Matrix Elements of β-Decay in jj Coupling

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Matrix elements of various allowed β-transitions are calculated in the jj coupling scheme. It is found that for Gamow-Teller matrix elements there is no distinction between favored and unfavored transitions, which appears to be in contrast with the experimental facts.

THE success of the shell model in explaining the experimental facts about the low-lying levels of nuclei is often attributed to the jj coupling scheme. Yet one could think of a model in which the order of the single nucleon levels is that of, say, the harmonic oscillator potential well but in which the order of the different states is determined by short-range attractive forces in the LS coupling scheme; this model would successfully explain the spins and parities of the low-lying nuclear levels. Whereas in the jj coupling model the strong spin-orbit interaction accounts for the magic numbers, in an LS coupling model some assumption is required which will lead to a break in the energy of the l shell after the filling of the first $2l+2$ (identical) nucleons. Also, spin-orbit interaction does not lead to jj coupling unless it is large as compared to the mutual interaction. In the light of these facts it seems interesting to test the jj coupling wave functions by using them to calculate matrix elements of allowed β-transitions in light nuclei. Although it might be argued that in these cases jj coupling is not fully manifested, it is not unreasonable to assume that it can already serve as a good approximation. The results show that jj coupling is not an adequate scheme in the mass range in which image transitions occur.

In the supermultiplet theory, which yields an LS coupling scheme, β-transitions are allowed only between states which belong to the same supermultiplet.[1] If small deviations from Wigner's first approximation[2] take place, supermultiplets will mix and transitions which were forbidden will occur; yet their matrix elements will be smaller than those of the transitions which were allowed. This can explain the occurrence of distinct favored and unfavored transitions among those which are allowed according to the spin, parity, and isotopic spin selection-rules. (These allowed transitions are separated into distinct favored and unfavored groups, characterized by widely separated values of the ft values.) In jj coupling there are no supermultiplets but only charge multiplets, and one cannot see any selection-rule that will make a distinction between the Gamow-Teller matrix elements of favored and unfavored transitions. The Fermi matrix elements between two states will vanish also in jj coupling, unless the two states have the same value of the isotopic spin T and belong to the same charge-multiplet, and will, therefore, contribute to transitions between ground states only in the case of mirror nuclei. It can be said that in jj coupling there is a strong mixing of supermultiplets and this destroys the distinction between favored and unfavored transitions, yet it could be hoped that the complexity (in this sense) of the jj coupling wave functions would yield big matrix elements for the actually observed favored transitions and small matrix elements for the unfavored.[3] The cases in which matrix elements were calculated before[4] did not justify such a hope, and the calculations presented here show more clearly that matrix elements of transitions which are found experimentally to be favored are not essentially bigger (and even smaller in some cases) than those of transitions with much higher ft values. Also, Feenberg[5] has carried out similar calculations and arrived at the same results. This shows that the jj coupling scheme does not give a good approximation to the wave functions at least in the region where the distinction between favored and unfavored transitions is observed. The highest A for which this is found in $A=43$, which is already in the region of what is described as the $f_{7/2}$ shell. For higher A, image transitions are not observed and no definite conclusions can be drawn.

In the following, we shall use the isotopic spin formalism and the method of tensor operators[6] in the calculation of the β-decay matrix elements. The operator which causes the transitions is a double tensor and has the form

$$\mathfrak{M}_q^{(k)}=\sum_i \tau_{\eta i} B_{qi}^{(k)}, \tag{1}$$

where $\tau_\eta=\sqrt{2}(t_{+1}+t_{-1})$ and $B_q^{(k)}$ is a tensor operator of

[1] E. P. Wigner, Phys. Rev. **56**, 519 (1939).
[2] E. P. Wigner, Phys. Rev. **51**, 106 (1937).
[3] One has in mind the case of the spin of the ground state which is the same in the case of short-range forces in LS coupling and in jj coupling and the example of the magnetic moment of the ground state in the case of n (odd) identical nucleons. This has the Schmidt value either in the ground state j^n, $J=j$ or in the ground state of l^n, ${}^2l_j$. In this case the magnetic moment is an odd tensor operator (being simply a vector) and its value depends on the seniority of the state but is independent of n, and as the case of $v=1$ in both coupling schemes is the case of a single nucleon, the values of the magnetic moments are equal for $v=1$ in LS coupling and in jj coupling. The Gamow-Teller operator is, however, an even double tensor (as a part of the magnetic moment is in the general case) and therefore not independent of n, so that the property described here should not necessarily hold.
[4] E. P. Wigner, "The jj Coupling Shell Model for Nuclei," Wisconsin Lecture Notes, 1952 (unpublished).
[5] E. Feenberg, *Nuclear Shell Structure* (Princeton University Press, Princeton, to be published).
[6] G. Racah, Phys. Rev. **62**, 438 (1942). The notation defined there will be used in this paper.

degree k ($k=0$, $B_0^{(0)}=1$ for Fermi interaction, whereas $k=1$, $B_q^{(1)}=\sigma_q$ for Gamow-Teller interaction). In order to illustrate the method of calculation we derive here the matrix elements for the case of a single nucleon β-decay in this formalism.

The matrix element of (1) between the state characterized by l, j, m, $t=\frac{1}{2}$, $t_z=\frac{1}{2}$ and the state l, j', m', $t'=\frac{1}{2}$, $t_z'=-\frac{1}{2}$ is given by

$$\begin{aligned}(\mathfrak{M}_q^{(k)})_{mm'}&=(ljm\tfrac{1}{2}\tfrac{1}{2}|\tau_\eta B_q^{(k)}|lj'm'\tfrac{1}{2}-\tfrac{1}{2})\\&=(-1)^{j+m+\frac{1}{2}+\frac{1}{2}}(lj\tfrac{1}{2}\|\sqrt{2}tB^{(k)}\|lj'\tfrac{1}{2})\\&\quad\times V(jj'k;-mm'q)V(\tfrac{1}{2}\tfrac{1}{2}1;-\tfrac{1}{2}-\tfrac{1}{2}1)\\&=-(-1)^{j+m}\sqrt{2}(\tfrac{1}{2}\|t\|\tfrac{1}{2})V(\tfrac{1}{2}\tfrac{1}{2}1;-\tfrac{1}{2}-\tfrac{1}{2}1)\\&\quad\times(lj\|B^{(k)}\|lj')V(jj'k;-mm'q),\qquad(2)\end{aligned}$$

as in this case only t_{+1} contributes to the transition. Introducing the values

$$(\tfrac{1}{2}\|t\|\tfrac{1}{2})=(\tfrac{3}{2})^{\frac{1}{2}}\quad\text{and}\quad V(\tfrac{1}{2}\tfrac{1}{2}1;-\tfrac{1}{2}-\tfrac{1}{2}1)=(3)^{-\frac{1}{2}},$$

we obtain

$$(\mathfrak{M}_q^{(k)})_{mm'}=-(-1)^{j+m}(lj\|B^{(k)}\|lj')\times V(jj'k;-mm'q).\qquad(3)$$

In order to compute the transition probability, we have to sum $|(\mathfrak{M}_q^{(k)})_{mm'}|^2$ over q and m' for a fixed value of m (it should then be independent of m). Instead it is more convenient to sum $|(\mathfrak{M}_q^{(k)})_{mm'}|^2$ over q, m, and m' and divide by $2j+1$. Using the properties of the V's we obtain

$$\begin{aligned}&\frac{1}{2j+1}\sum_q\sum_{mm'}|(\mathfrak{M}_q^{(k)})_{mm'}|^2\\&=\frac{1}{2j+1}(lj\|B^{(k)}\|lj')^2\sum_q\sum_{mm'}V(jj'k;-mm'q)^2\\&=\frac{1}{2j+1}(lj\|B^{(k)}\|lj')^2\sum_q\frac{1}{2k+1}\\&=\frac{1}{2j+1}(lj\|B^{(k)}\|lj')^2.\qquad(4)\end{aligned}$$

This is the quantity usually written as $|\int 1|^2$ or $|\int\sigma|^2$. The connection with the usual formulation is given by

$$\begin{aligned}(lj\|1\|lj')^2&=(2j+1)\delta_{jj'},\\(l,l+\tfrac{1}{2}\|\sigma\|l,l+\tfrac{1}{2})^2&=(2j+1)(j+1)/j\\&=2(l+1)(2l+3)/(2l+1),\\(l,l-\tfrac{1}{2}\|\sigma\|l,l-\tfrac{1}{2})^2&=(2j+1)j/(j+1)\\&=2l(2l-1)/(2l+1),\\(l,l\pm\tfrac{1}{2}\|\sigma\|l,l\mp\tfrac{1}{2})^2&=8l(l+1)/(2l+1).\end{aligned}\qquad(5)$$

In the treatment of wave functions of many nucleons we shall assume that the total isotopic spin T is a good quantum number. This assumption is certainly justified in the region $A<50$ as in this region the protons and neutrons occupy the same shells and highly favored image transitions occur (charge symmetry is not enough to account for the big matrix elements between mirror nuclei). The easiest way to calculate the matrix elements is to express the totally antisymmetrized wave function of n nucleons by means of the coefficients of fractional parentage (c.f.p.). These are defined by[7]

$$\psi(j^n\alpha TT_zJM)=\sum_{\alpha'T'J'}\psi(j^{n-1}(\alpha'T'J')jTT_zJM)\times(j^{n-1}(\alpha'T'J')jTJ|\}j^n\alpha TJ).\qquad(6)$$

In the case $n=2$ the c.f.p. are given by means of trivial considerations, but in the case $n=3$, which will be treated in the following, the evaluation of the c.f.p. requires some work. Most of the c.f.p. that will be used below are given by Edmonds and Flowers.[8] The matrix elements can be calculated in terms of the c.f.p. by using the analogs of the formulas (in Sec. 5 of reference 7). The l, L, and S appearing in those formulas should be properly replaced by j, J, and T.

We shall compare the matrix element of the transition I from the state j^3, J, $T=\frac{3}{2}$ to the state j^3, J', $T=\frac{1}{2}$, to those of the image transition II from j^3, J, $T=\frac{1}{2}$, $T_z=\frac{1}{2}$ to j^3, J, $T'=\frac{1}{2}$, $T_z'=-\frac{1}{2}$. The matrix elements for the case of transitions within the j shell are given by

$$\begin{aligned}(\mathfrak{M}_q^{(k)})_{MM'}&=(j^n\alpha TT_zJM|\textstyle\sum_i\tau_{\eta i}B_{qi}^{(k)}|j^n\alpha'T'T_z'J'M')\\&=n\sum_{\alpha_1T_1J_1}(j^n\alpha TJ\{|j^{n-1}(\alpha_1T_1J_1)jTJ)\\&\quad\times(T_1J_1\tfrac{1}{2}j_nTT_zJM|\tau_{\eta n}B_{qn}^{(k)}|T_1J_1\tfrac{1}{2}j_nT'T_z'J'M')\\&\quad\times(j^{n-1}(\alpha_1T_1J_1)jT'J'|\}j^n\alpha'T'J'),\qquad(7)\end{aligned}$$

where α stands for the additional quantum numbers necessary to define the state. Using (44) of reference 6, we obtain

$$\begin{aligned}(\mathfrak{M}_q^{(k)})_{MM'}&=n\sum_{\alpha_1T_1J_1}(j^n\alpha TJ\{|j^{n-1}(\alpha_1T_1J_1)jTJ)\\&\quad\times(j^{n-1}(\alpha_1T_1J_1)jT'J'|\}j^n\alpha'T'J')\\&\quad\times V(TT'1;-T_zT_z',(T_z-T_z'))V(JJ'k;-MM'q)\\&\quad\times(-1)^{J_1+k-j-J'+T_1+1-\frac{1}{2}-T'}\sqrt{2}(\tfrac{1}{2}\|t\|\tfrac{1}{2})(lj\|B^{(k)}\|lj)\\&\quad\times[(2T+1)(2T'+1)(2J+1)(2J'+1)]^{\frac{1}{2}}\\&\quad\times W(\tfrac{1}{2}T\tfrac{1}{2}T';T_11)W(jJjJ';J_1k).\qquad(8)\end{aligned}$$

This expression becomes, upon putting $n=3$, $T'=\frac{1}{2}$,

[7] G. Racah, Phys. Rev. **63**, 367 (1943).
[8] A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **214**, 515 (1952).

$T_z'=\frac{1}{2}$, and inserting the value $(\frac{1}{2}\|t\|\frac{1}{2})=(\frac{3}{2})^{\frac{1}{2}}$,

$$(\mathfrak{M}_q^{(k)})_{MM'}$$
$$=(-1)^{k-j-J'}3(6)^{\frac{1}{2}}(2T+1)^{\frac{1}{2}}V(T\tfrac{1}{2}1;-T_z,\tfrac{1}{2},T_z-\tfrac{1}{2})$$
$$\times[(2J+1)(2J'+1)]^{\frac{1}{2}}(lj\|B^{(k)}\|lj)V$$
$$\times(JJ'k;-MM'q)\sum_{T_1J_1}(j^3\alpha TJ\{|j^2(T_1J_1)jTJ)$$
$$\times(j^2(T_1J_1)j\tfrac{1}{2}J'|\}j^3\alpha'\tfrac{1}{2}J')$$
$$\times W(\tfrac{1}{2}T\tfrac{1}{2}\tfrac{1}{2};T_11)W(jJjJ';J_1k). \quad (9)$$

In order to compare the image transition to the other, let us put $J=J'=j$, then we obtain for the two cases

I. $T=\frac{3}{2}$, $T_z=\frac{3}{2}$:

$$(\mathfrak{M}_q^{(k)})_{MM'}=-(-1)^k3(6)^{\frac{1}{2}}(2j+1)(lj\|B^{(k)}\|lj)$$
$$\times V(jjk;-MM'q)\sum_{J_1}(j^3\alpha\tfrac{3}{2}j\{|j^2(1J_1)j\tfrac{3}{2}j)$$
$$\times(j^2(1J_1)j\tfrac{1}{2}j|\}j^3\alpha'\tfrac{1}{2}j)W(jjjj;J_1k)W(\tfrac{1}{2}\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2};11) \quad (10)$$

II. $T=\frac{1}{2}$, $T_z=-\frac{1}{2}$:

$$(\mathfrak{M}_q^{(k)}{}_{MM'}=-(-1)^k6(2j+1)(lj\|B^{(k)}\|lj)$$
$$\times V(jjk;-MM'q)\sum_{T_1J_1}(j^3\alpha\tfrac{1}{2}j\{|j^2(T_1J_1)j\tfrac{1}{2}j)$$
$$\times(j^2(T_1J_1)j\tfrac{1}{2}j|\}j^3\alpha'\tfrac{1}{2}j)W(jjjj;J_1k)W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2};T_11).$$

The α characterizing the state should now be specified. In some of the cases to be considered the total spin J and the total isotopic spin T are sufficient to characterize the state uniquely. In other cases, however, in which more than one state with the same J and T are found in the configuration considered (as can be the case for j^3, $j\geqslant 5/2$) we shall take for the ground state the one which is the lowest in the case of short-range attractive (spin independent) forces. This is the state with $J=j$ (for $T=\frac{3}{2}$ or $T=\frac{1}{2}$) which belongs to the irreducible representation of the symplectic group in $2j+1$ dimensions characterized by the partition (10). This state may be said to have the lowest seniority. Inserting in (10) the proper c.f.p. of the ground states thus specified and summing the square of $(\mathfrak{M}_q^{(k)})_{MM'}$ over q, M, and M' and dividing by $2j+1$, we obtain the appropriate transition probabilities. These can be naturally written down as the product of (4) of the single nucleon transition $j\rightarrow j$ and a certain factor which is found to be, for the two cases considered, as follows: $(2j-1)/3(j+1)$ in case I and $[(j+4)/3(j+1)]^2$ in case II. These expressions have the following values (for $k=1$):

j	1/2	3/2	5/2	7/2
I	...	4/15	8/21	4/9
II	1	121/225	169/441	25/81

We see that for $j>\frac{1}{2}$ these two expressions are of the same order of magnitude; for $j=5/2$ they are almost equal, and for $j\geqslant 7/2$ the transition I should be more favored than the image transition II. The result shows clearly that jj coupling cannot account for the observed favored transitions, it leads to the conclusion that image transitions should not be more favored than the others. The situation is not essentially changed if we also take into account the Fermi matrix elements, as it is usually assumed that these have an almost equal weight to that of the Gamow-Teller matrix elements; their addition will multiply the $|(\mathfrak{M}_q^{(k)})_{MM'}|^2$ of the image transitions by a factor ~ 2 which is far less than the observed ratios.

The cases to which this formula apply are A^{35} (log $ft\geqslant 3.53$) as compared to S^{35} (log $ft=4.98$) in the $d_{7/2}$ shell, and Ti^{43} (log $ft\sim 3.40$) as compared to Sc^{43} (log $ft\geqslant 4.77$) in the $f_{7/2}$ shell.

Short-range attractive forces (and also attractive tensor forces) predict that the ground state of the configuration j^n, n odd, shall have the spin $J=j$. Therefore, if in an actual case the ground states (of the nuclei involved in the transition) have other spins, we must conclude that other interactions take here place and that the wave functions which transform according to the irreducible representations of Sp$(2j+1)$ (and which diagonalize the energy in the case of short-range forces) are not adequate to the description of the states. Only if we assume that the spin of the ground state of Na^{25} is 5/2 (unlike Na^{23}) does it make sense to apply our previous results for $j=5/2$ to a comparison of its β-decay (log $ft=5.25$) with the image transition of Al^{25} (log $ft\sim 3.53$). In the case of $A=23$ the ground state of Na^{23} has a spin $\frac{3}{2}$ and therefore we do not know what are the adequate quantum numbers which characterize this state (there are three independent states of $(d_{\frac{5}{2}})^7$ with $J=\frac{3}{2}$, $T=\frac{1}{2}$).

In the case of $A=19$ one finds that the ground state of F^{19} has $J=\frac{1}{2}$ and that of O^{19} probably has $J=\frac{3}{2}$. These spins of ground states in the $(d_{\frac{5}{2}})^3$ configuration cannot be explained by assuming short-range forces (or tensor forces) alone, and one might think that either other forces are present or that configuration interaction (probably with $s_{\frac{1}{2}}$ orbits) is important in this case. However, if the configuration considered is to a large extent pure $(d_{\frac{5}{2}})^3$ we can still calculate the transition probability of O^{19} to the ground state of F^{19} without saying anything about the interaction, as in this configuration there is only one state with $J=\frac{3}{2}$, $T=\frac{3}{2}$ and only one state with $J=\frac{1}{2}$, $T=\frac{1}{2}$. Using (9), we obtain for the transition probability of O^{19} the factor, used above, equal to 4/7 (experimentally, log $ft=5.57$) as compared with the case of the image transition of Ne^{19} $[(d_{\frac{5}{2}})^3, J=\frac{1}{2}]$ for which the factor is equal to 121/105 (for this transition log $ft=3.23$).

If there are only two j nucleons outside closed shells, the matrix element of the β-transition between the odd-odd and the even-even nucleus can be easily written down. We also give here the factor by which the transition probability of the single j nucleon should be multiplied in order to obtain the probability of the transition considered. The even-even nucleus has $T=1$ and $T_z=1$, say, whereas for the odd-odd nucleus $T_z=0$ and T is either 1 or 0; the transition probability is the same for the two cases. The factor defined above is

$$2(2j+1)(2J+1)W(jJjJ';jk)^2, \qquad (11)$$

where J' is of the initial state and J is of the final state. In case one of the J, J' vanishes the other must be equal to 1 (if $k=1$) and the result is, for initial $J'=1$, final $J=0$: $2(2j+1)/3(2j+1)=\frac{2}{3}$, and for initial $J'=0$, final $J=1$: $2(2j+1)3/3(2j+1)=2$. Both of these factors are of order of magnitude of unity and all such transitions should be favored. A similar conclusion is also the result of the supermultiplet theory, and both these theories account for the cases in which the ft value is low (He^6, C^{10}, and F^{18})[9] and both have difficulty in explaining the high ft values of such transitions as of P^{30}, Cl^{34}, and K^{38}.*

The other cases of odd-odd to even-even transitions, where $N\neq Z$ in the odd-odd nucleus, are not favored and should be forbidden according to the supermultiplet theory. It is interesting to calculate the matrix elements in jj coupling in the case that in the odd-odd nucleus there are a j' nucleon and a j hole. The ratio between the transition probability in this case to the probability of the single nucleon transition $j'\rightarrow j$ is given by

$$n\tfrac{1}{2}(2j'+1)(2J+1)W(jJj'J';jk)^2, \qquad (12)$$

where n is the number of nucleons in the closed j shell $-2(2j+1)$. J must vanish and J' must be equal to 1 (for $k=1$) in which case the factor (12) becomes simply $(2j'+1)/3$. The only cases to which this result is applicable are B^{12} and N^{12} for which $j=\frac{3}{2}$ and $j'=\frac{1}{2}$ and the correction factor is $\frac{2}{3}$. This is not enough to explain the high ft values of these transitions ($\log ft=4.17$ and >4.3, respectively).

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[9] And also Al^{26} for which neither J nor J' are known; in this case the correcting factors for the reasonable possibilities are as follows:

$J'\diagdown J$	0	2	4
1	2/3	128/105	
3		162/245	32/49
5			2/7

* *Note added in proof.*—A super-allowed transition between the ground states of Cl^{34} and S^{34} is reported to have been found in the E.T.H., Zurich.

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Thermal Neutron-Proton Capture

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Measurement of the thermal neutron capture cross section of hydrogen has been effected by a comparison with boron using the technique of pile oscillation. The 2200-m/sec value obtained, 0.332 b, has a 2 percent uncertainty resulting primarily from the effect of neutron moderation by hydrogen. However, the result indicates an exchange moment contribution to the cross section of $\sim 6\pm 3$ percent.

INTRODUCTION

THE thermal $n-p$ capture cross section is of theoretical interest[1–5] in studying the nucleon-nucleon interaction; $n-p$ scattering data from thermal energies to ~ 5 Mev in conjunction with the deuteron binding energy allow one to fix the so-called "effective singlet and triplet ranges."[6] It is the near equality of $n-p$ and $p-p$ singlet ranges ($2.4\pm 0.3\times 10^{-13}$ cm and $2.7\pm 0.2\times 10^{-13}$ cm, respectively) that suggests charge independence of nuclear forces. $n-p$ capture, in addition to involving the singlet and triplet range, also involves an exchange moment.[7] This has the effect of increasing the $n-p$ capture cross section by ~ 5 percent. Owing primarily to the large ratio of scattering to absorption (~ 150) the uncertainty in the measured value of the $n-p$ capture cross section[8] is also ~ 5 percent. It is the object of the present work to improve on the accuracy of this measurement in order to better estimate the exchange moment contribution to the $n-p$ capture cross section. An improved technique has become available for this purpose since the recent refueling of the heavy-water reactor at the Argonne National Laboratory.

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