The Theory of Finite Displacement Operators and Fundamental Length

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The general finite displacement operator in an n-dimensional complex continuum is defined as an arbitrary superposition of exponential Taylor operators (a Taylor operator yields a Taylor's series). On restriction to the space-time continuum and four-dimensional time-like intervals of constant length ω, the corresponding finite displacement operator may be considered as an ordinary function of the operators u_σ (the partial derivative with respect to x_σ), and must satisfy a Klein-Gordon type equation in u space. This equation possesses relativistic invariant and four-vector solutions that in the limit $\omega\rightarrow 0$ reduce to 1 and u_σ, respectively. These operators are combined with the Compton wavelength k and the Dirac or Duffin γ_σ, respectively, to produce a relativistically invariant correspondence type finite-displacement operator generalization of the Dirac-Duffin equation. If the fields are charged, the electromagnetic potentials may be introduced in a manner which leaves the mass spectrum unaltered. The relationship to other nonlocal theories and to the reciprocity theory of Born is briefly considered.

INTRODUCTION

THE purpose of this paper is to give a new mathematical formulation of a theory of fundamental length first proposed in a preceding paper.[1] The approach taken in the present paper arises out of a recent observation of E. Schrödinger, kindly called to our attention by Freistadt,[2] that our scalar averaging operator, Eq. (8), considered as a function of the variables u_σ, is a solution of the Klein-Gordon equation (2). No knowledge of the former mathematical methods is required in order to follow those used here, since the latter methods are entirely independent of the former. Furthermore, because of the great dissimilarity of the methods it seems desirable to restate the necessary basic postulates of the theory and to make the development from them *ab initio*. We also treat briefly the relationship of this theory to some proposed by others. However, general field theoretic questions will be left for a later paper.

For the present purposes, it is appropriate to take as our point of departure the following set of postulates:[1]

A. *One* new fundamental real constant ω with the dimensions of length shall be introduced into the theory; we shall call it a fundamental length.

B. The usual differential laws (field equations) shall be replaced by finite displacement laws such that the field equations at *any* given point involve the field quantities at only those points whose "distance" from the given point satisfies the time-like condition

$$\Sigma_\sigma\, \Delta x_\sigma^2+4\omega^2=0.$$

C. There shall be a *correspondence principle* such that in the limit as ω approaches zero the new laws reduce to the old differential laws (in which only the infinitesimal neighborhood of any given point enters the laws).

D. The laws shall be relativistically invariant.

With regard to postulate B, we must emphasize that we use a continuum and not a lattice or any other quantization of space. This continuum is the space-time continuum of special relativity, and we shall label its points by x_σ ($\sigma=1$ to 4), where x_k ($k=1$ to 3) are the usual space coordinates and are purely real, while $x_4=ix_0=ict$ is purely imaginary, and t is the time. The wave function $\psi(x_\sigma)$ satisfies a certain infinite-order differential equation (9) at all points of the space-time continuum. It is shown in Sec. I that in virtue of postulate B the infinite-order differential operators, $D(u_\lambda)$, which we call finite-displacement operators, by means of which the wave equation (9) is constructed, must be solutions of Eq. (3), *viz.*,

$$[\sum \Delta x_\sigma^2+4\omega^2]D(u_\lambda)=0;$$

in this equation $D(u_\lambda)$ is to be treated as an ordinary function of the independent variables u_λ, and the Δx_σ are symbolic notations for the operators $\partial/\partial u_\sigma$, i.e., $\Delta x_\sigma=\partial/\partial u_\sigma$ in (3). We write (3) in this form to bring out the intimate connection between (1) and (3), which is such that by setting $\Delta x_\sigma=\partial/\partial u_\sigma$ in (1), it becomes the operator on $D(u_\lambda)$ in (3). Any solution of (3) furnishes us with a differential operator on replacing the u_σ by $\partial/\partial x_\sigma$. For the formulation of the theory, then, we may replace postulate B by the postulate that the finite-displacement operators are solutions of Eq. (3).

According to postulates C and D, we seek relativistic scalar and four-vector solutions of (3), $f(u)$ and $g(u)u_\sigma$, respectively, to combine with the scalar $k(=mc/\hbar)$ and the Dirac or Duffin matrices γ_σ to form the wave equation,

$$[\gamma_\sigma g(u)u_\sigma+kf(u)]\psi=0,$$

which in the limit $\omega\rightarrow 0$ reduces to the usual differential laws,

$$(\gamma_\sigma u_\sigma+k)\psi=0.$$

This is carried out in Sec. II, where it is shown that the above postulates uniquely determine the form of the wave equation [see Eq. (9)]. The functions $f(u)$

[1] B. T. Darling, Phys. Rev. 80, 460 (1950).
[2] H. Freistadt, Compt. rend. 235, 23 (1952).

and $g(u)$ are even functions of u, where

$$u^2=\sum_{\sigma=1}^{4}\frac{\partial^2}{\partial x_\sigma^2}=\sum_{k=1}^{3}\frac{\partial^2}{\partial x_k^2}-\frac{\partial^2}{\partial x_0^2}.$$

The wave equation (9) of Sec. II possesses a mass spectrum in virtue of the time-like intervals assumed in postulate B.[1] That a spectrum of masses results may perhaps be expected from the uncertainty principle $\Delta E\Delta t\cong\hbar$ which, for an essential uncertainty in the proper time, $\Delta\tau=\omega/c$, reduces to $\Delta m\Delta\tau\cong\hbar/c^2$.

In the case of a charged field, the electromagnetic potentials are introduced so as not to disturb the mass quantization, and the u_σ are replaced with $u_\sigma-(ie/\hbar c)A_\sigma$ as usual.

I. FINITE DISPLACEMENT OPERATORS

Let x_σ ($\sigma=1,\cdots n$) be the coordinates of a point in an n-dimensional space; this space is assumed to be a *continuum* and may in general be *complex*. The individual x_σ may range over all complex numbers, or some may be confined to purely real numbers while others may be confined to purely imaginary numbers, etc. Such restrictions will not invalidate the argument of this section. Consequently the derivations will then be immediately applicable to the case of the space-time continuum for which the space coordinates x_σ ($\sigma=1$, 2, 3) are purely real, while x_4 ($\sigma=4$) is purely imaginary since we set $x_4=ix_0=ict$ where t is the time.

Now let Δx_σ be the components of some finite vectorial displacement of this space on itself, so that every point x_σ is displaced to another point $x_\sigma+\Delta x_\sigma$. Let $F(x_\sigma)$ be a function defined throughout this space. Then the value of the function at any subsequent position may be expressed in terms of the value of the function at the corresponding former position by means of Taylor's theorem, which we express in the well-known operator form

$$F(x_\sigma+\Delta x_\sigma)=\exp(\Delta x_\sigma u_\sigma)F(x_\sigma),$$

where $u_\sigma=\partial/\partial x_\sigma$ denotes the partial derivative with respect to x_σ, and the summation convention of repeated index is understood. We shall speak of $\exp(\Delta x_\sigma u_\sigma)$ as the finite displacement operator corresponding to the displacement Δx_σ. More generally, any linear combination of such operators corresponding to different displacements Δx_σ will be spoken of as a finite displacement operator; in particular, this combination may consist of a continuous distribution of displacements. With these definitions it is to be understood that the functions are defined throughout the space, and that the operators may be applied to them at every point.

In the case of one-dimension, consider the following two finite displacement operators f_ω and g_ω:

$$f_\omega\psi(x)=[\psi(x+\omega)+\psi(x-\omega)]/2,$$

and

$$g_\omega\psi(x)=[\psi(x+\omega)-\psi(x-\omega)]/2\omega,$$

associated with the displacements $\Delta x=+\omega$ and $\Delta x=-\omega$. It is clear that the averaging operator f_ω and the differencing operator g_ω have the properties

$$\lim_{\omega\to 0}f_\omega=1,\quad \lim_{\omega\to 0}g_\omega=u=d/dx,$$

and that

$$f_\omega=[e^{\omega u}+e^{-\omega u}]/2=\cosh(\omega u)$$
$$g_\omega=[e^{\omega u}-e^{-\omega u}]/2\omega=\sinh(\omega u)/\omega.$$

The operators g_ω and f_ω are related by

$$g_\omega=(1/\omega^2)df_\omega/du,$$

where the derivative of f_ω on the right side is to be taken in the ordinary way considering f_ω as a function of u. Both f_ω and g_ω considered as functions of u are solutions of

$$[d^2/du^2-\omega^2]D(u)=0.$$

Returning to the general case, let the displacement Δx_σ satisfy

$$\textstyle\sum_\sigma \Delta x_\sigma^2+4\omega^2=0, \tag{1}$$

where ω is a constant (in general complex). Then the finite displacement operator $e^{\Delta x_\sigma u_\sigma}$ corresponding to Δx_σ, when considered as a function of the u_σ treated as independent variables, is such that

$$\sum_\sigma \frac{\partial^2}{\partial u_\sigma^2}e^{\Delta x_\lambda u_\lambda}=\sum_\sigma \Delta x_\sigma^2 e^{\Delta x_\lambda u_\lambda}=-4\omega^2 e^{\Delta x_\lambda u_\lambda}.$$

This result makes it apparent that any linear combination $D(\mathbf{u})$ of operators corresponding to displacements satisfying (1), when considered as a function of the independent variables u_σ, must obey the equation

$$[\textstyle\sum_\sigma \partial^2/\partial u_\sigma^2+4\omega^2]D(\mathbf{u})=0. \tag{2}$$

Here $\mathbf{u}$ is the vector with components u_σ. The last equation takes a more suggestive form if we convert the left side of the equation (1) into an operator by the identification $\Delta x_\sigma\to\partial/\partial u_\sigma$, whence (1) is replaced by (2) in the form

$$[\textstyle\sum_\sigma \Delta x_\sigma^2+4\omega^2]D(\mathbf{u})=0. \tag{3}$$

Thus Eq. (1), defining the finite displacements, Δx_σ, is at the same time the operational form of Eq. (3) which the corresponding finite displacement operators must satisfy.

From now on we shall be concerned only with the space-time continuum, for which x_σ ($\sigma=1$, 2, 3) are purely real while x_4 is purely imaginary. It is clear now that if there are to be any finite displacement laws in physics involving one new fundamental constant ω in accordance with postulates A and B, they must be formed from finite-displacement operators $D(\mathbf{u})$ operating on $\psi(x_\sigma)$ defined over the space-time continuum of special relativity, and the $D(\mathbf{u})$ must be solutions of

(2). From now on the constant ω in (1) will also be understood to be a positive real number. On setting $\Delta x_4 = i\Delta x_0$, Eq. (1) takes the familiar form

$$\Delta x_0^2 - \sum_{k=1}^{3} \Delta x_k^2 = 4\omega^2, \tag{4}$$

from which it is apparent that the intervals are time-like as required for mass quantization.

II. COVARIANT FINITE DISPLACEMENT OPERATORS AND THE GENERALIZED LAWS OF MOTION

The operator in the Dirac-Duffin-Kemmer equation, which we seek to generalize in accordance with the postulates A to D, may be written

$$\gamma_\sigma u_\sigma + k. \tag{5}$$

It consists of the sum of two scalars k and $\gamma_\sigma u_\sigma$, the latter consisting of the scalar product of the 4-vector differential operator, $u_\sigma = \partial/\partial x_\sigma$, with the Dirac or Duffin matrices denoted indifferently by the γ_σ which, of course, must satisfy the appropriate commutation relations for the respective cases. In order to convert (5) into a finite displacement operator over the time-like intervals defined by (1), we must seek a scalar and vector solution of (2) with the properties:

(a) The scalar operator $f(u)$ shall be relativistically invariant and shall satisfy the correspondence requirement $\lim_{\omega\to 0} f(u) = 1$.

(b) The vector operator $g(u)u_\sigma$ shall be a 4-vector and $\lim_{\omega\to 0} g(u)u_\sigma = u_\sigma$.

We shall see that the conditions (a) and (b) uniquely determine the operators.

(a) The Scalar Finite-Displacement Operator

It is a well-known fact that the only Lorentz invariant functions of u_σ are functions of $u = [\sum_\sigma u_\sigma^2]^{\frac{1}{2}}$ alone.[3] Consequently our scalar operator is a function $f(u)$ of u alone. On substituting $f(u)$ into the differential equation (2) there results the ordinary differential equation,

$$f'' + (3/u)f' + 4\omega^2 f = 0, \tag{6}$$

in which the primes denote first and second derivatives of $f(u)$ with respect to u. By changing the independent variable to $z = 2\omega u$ and setting $f = J(z)/z$, we find that J satisfies Bessel's differential equation,

$$J'' + (1/z)J' + (1 - 1/z^2)J = 0, \tag{7}$$

for Bessel's functions of order one, the primes now denoting derivatives with respect to z. The correspondence condition limits J to the solution regular at the origin and we find

$$f(u) = 2J_1(z)/z, \tag{8}$$

where $J_1(z)$ is the usual Bessel's function of order one.[4] Thus we see that Lorentz invariance and the correspondence limit alone uniquely determine $f(u)$.

(b) The Vector Finite-Displacement Operator

Just as the finite difference operator g_ω of Sec. I is related to the averaging operator f_ω for the one-dimensional case, so here the vector operator satisfying (2) is related to the scalar operator deduced under (a) in the same way. Thus

$$-\frac{1}{\omega^2}\frac{\partial f}{\partial u_\sigma} = \frac{8J_2(z)}{z^2} u_\sigma$$

satisfies (2) and is such that

$$\lim_{\omega\to 0} \frac{8J_2(z)}{z^2} u_\sigma = u_\sigma,$$

as required by the correspondence principle. Again the relativistic property, together with the correspondence requirement, uniquely determines the solution. It may be mentioned that the scalar and vector operators and the relationship between them are precisely the same as obtained in the first paper.

We may sum up the considerations thus far in the following theorem:

The only finite displacement operators that can be combined with the scalar k and the four-vector γ_λ to form the invariant law of motion in which the displacements Δx_σ are bound by the time-like relation,

$$\sum_\sigma \Delta x_\sigma^2 + 4\omega^2 = 0,$$

involving the introduction of *one* new fundamental (real) constant ω having the dimensions of a length, are the relativistic scalar and vector solutions of

$$[\sum_\sigma \Delta x_\sigma^2 + 4\omega^2] D(u) = 0,$$

where $\Delta x_\sigma = \partial/\partial u_\sigma$; the correspondence principle that $\omega\to 0$ yield the Dirac-Duffin-Kemmer equation restricts the operators to the regular solutions, and we obtain, uniquely, the generalized wave equation

$$\left[-\frac{1}{\omega^2}\gamma_\sigma\frac{\partial}{\partial u_\sigma} + k\right]\frac{2J_1(z)}{z}\cdot\psi = 0, \tag{9}$$

where $z = 2\omega u$, $u = [\sum u_\sigma^2]^{\frac{1}{2}}$, and $u_\sigma = \partial/\partial x_\sigma$.

The wave equation (9) may be written (on remembering $\Delta x_\sigma = \partial/\partial u_\sigma$) in the form

$$[-\gamma_\sigma \Delta x_\sigma + k\omega^2] f(z)\cdot\psi = 0, \tag{10}$$

where $f(z) = 2J_1(z)/z$ is the scalar solution of

$$[\sum_\sigma \Delta x_\sigma^2 + 4\omega^2] f(z) = 0, \tag{11}$$

[3] H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, 1939), first edition, p. 27.

[4] E. Schrödinger, Proc. Roy. Irish Acad. **47A** 1 (1941).

and satisfies

$$\lim_{\omega\to 0} f(z)=1.$$

The dot in (9) and (10) indicates that the operations are to be performed on $f(z)$ first and then the result converted to an operator on ψ by the identification $u_\sigma=\partial/\partial x_\sigma$.

III. INTRODUCTION OF THE ELECTROMAGNETIC POTENTIALS

When we come to introduce the electromagnetic potentials into Eqs. (9) and (10) in case the fields are charged, two conditions presumably should be satisfied. First, the resulting wave equation should be gauge invariant because of the well-known arbitrariness of the electromagnetic potentials under gauge transformations; and second, it should not interfere with the quantization of the mass. The gauge invariance is obtained in the usual manner by replacing u_σ by $U_\sigma=u_\sigma-(ie/\hbar c)A_\sigma$. Whereas, the invariance of mass quantization is achieved by replacing $z=2\omega u$ in the operator of Eqs. (9) or (10) by $z=-2\omega\gamma_\lambda u_\lambda$. That this is so may be seen by noting that the operator in (9) or (10) is then a function of z alone. Indeed, either equation after carrying out the indicated operations and trivial multiplications or divisions by ω may be written

$$D(z)\psi=0, \tag{12}$$

with

$$D(z)=[-2J_2(z)+\bar{k}J_1(z)]/z, \tag{13}$$

in which $z=-2\omega\gamma_\lambda u_\lambda$ and $\bar{k}=\omega k=\beta e^2/\hbar c$.[5] The dimensionless quantity β is defined by setting $\omega=\beta e^2/mc^2$ and is of order unity. If we define an operator Z by setting $Z=-2\omega\gamma_\lambda U_\lambda$, then the wave equation including the potentials and satisfying all requirements is

$$D(Z)\psi=0. \tag{14}$$

It is apparent that if z_n (a real number)[6] is a root of $D(z_n)=0$, and if ψ_n is a solution of $(Z-z_n)\psi_n=0$, then ψ_n is also a solution of Eq. (14).

IV. RELATION TO OTHER THEORIES

It has been proposed at various times to make up operators by taking finite or infinite products of Dirac operators (5) with any arbitrarily chosen spectrum of masses, or by introducing (arbitrarily or by other means) transcendental functions of $\gamma_\lambda u_\lambda$.[7] Obviously, in the first type of theory there does not exist a correspondence principle. It is furthermore apparent that there is no means of fixing upon any spectrum of masses—this being left arbitrary. But more important, both types of theories cannot be considered as theories of fundamental length as such, for they consist of introducing not just one new fundamental constant length, but rather, as we shall see, a whole spectrum of lengths specified by a weight function. In still other theories[8] smearing functions of various quantities are introduced. Here also a spectrum of lengths specified by an arbitrary function is introduced.

Where smearing functions are used, it is usually apparent on simple inspection of the function that a spectrum of lengths is involved, with the smearing function itself playing the part of the weight function. In other cases we may express them as Fourier-Bessel integrals over ω of the averaging operator (8). We may then forthwith turn our attention to the first-mentioned theories.[7]

We shall find it convenient to feature the dependence of the $D(z)$ of Eq. (13) on ω. Let $x=-\gamma_\lambda u_\lambda$, then $z=\omega x$, and

$$D(z)=D(\omega x)=[-2J_2(\omega x)+\bar{k}J_1(\omega x)]/\omega x. \tag{15}$$

Since no correspondence limit $\omega\to 0$ will be involved in the following we may hold $\bar{k}$ fixed at its value $\beta(e^2/\hbar c)$. Our problem then is to express an operator $g(x)$ of one of the above theories in the form

$$g(x)=\int_0^\infty h(\omega)D(\omega x)d\omega. \tag{16}$$

The weight function, $h(\omega)$, specifying the spectral composition of the lengths ω involved in the operator $g(x)$, may be found by means of Mellin transforms.[9] If $G(s)$ is the Mellin transform of $g(x)$, then

$$G(s)=\int_0^\infty g(x)x^{s-1}dx, \tag{17}$$

and

$$g(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)x^{-s}ds. \tag{18}$$

In the latter integral the path of integration is along a line parallel to the imaginary axis and at some suitably chosen distance c to the right of that axis. Taking the transform of (16) and reversing the order of integra-

[5] We wish to call attention to an error in footnote (25) of the first paper, which incorrectly set $z=2\omega\gamma_\lambda u_\lambda$ and gives an incorrect sign to the J_2 term of Eq. (13) above.

[6] It is not difficult to prove that $J_{\nu+1}(z)+kJ_\nu(z)=0$, where k is real and ν is real and greater than minus one, has only real roots.

[7] F. Bopp, Ann. Physik **38**, 345 (1940); Z. Naturforsch. **1**, 53 (1946). A. Landé and L. H. Thomas, Phys. Rev. **60**, 121, 514 (1940); **65**, 175 (1944). B. Podolsky and C. Kikuchi, Phys. Rev. **62**, 68 (1942); **65**, 228 (1944). D. Blokhinzev, J. Phys. U.S.S.R. **11**, 72 (1947). D. J. Montgomery, Phys. Rev. **69**, 117 (1947). A. Green, Phys. Rev. **72**, 628 (1947). B. Podolsky and P. Schwed, Revs. Modern Phys. **20**, 40 (1948). L. de Broglie, Compt. rend. **229**, 157, 269, 401 (1949). A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950). W. Heisenberg, Z. Naturforsch. **5a**, 251, 367, 373 (1950).

[8] F. Bopp, Ann. Physik **42**, 573 (1943). R. P. Feynman, Phys. Rev. **74**, 939 (1948). R. Peierls and H. McManus, Proc. Roy. Soc. (London) **A195**, 323 (1948). J. Irving, Proc. Phys. Soc. (London) **A62**, 780 (1949).

[9] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford, 1937), first edition, pp. 7 and 315.

tions, we obtain

$$G(s)=\int_0^\infty h(\omega)d\omega\int_0^\infty x^{s-1}D(\omega x)dx.$$

On changing the variable x to $v=\omega x$, this becomes

$$G(s)=\int_0^\infty h(\omega)\omega^{-s}d\omega\int_0^\infty v^{s-1}D(v)dv.$$

From the form of (17) we see that

$$G(s)=H(1-s)\mathfrak{D}(s), \tag{19}$$

where

$$H(s)=\int_0^\infty h(\omega)\omega^{s-1}d\omega,$$

and

$$\mathfrak{D}(s)=\int_0^\infty D(v)v^{s-1}dv,$$

are the Mellin transforms of h and D. Solving (19) for H and making use of (18) we have

$$h(\omega)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{G(1-s)}{\mathfrak{D}(1-s)}\omega^{-s}ds,$$

the desired solution.

The Mellin transform of $D(v)$ may be obtained by use of a formula from the theory of Bessel functions,[10]

$$\int_0^\infty\frac{J_\nu(t)dt}{t^{\nu-\mu+1}}=\frac{\Gamma(\frac{1}{2}\mu)}{2^{\nu-\mu+1}\Gamma(\nu-\frac{1}{2}\mu+1)},$$

valid when $0<\mathrm{R}(\mu)<\mathrm{R}(\nu)+\frac{1}{2}$, where R means the real part. By straightforward application of this formula we find

$$\mathfrak{D}(s)=-\frac{2\Gamma[(s+1)/2]}{2^{2-s}\Gamma[(5/2)-(s/2)]}+k\frac{\Gamma(s/2)}{2^{2-s}\Gamma(2-\frac{1}{2}s)},$$

valid when $0<\mathrm{R}(s)<\frac{3}{2}$. The constant c in (18) must of course be chosen in this range.

We next turn our attention to the reciprocity theory of Born.[11] His position is that the basic laws of physics are self-reciprocal between coordinate and momentum space, a self-reciprocal function being defined in two different not exactly equivalent ways[12] as either functions that are their own Fourier transforms or eigenfunctions $F(p)$ of

$$S(x,p)F(p)=sF(p),$$

where

$$S(x,p)=S(p,-x)$$

is a reciprocal invariant and $x_k\rightarrow -i\partial/\partial p_k$. The basic reciprocal invariant function which he introduces is the eight-dimensional distance in phase space,

$$S=x_kx^k+p_kp^k, \tag{20}$$

which we may write in our notation as

$$\sum\Delta x_\sigma{}^2-\sum\Delta u_\sigma{}^2,$$

since $p_k=-i\Delta u_k$. Note that the x and p in (20) are dimensionless quantities (see later).

At this point it is clear that our operator (15) is not self-reciprocal in either sense and that while the so-called "reciprocal" Klein-Gordon equation (3) plays an important role in the theory of fundamental length, it is not reciprocally invariant. We treat the coordinate space on an entirely different footing than the momentum space. This divergence between the theories becomes more apparent when we proceed further. For, contrary to determining a finite displacement operator in the manner we do from Eq. (1), Born determines an operator $F(p)$ from the eigenvalue problem,[13]

$$[-\partial^2/\partial p_k\partial p^k+p_kp^k]F_l(p_\lambda)=s_lF_l(p_\lambda).$$

Thus, the eight-dimensional distance (20) takes on a spectrum of eigenvalues s_l with their corresponding eigenoperators F_l. Consequently, Born's theory in our opinion is not a theory of fundamental length, but is rather a spectral theory of "distance" in phase space.

Finally, in Born's theory there is no correspondence principle such as our $\omega\rightarrow 0$ limit. The x and p in formula (20) are dimensionless quantities in terms of a length a and momentum b, where $ab=\hbar$. The masses are given by

$$\mu=bk/c=\hbar k/ac,$$

where c is the velocity of light and $k=(p_\lambda p^\lambda)^{\frac{1}{2}}$ is a root of $F_l(p_\lambda)=0$. Then as $a\rightarrow 0$ (corresponding to our $\omega\rightarrow 0$) all $\mu\rightarrow\infty$. The electron is treated on a different footing in Born's theory and, so to speak, stands outside his mass quantization procedure.

We conclude this section with the mention of a theory of Landé[14] which makes use of time-like intervals obeying the form (1). However, he makes use of (1) in an entirely different manner than we do, with the consequence that his theory leads to a spectrum of charges with one mass whereas we obtain a spectrum of masses with one charge.

ACKNOWLEDGMENT

We wish to thank Dr. E. Schrödinger for his valuable advice and criticism in the preparation of this paper.

[10] G. N. Watson, *Theory of Bessel Functions* (University Press, Cambridge, 1922), second edition, p. 391.
[11] M. Born, Revs. Modern Phys. **21**, 463 (1949).
[12] E. Schrödinger, Proc. Roy. Irish Acad. **55A2**, 29 (1952).
[13] One should consult the paper of Schrödinger (reference 12) for a careful consideration of this eigenvalue problem as well as the problem of Fourier reciprocity.
[14] A. Landé, J. Franklin Inst. **229**, 767 (1940).