level shifts has met with very great success. The more precise measurements make possible a more severe test of the theory.

A crude estimate of the order of magnitude to be expected is easily obtained. The main part of the second order level shift is given by the term[1]

$$(\alpha^3Z^4/n^3)\ \mathrm{Ry}\ \ln(mc^2/\bar{E}).$$

The leading term of the fourth-order contribution may be expected to be of the form

$$(\alpha^4Z^4/n^3)\ \mathrm{Ry}\ \ln(mc^2/\bar{E}),$$

or simply

$$(\alpha^4Z^4/n^3)\ \mathrm{Ry},$$

according as there is or is not a logarithm.[2] Thus the fourth-order effect might amount to a part in a hundred or a part in a thousand of the second-order effect, and these are experimentally significant magnitudes.

METHOD OF CALCULATION

In the evaluation of second-order radiative corrections to atomic energy levels two approaches have been used. One of these consists of an evaluation of a formula for the self energy obtained in a straightforward manner from second-order perturbation theory, while the other consists of an evaluation of the second-order radiative corrections to elastic scattering, from which one infers the level shift. The relationship between the two methods is quite apparent if one compares the formulas of French and Weisskopf[3] with those of Feynman.[4] Thus, for the intermediate states of the electron, French and Weisskopf use plane waves corrected by a single Born approximation rather than the exact states of the external potential. This approximation corresponds essentially to the scattering approximation. For transitions involving low-energy photons both the scattering approximation and, accordingly, the above-mentioned treatment of intermediate states fail in the second-order calculation. This failure manifests itself in the appearance of an infrared catastrophe, and makes necessary a modified treatment for the low energy photons.

The scattering method, while possibly less straightforward, is actually much the simpler of the two, particularly for the higher-order corrections. Accordingly, this paper will be devoted to a discussion and evaluation of the fourth-order radiative corrections to elastic scattering in first Born approximation. It is easily seen that the result may be represented as a modification in the interaction energy density of the electron with the external potential of the form:

$$C_1\bar{\psi}(x)\sigma_{\mu\nu}\psi(x)\int f_1(x-x')F_{\mu\nu}{}^e(x')d^4x'$$
$$+C_2\left(-ie\bar{\psi}(x)\gamma_\mu\psi(x)\Box^2\int f_2(x-x')A_\mu{}^e(x')d^4x'\right)$$
$$\int f_1(x-x')d^4x'=\int f_2(x-x')d^4x'=1. \qquad (1)$$

* The results of this calculation have been reported previously by Bersohn, Weneser, and Kroll, Phys. Rev. **86**, 596 (1952).
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[1] H. Bethe, Phys. Rev. **72**, 339 (1947).

[2] An investigation of the fourth-order level shift using non-relativistic electrodynamics is relatively simple and indicates that there is no logarithm.

[3] J. B. French and V. F. Weisskopf, Phys. Rev. **75**, 1240 (1949).

[4] R. P. Feynman, Phys. Rev. **74**, 1430 (1948).

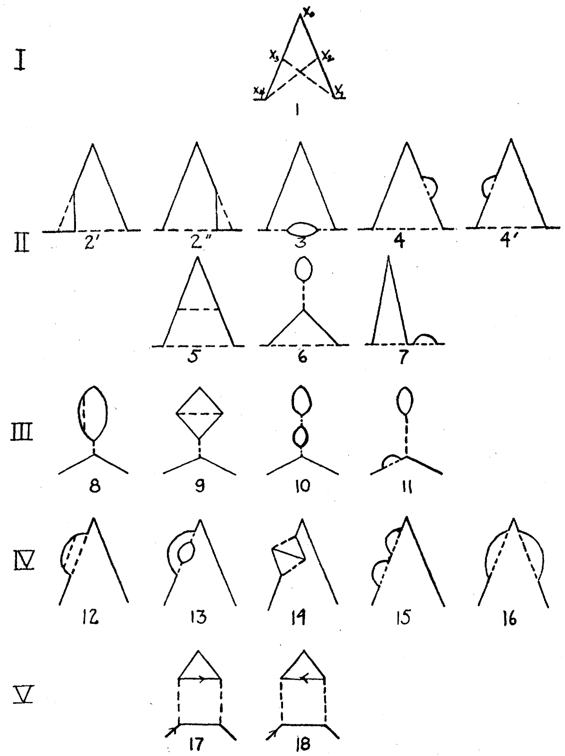

FIG. 1. The Feynman diagrams corresponding to $U_1^{(4)}$.

Expressed as in Eq. (1), the two terms correspond to an extra magnetic moment interacting with a modified external field and the interaction of the electron current with a modified external potential. The level shift may be obtained by including Eq. (1) in the interaction energy of the bound electron and evaluating its effect using first-order perturbation theory. The scattering approximation is expected to be valid to order $\alpha^4 Z^4$ Ry, at least for high energy photons. The functions $f_1(x-x')$, $f_2(x-x')$ are characterized by a range of the order of the electron's Compton wavelength, and will be replaced by δ functions. The effect on the level shift of this replacement is at most of order $\alpha^7 Z^5$ Ry. In view of the approximation involved in the use of the scattering method, a more precise representation of these functions is, in fact, unwarranted.

The evaluation of the fourth-order level shift requires, therefore, merely the calculation of the constants C_1 and C_2. C_1 represents the fourth-order contribution to the anomalous moment and has already been calculated. We shall be concerned here with the closely related but considerably more involved problem of evaluating C_2. It should be mentioned that C_2 turns out to be finite in the infrared. This suggests that for the fourth-order corrections the scattering approximation is valid to order $\alpha^4 Z^4$ Ry even for the low-energy photons.[5]

THE SCATTERING CALCULATION

The calculation was carried out using the Dyson S-matrix formulation of quantum electrodynamics and the Dyson renormalization program.[6] We are here concerned only with the fourth-order, single-interaction with the external field, one-electron part of the S-matrix. This has been discussed by Karplus and Kroll,[7] and called by them $U_1^{(4)}$.

The Feynman diagrams that describe $U_1^{(4)}$ are repeated in Fig. 1. The sum of the diagrams 17 and 18 is just zero by Furry's theorem. The diagrams of group IV and diagram 7 give rise only to renormalizations of lower order scattering processes, and so can be dropped from further consideration. The diagrams of class III are vacuum-polarization effects; these terms have been calculated by Baranger, Dyson, and Salpeter.[8] This leaves the diagrams 1 through 6, and it is seen later that only 1 through 5 actually contribute.

The integrals corresponding to diagrams 1 through 6 are, in fact, given explicitly by Karplus and Kroll. The procedure for reducing these expressions to integrals over Feynman's auxiliary variables has been fully discussed and will be reviewed only briefly. The essential problem here is one of organizing the calculations in such a way as to keep the computational labor involved within manageable limits. Therefore, in the discussion to follow, emphasis will be placed upon the procedure used. As the procedure differs from diagram to diagram, the various diagrams will be discussed separately. Diagram 1, being irreducible, is the best organized and so the easiest to discuss. It is also the most complicated one to evaluate, and, therefore the one for which organization is most necessary. Therefore, we consider it first and in some detail.

The Evaluation of the Irreducible Diagram

Diagram 1 (see Fig. 1 for the labeling) gives rise, immediately on application of Dyson's prescription, to the integral

$$M_1 = -\frac{e}{\hbar c}\frac{\alpha^2\pi^2}{4}\int_{-\infty}^{+\infty} d^4x_0 d^4x_1 d^4x_2 d^4x_3 d^4x_4$$
$$\times\{A_\mu^e(x_0)D_F(x_3-x_1)D_F(x_4-x_2)\bar{\psi}(x_1)$$
$$\times\gamma_\nu S_F(x_2-x_1)\gamma_\lambda S_F(x_0-x_2)\gamma_\mu S_F(x_3-x_0)$$
$$\times\gamma_\nu S_F(x_4-x_3)\gamma_\lambda\psi(x_4)\}. \quad (2)$$

On inserting the Fourier transforms and carrying out

[5] An explicit proof that the fourth-order level shift is given correctly to order $\alpha^4 Z^4$ Ry by the scattering approximation has been given by one of the authors (NK) and R. L. Mills. The corrections to the scattering approximation are of order $\alpha^5 Z^5$ Ry. A note on this subject is in preparation.

[6] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

[7] R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950).

[8] Baranger, Dyson, and Salpeter, Phys. Rev. **88**, 680 (1952).

the trivial integrations, M_1 becomes

$$M_1=(e/\hbar c)(\alpha^2/\pi^2)\int d^4kd^4k'd^4p_1d^4p_2A_\mu^e(p_1-p_2)$$
$$\times\frac{\bar\psi(p_1)\gamma_\nu[i\gamma\cdot(p_1-k)-\kappa]\gamma_\lambda[i\gamma\cdot(p_1-k-k')-\kappa]\gamma_\mu[i\gamma\cdot(p_2-k-k')-\kappa]\gamma_\nu[i\gamma\cdot(p_2-k')-\kappa]\gamma_\lambda\psi(p_2)}{[(p_1-k)^2+\kappa^2][(p_1-k-k')^2+\kappa^2][(p_2-k-k')^2+\kappa^2][(p_2-k')^2+\kappa^2][k^2+\lambda^2][k'^2+\lambda^2]}. \quad (3)$$

A photon rest-mass has been used to control the infrared divergence; it will be considered later. Denominators are combined by the Feynman identity

$$\frac{1}{abcdef}=5!\int_0^1\frac{duzdzw^2dwx^3dxy^4dy}{[(a-b)yxwzu+(b-c)yxwz+(c-d)yxw+(d-e)yx+(e-f)y+f]^6}. \quad (4)$$

The form of the final integrals over the auxiliary variables u, z, w, x, y depends on the choice of assignment of denominators in this identity. The assignment used here—and by Karplus and Kroll in the magnetic moment calculation—is

$$\begin{aligned} a&=[(p_2-k-k')^2+\kappa^2], & d&=[k^2+\lambda^2],\\ b&=[(p_1-k-k')^2+\kappa^2], & e&=[(p_2-k')^2+\lambda^2],\\ c&=[(p_1-k)^2+\kappa^2], & f&=[k'^2+\lambda^2]. \end{aligned} \quad (5)$$

This choice of a and b, together with the fact that $a=b$ for $p_1=p_2$, will be seen later to make one of the auxiliary variable integrations, the u integration, trivial. Then

$$M_1=120\left(\frac{e}{\hbar c}\right)\left(\frac{\alpha^2}{\pi^2}\right)\int d^4kd^4k'd^4p_1d^4p_2A_\mu^e(p_1-p_2)\int_0^1 duzdzw^2dwx^3dxy^4dy$$
$$\times\frac{\bar\psi(p_1)\gamma_\nu[i\gamma\cdot(p_1-k)-\kappa]\gamma_\lambda[i\gamma\cdot(p_1-k-k')-\kappa]\gamma_\mu[i\gamma\cdot(p_2-k-k')-\kappa]\gamma_\nu[i\gamma\cdot(p_2-k')-\kappa]\gamma_\lambda\psi(p_2)}{\{k^2xy+k'^2(1-xy+xywz)+2k\cdot(-\Delta pyxwzu-p_1yxw +2k\cdot k'yxwz+2k'\cdot(-\Delta pyxwzu-p_1yxwz-p_2y+p_2xy)+\lambda^2(1-y+xy(1-w))\}^6}, \quad (6)$$

with $\Delta p=(p_2-p_1)$, where use has been made of $p_1^2+\kappa^2=p_2^2+\kappa^2=0$.

Further manipulation is greatly facilitated by introducing new variables K, K', linearly related to the k, k', that diagonalize the denominator to the form[9]

$$\{\alpha K^2+\beta(K')^2+\gamma\kappa^2+\delta(\Delta p)^2+\epsilon\lambda^2\}, \quad (7)$$

where α, β, γ, δ, ϵ are

$$\alpha=xy,$$
$$\beta=[1-xy(1-wz+w^2z^2)],$$
$$\gamma=w^2xy+y^2\frac{[1-x+xwz(1-w)]^2}{[1-xy(1-wz+w^2z^2)]},$$
$$\delta=w^2xyuz(1-uz)+y^2xwz\frac{[1-x+uxwz(1-w)][1-w-u(1-wz)]}{[1-xy(1-wz+w^2z^2]},$$
$$\epsilon=1-y+xy,$$

and where further use has been made of $p_1^2+\kappa^2=p_2^2+\kappa^2=0$. In terms of these new variables, K and K', the numerator can be brought into the form:

$$\begin{aligned}\bar\psi(p_1)\{&(\ )(K')^4\gamma_\mu+(\ )K^2(K')^2\gamma_\mu+(\ )K^2\kappa^2\gamma_\mu+(\ )\kappa^4\gamma_\mu\\ &+(\ )(K')^2(\Delta p)^2\gamma_\mu+(\ )K^2(\Delta p)^2\gamma_\mu+(\ )\kappa^2(\Delta p)^2\gamma_\mu\\ &+(\ )K^2\kappa\sigma_{\mu\nu}\Delta p_\nu+(\ )(K')^2\kappa\sigma_{\mu\nu}\Delta p_\nu+(\ )\kappa^3\sigma_{\mu\nu}\Delta p_\nu\\ &+(\ )\kappa\sigma_{\mu\nu}\Delta p_\nu(\Delta p)^2+(\ )\gamma_\mu(\Delta p)^4+(\ )K^2\kappa\Delta p_\mu\\ &\qquad+(\ )(K')^2\kappa\Delta p_\mu+(\ )\kappa^3\Delta p_\mu\}\psi(p_2)A_\mu^e,\end{aligned} \quad (8)$$

where the factors are functions of the auxiliary variables. Terms containing K, K' to odd powers have been dropped since they vanish on integration over K, K'.

On separating off the renormalizations and carrying out the K, K^1 integrations, the result immediately takes the form:

$$\begin{aligned}\bar\psi(p_1)\{&f((\Delta p/\kappa)^2)A_\mu^e(\Delta p)\sigma_{\mu\nu}\Delta p_\nu/\kappa\\ &+g((\Delta p/\kappa)^2)\gamma_\mu A_\mu^e(\Delta p)(\Delta p)^2/\kappa^2\\ &+h((\Delta p/\kappa)^2)\Delta p_\mu A_\mu^e(\Delta p)\\ &+j((\Delta p/\kappa)^2)\sigma_{\mu\nu}\Delta p_\nu(\Delta p)^2A_\mu^e(\Delta p)\\ &\qquad+l((\Delta p/\kappa)^2)\gamma_\mu(\Delta p)^4A_\mu^e(\Delta p)\}\psi(p_2).\end{aligned} \quad (9)$$

In this calculation terms of higher order in $(\Delta p/\kappa)$ than the second are discarded. This corresponds to replacing the functions $f_1(x_1-x_2)$, $f_2(x_1-x_2)$ appearing in Eq. (1) by δ functions, as previously discussed. This leaves the

[9] Note that the definition of δ,γ is such that $(\delta/\gamma)\leq 1$. This means that after the K, K' integration an expansion in $\Delta p/\kappa$, for small $(\Delta p/\kappa)$, is valid for all values of the auxiliary variables.

terms

$$\bar{\psi}(p_1)\{f(0)A_\mu^e(\Delta p)\sigma_{\mu\nu}\Delta p_\nu/\kappa+g(0)\gamma_\mu A_\mu^e(\Delta p)(\Delta p/\kappa)^2 +h(0)(\Delta p_\mu/\kappa)A_\mu^e(\Delta p)\}\psi(p_2). \quad (10)$$

Now, the term $h(0)\Delta p_\mu/\kappa A_\mu^e(\Delta p)$ is gauge variant; it is, then, to be expected that if all terms of this form are carefully collected, the coefficient, $h(0)$, will vanish. In this problem $A_\mu^e=(0,0,0,V(r))$, and so $\Delta p_\mu A_\mu^e(\Delta p)=0$ in any case. The $f(0)A_\mu^e(\Delta p)\sigma_{\mu\nu}\Delta p_\nu/\kappa$ term is just the fourth-order contribution to the anomalous moment of the electron.[4] The problem here is to calculate the $g(0)$.

The terms in the numerator, (8), that contribute to $g(0)$ are those in which Δp appears as $(\Delta p)^0$ or $(\Delta p)^2$. The terms contributing to $f(0)$ are the significantly less populous group involving only the terms $\kappa K^2\sigma_{\mu\nu}\Delta p_\nu$, $\kappa(K')^2\sigma_{\mu\nu}\Delta p_\nu$, $\kappa^3\sigma_{\mu\nu}\Delta p_\nu$. The various spinor products appearing originally in the numerator are reduced to the form Eq. (8) using a number of identities, of which a particularly simple example is

$$\bar{\psi}(p_1)\gamma_\nu\gamma\cdot p_1\gamma_\mu\gamma\cdot p_2\gamma_\nu\psi(p_2) =\bar{\psi}(p_1)\{-2\kappa^2\gamma_\mu-2\gamma_\mu(\Delta p)^2\}\psi(p_2).$$

In obtaining these identities free use has been made of the fact that $\bar{\psi}(p_1)$, $\psi(p_2)$ are free particle spinors; all terms depending on Δp otherwise than as $(\Delta p)^0$ or $(\Delta p)^2$ have been dropped.

As written, $g(0)$ is a sum of integrals over the five auxiliary variables, and is a function of λ, the photon mass. The result is of physical interest only in the limit $\lambda\rightarrow 0$. In the very great majority of integrals, called group g', there is convergence uniform with respect to the λ. In these terms it is, then, quite correct to put $\lambda\equiv 0$; in doing this, only terms that go to zero as λ goes to zero are dropped. However, in a particular group of integrals, putting $\lambda\equiv 0$ makes the integrals divergent; these are discussed later, calling this group g''.

The set of integrals g' is discussed first. A particular system of classification was used; this classification appeared to us to be crucial to the feasibility of the calculation. The g' can be written—after the trivial integration—as

$$g'=\Sigma c(k,m,n,s,t)I(k,m,n,s,t), \quad (11)$$

where the c's are numerical factors and

$$I(k,m,n,s,t)=\int_0^1 dzdwdxdy\frac{z^{s+1}w^{t+2}x^{n+2k-3-m}y^{n+2k-2}}{\beta^{n+k}\gamma^k}[1-x(1-wz+w^2z)]^m$$
$$=\int_0^1 dzdwdxdy\frac{z^{s+1}w^{t+2}x^{n+2k-3-m}y^{n+k-2}[1-x(1-wz+w^2z)]^m}{[1-xy(1-wz+w^2z^2)]^n[w^2x+y\{(1-x+xwz(1-w))^2-w^2x^2(1-wz+w^3z^2)\}]^k}, \quad (12)$$

with k, m, n, s, t positive integers such that $0\leq m\leq n+2k-3$. Terms that actually appeared are given by

$k=1$	$n=1\{m=0\dagger$	$n=2\begin{cases}m=1\\ m=0\dagger\end{cases}$	$n=3\begin{cases}m=1\\ m=0\dagger\end{cases}$	$n=4\begin{cases}m=1^*\\ m=0^*\end{cases}$	
$k=2$	$n=0\{m=0\dagger$	$n=1\begin{cases}m=1\dagger\\ m=0\dagger\end{cases}$	$n=2\begin{cases}m=2\dagger\\ m=1\dagger\\ m=0\end{cases}$	$n=3\begin{cases}m=3\dagger\\ m=2\dagger\\ m=1\end{cases}$	$n=4\begin{cases}m=3^*\\ m=2^*\end{cases}$
$k=3$	$n=-1\{m=0$	$n=0\begin{cases}m=2\\ m=1\end{cases}$	$n=1\begin{cases}m=2\\ m=1\end{cases}$	$n=2\begin{cases}m=3\\ m=2\end{cases}$	various higher n

each with a range of values for s, t.

This system of classification in terms of the I's was first used by Karplus and Kroll in their magnetic moment calculation; in the terms marked by † the expressions after the x, y integrations were furnished us by these authors. It is not surprising that the same type of integrals appear in $g(0)$ as appear in $f(0)$. However, the evaluation of $g(0)$ is a much more formidable task than that of $f(0)$. First, there are simply a greater number of terms present after the K-space integration that must be rearranged into this classification system. We have already seen that the magnetic-moment terms came from a smaller group of terms in Eq. (8). The second and more important source of difficulty is the appearance of terms in $g(0)$ of higher k, n than are present in $f(0)$. For given k, n the labor involved in evaluating I increases sharply with decreasing m. There is an even sharper increase with increase of k and n. The appearance of increased difficulties can be seen directly from Eq. (7). The terms

$$(K')^2(\Delta p)^2\gamma_\mu,\quad K^2(\Delta p)^2\gamma_\mu,\quad \kappa^2(\Delta p)^2\gamma_\mu,$$

will give rise to I's of k equal to those arising from

$$(K')^2\kappa\sigma_{\mu\nu}\Delta p_\nu,\quad K^2\kappa\sigma_{\mu\nu}\Delta p_\nu,\quad \kappa^3\sigma_{\mu\nu}\Delta p_\nu.$$

However, in addition there are the other terms that contribute to $g(0)$. Thus, $\kappa^4\gamma_\mu$ terms are the $k=3$ integrals. The $(K')^2\kappa^2\gamma_\mu$, $K^2\kappa^2\gamma_\mu$ give rise to $k=2$ integrals as compared to the $k=1$ integrals coming from the $(K')^2(\Delta p)^2\gamma_\mu$, $K^2(\Delta p)^2\gamma_\mu$. Further, even when the two terms being compared give integrals of equal k the coefficients are much more complicated in the terms associated with $g(0)$. Also, even for equal k, $g(0)$ contains I's of higher n, lower m. Finally, the appearance of the higher s, t terms multiplies the labor.

The I's are computed by carrying out the integrations in the order x, y, z, w. Actually, it was found sufficient to carry out the x, y integration; there then remain for any I a small number of double integrals. A series of tables for these double integrals then completes the evaluation.[10] The values of a large number of I's for the k, n m indicated were obtained and are available. All of the integrals can be evaluated analytically and can be expressed in terms of the same set of transcendentals as those of Karplus and Kroll.

The integrals marked by an * were not evaluated. Instead the terms that contained these I's were estimated by *rigorous* upper and lower bounds. This was done for the I's of higher k, n, lower m where the labor was very large and the computed contributions quite small. The fact that these quantities were not evaluated exactly leads to a small uncertainty in the final result; the stated uncertainty represents outside limits; the actual error is probably much less.

[10] These tables coincide in part with those developed by Karplus and Kroll, and these parts were made available to us.

Next the set g'' is discussed. All the terms that are λ-divergent come from the quantity obtained on putting k, k' equal to zero in the numerator of Eq. (5); after the removal of the renormalization terms and integration over the photon momenta, this gives rise to the integral

$$\int_0^1 dwdzdxdy \left\{ \frac{2zw^2xy^2}{\{w^2xy+y^2([1-x+xwz(1-w)]^2-w^2x^2[1-wz(1-wz)])+\lambda^2(1-y+xy)[1-xy(1-wz+w^2z^2)]\}^2} \right.$$
$$\left. \times \frac{-4zw^2xy^2\int_0^1 du[uzw^2(1-uz)xy][1-xy(1-wz+w^2z^2])+xwzy^2[1-u-w(1-uz)][1-x+xwzu(1-w)]}{\{w^2xy+y^2([1-x+xwz(1-w)]^2-w^2x^2[1-wz(1-wz)])+\lambda^2(1-y+xy)[1-xy(1-wz+w^2z^2]\}^3} \right\}. \quad (13)$$

A deficiency in powers of z and w makes the integrals divergent if we put $\lambda \equiv 0$. The presence of the λ makes evaluation appreciably more difficult, as the above discussed procedure fails. However, a more roundabout procedure proved sufficiently simple to be carried out. Thus, writing the integrals as

$$\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_0^1 dwF(x, y, z, w, \lambda/\kappa), \quad (14)$$

the identity

$$\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_0^1 dwF(x, y, z, w, \lambda/\kappa)$$
$$=\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_{(\lambda/\kappa)}^1 dwF(x, y, z, w, 0)$$
$$+\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_{(\lambda/\kappa)}^1 dw\{F(x, y, z, w, \lambda/\kappa)$$
$$-F(x, y, z, w, 0)\}+\int_0^1 dx\int_0^1 dy\int_0^1 dz\int_0^{(\lambda/\kappa)} dw$$
$$\times F(x, y, z, w, \lambda/\kappa)$$

provides a useful separation. The first term is easily integrated over x, y, z by the same procedure that was used in the I's. The remaining w integration is also easily carried through, since, as can be shown rigorously, it is correct to put $\lambda=0$ throughout except for the integration of a simple rational fraction. It is only this first term that gives a λ-divergent dependence. The second and third terms give a result that is finite as λ goes to zero. For these terms it is very useful to use the new variables

$$w=(\lambda/\kappa)W, \quad 1-x=(\lambda/\kappa)X.$$

The explicit dependence on λ occurs in two ways: λ still appears in the integrand after cancelling out common powers of λ between numerator and denominator, and λ appears in the limits as κ/λ. It can be rigorously shown that it is correct to put $\lambda=0$ in both places simultaneously. There is left, then, only a definite integral without any dependence on parameters. It is possible to carry these integrals out more or less straightforwardly. Only terms that go to zero with λ have been dropped in this whole procedure.

The final result for this diagram is

$$M_1=-8\pi^2\left(\frac{e}{\hbar c}\right)\alpha^2\int d^4p_1d^4p_2\bar{\psi}(p_1)\gamma_\mu A_\mu{}^e(\Delta p)$$
$$\times(\Delta p/\kappa)^2\psi(p_2)m_1,$$
$$m_1=-(13/9)\ln\lambda-4.24\pm0.10.$$

The Evaluation of the Reducible Diagrams

The other diagrams are all reducible; these require the use of the modified $\bar{S}_F$, $\bar{D}_F$, or $\bar{\Gamma}_\mu$ functions. Since the form of the modified functions is sufficiently more complex than that of S_F, D_F, γ_μ, the organization of the terms is less unified. This tends to prevent a simple description of the calculations. Since only diagram 2 is of appreciable difficulty, further comments are limited to a few remarks on this diagram.

The diagrams 2′ and 2″ lead to equal contributions and we understand M_2 to mean $2M_{2'}$. Then:

$$M_2=8\pi i\left(\frac{e}{\hbar c}\right)\alpha\int d^4p_1d^4p_2A_\mu{}^e(\Delta p)\int d^4k$$
$$\times\bar{\psi}(p_1)\gamma_\nu\frac{i\gamma(p_1-k)-\kappa}{(p_1-k)^2+\kappa^2}\gamma_\mu\{\bar{\Gamma}_\mu(p_1-k, p_2-k)\}\frac{1}{k^2+\lambda^2}\psi(p_2). \quad (15)$$

The quantity in the brackets is the contribution of the corner vertex, the renormalization term having already been removed. This modified function is obtained explicitly in Karplus and Kroll.[11]

The prescription of the choice of factors in the denominator combination identity cannot be given simply here. However, by

[11] There is a transcription error in Karplus and Kroll Eq. (31); the definition of K_μ should read:

$$K_\mu(p', p''; u, v)$$
$$=(1-u)(i\gamma\cdot p'+\kappa)\gamma_\mu(i\gamma\cdot p''+\kappa)$$
$$-(i\gamma\cdot p'+\kappa)[\kappa(1-u^2)\gamma_\mu+i(1-u)(1-uv)(p'+p'')_\mu$$
$$-i(1-u+2uv)(1-uv)(p'-p'')_\mu]$$
$$-[\kappa(1-u^2)\gamma_\mu+i(1-u+uv)(1-u)(p'+p'')_\mu$$
$$+i(1-u+uv)(1+u-2uv)(p'-p'')_\mu](i\gamma\cdot p''+\kappa)$$
$$-(1-u)\gamma_\mu[(p'^2+\kappa^2)(1-uv)+(p''^2+\kappa^2)(1-u+uv)]$$
$$-i\kappa(p'-p'')_\mu u(1+u)(1-2v)+\gamma_\mu(p'-p'')^2[1-u+u^2v(1-v)]$$
$$+\kappa\sigma_{\mu\nu}(p'-p'')_\nu u(1-u).$$

suitable choices of the order of the factors and by suitable simple transformations, the auxiliary variable integrals can be made to fall into a useful classification system. Again, in the great majority of integrals it is correct simply to put $\lambda\equiv 0$. In those integrals in which λ must be kept, the procedure outlined for the analogous integrals of diagram 1 is applicable here; no further discussion of these is necessary. Of the remaining integrals, with $\lambda\equiv 0$, the group containing almost all the integrals can be written as

$$\Sigma c(k,m,n,s,t)J(k,m,n,s,t),$$

where the c's are constants and J is given by

$$J(k,m,n,s,t)=\int^1 du\int^1 dv\frac{u^s v^t}{(1-uv)^{n+k-m}}\times\int_0^1 dx\int_0^1 dy\frac{y^n x^{n+k-1-m}(1-ux(1-v)(1-uv))^m}{[yv(1-ux(1-v)/(1-uv))^2+ux/(1-uv)]^k}; \quad (16)$$

the k, m, n, s, t are positive integers such that $0\leq m\leq n+k-1$. The integrals that actually occurred are given by

$k=1^*$,
$k=2$, $n=1^*$, 2, 3,
$k=3$, $n=$ 2^*, 3, 4.

Those marked by an asterisk were evaluated before this system of classification was devised; the others are available. There is a very great simplification brought about by this system of classification, especially for the higher k. The use of a more naive system —say, according to powers of x in the numerator, rather than the $x^{n+k-1-m}[1-ux(1-v)/(1-uv)]^m$ is almost prohibitively laborious. The I's are easily evaluated by integrating in the order y, x, u, v; after the y, x integrations the u, v functions are of the same form as the two-variable integrals met with in diagram 1, and the tables used there are immediately applicable here.

Not all the integrals fit into this classification. A few integrals are found in which the power of x in the numerator is greater than $(n+k)$. The simple program of integrating in the order y, x, u, v is then stopped after the y integration by the presence of the integral:

$$(\text{function of } u,v)\int_0^1 dx\frac{1}{ax+b}\ln(cx+d);$$

a, b, c, d are functions of u and v. Since the result of the x integration is not expressible in terms of elementary functions, a more roundabout and much more laborious technique of integration must be used. While these integrals can again be carried through and expressed in terms of the previously mentioned set of transcendentals, the smaller terms of this kind were in fact estimated by means of rigorous upper and lower bounds.

The final result for diagram 2 is

$$m_2=-\tfrac{2}{3}(\ln\lambda)^2-(1/18)\ln\lambda+4.86\pm 0.11.$$

The diagrams 3, 4, 5 were carried out easily and exactly. The integrals were sufficiently few and simple so that no system of classification was required. The calculations in these diagrams were not much more difficult than the analogous magnetic moment calculations; in the case of the diagrams 1 and 2 the tasks were of a different order of magnitude. The integrals that were λ dependent were more difficult here than in the magnetic moment case; however, the procedure discussed above is efficacious here too.

Diagram 6 is easily dismissed. It consists of two second-order pieces joined by a photon line. Since the photon part is proportional to $(\Delta p/\kappa)^2$, the vertex part to $(\Delta p/\kappa)$ at least, the diagram as a whole cannot contribute terms of order $(\Delta p/\kappa)^2$.

The results for the five diagrams are

$$M=-8\pi^2\left(\frac{e}{\hbar c}\right)\alpha^2\int d^4p_1d^4p_2\bar{\psi}(p_1)\gamma_\mu A_\mu{}^e(\Delta p)\times(\Delta p/\kappa)^2\psi(p_2)m,$$

$$m=\sum m_i;$$

$$m_1=-(13/9)\ln\lambda-4.24\pm 0.10,$$

$$m_2=-\tfrac{2}{3}(\ln\lambda)^2-(1/18)\ln\lambda+4.86\pm 0.11,$$

$$m_3=-(77/432)\pi^2+(1099/648)=-0.06,$$

$$m_4=\tfrac{2}{3}(\ln\lambda)^2+(1/18)(\ln\lambda)-(17/36)\pi^2+1109/864=\tfrac{2}{3}(\ln\lambda)^2+(1/18)\ln\lambda-3.38,$$

$$m_5=(13/9)(\ln\lambda)+(91/216)\pi^2-(355/432)=(13/9)(\ln\lambda)+3.34,$$

$$m=0.52\pm 0.21. \quad (17)$$

Note that the sum of the five diagrams is infrared convergent; the use of the λ can now be regarded as completely formal.

This result corresponds to an addition to the interaction-energy-density

$$-\tfrac{1}{2}ie\bar{\psi}\gamma_\mu\psi[\Box^2/\kappa^2A_\mu{}^e][(\alpha^2/2\pi^2)m].$$

Such a perturbation contributes to the level shift in a hydrogenic atom an energy:

$$\alpha^4Z^4/n^3\,\mathrm{Ry}[(4/\pi^2)m]\delta_{l,0}.$$

For the $2S$ level of hydrogen this is

$$0.24\pm 0.10 \text{ megacycle per second.}$$

Again it should be remarked that the error indicated arises from the fact that some of the integrals were estimated, using rigorous upper and lower bounds, instead of being evaluated exactly. The error stated represents the outside limits; the actual error is probably much smaller.

The result obtained is somewhat small in terms of the order-of-magnitude estimates given in the introduction. There does not seem to be any particular reason for anticipating a small coefficient. In the corresponding calculation of the fourth-order magnetic moment, the coefficient was large.

For completeness we note that the fourth-order magnetic moment contributes -0.70 Mc/sec to the $2S_{\frac{1}{2}}$, $+0.24$ Mc/sec to the $2P_{\frac{1}{2}}$ level, and -0.12 Mc/sec to the $2P_{\frac{3}{2}}$ levels in hydrogen.