The Electric Scattering of the Polarizable Deuteron*

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The effect of the electric polarizability of the deuteron upon its electric scattering is discussed for cases in which $Ze^2/\hbar v \gg 1$ and the classical closest distance of approach is so large that the effects of direct nuclear interaction are negligible. Under such circumstances, the chief departure from the Rutherford scattering law is due to the polarizability of the deuteron. The magnitude of the departure is calculated, and in particular, the polarizability contribution to backward scattering is estimated to be respectively about 3 percent and 4 percent for 8-Mev and 10-Mev deuterons scattered by bismuth and uranium.

I. INTRODUCTION

As has been shown in the previous paper,[1] the structure of the deuteron is such that it exhibits a polarizability when placed in an electric field. This phenomenon leads us to expect that the Rutherford cross section for electric scattering of a nucleus of charge Z will be somewhat modified.[2] Under ordinary circumstances this effect is likely to be obscured by nuclear interactions. Consequently, in this paper, we will confine our considerations to the energy regions where the nuclear effects are negligible, that is, the energy regions where the penetration of any part of the deuteron through the Coulomb barrier to the scattering nucleus is negligibly small. The approximate conditions for this are that the classical distance of closest approach be considerably larger than the nuclear radius and that the penetrability much beyond the classical distance of closest approach be small, which occurs when $Ze^2/\hbar v \gg 1$.

The condition $Ze^2/\hbar v \gg 1$ is also the condition in Coulomb scattering that the classical approximation be valid. This approximation can be successfully used in obtaining numerical results as is done below in Sec. III. This section, in conjunction with the preceding paper,[1] provides a complete procedure for calculating the polarizability anomaly of the deuteron. Section II has been included in order to exhibit the relation of this paper to the recent work of French and Goldberger,[3] and it is not essential for an understanding of Sec. III.

II. DEUTERON STRUCTURE AND POLARIZABILITY SCATTERING

In order to compare this calculation of the deuteron polarizability scattering with the French and Goldberger[3] treatment of the effects of the finite deuteron size, we can start with their wave equation[3] for a deuteron of incident momentum $\hbar\mathbf{k}$ which is electrically scattered by a nucleus of charge Z. We then have

$$\begin{aligned}[\tfrac{1}{2}(\hbar^2/m)(\nabla_R^2+k^2)-Ze^2R^{-1} \\ +\tfrac{1}{2}(\hbar^2/\mu)\nabla_r^2-V(\mathbf{r})-\epsilon]\Psi(\mathbf{R},\mathbf{r}) \\ =Ze^2[|\mathbf{R}+\tfrac{1}{2}\mathbf{r}|^{-1}-R^{-1}]\Psi(\mathbf{R},\mathbf{r}),\end{aligned} \quad (1)$$

where $\mathbf{R}=\frac{1}{2}(\mathbf{r}_p+\mathbf{r}_n)$, $\mathbf{r}=\mathbf{r}_p-\mathbf{r}_n$, m = deuteron mass, μ = deuteron reduced mass, $V(\mathbf{r})$ = neutron-proton potential, and ϵ = deuteron binding energy. If we expand $\Psi(\mathbf{R},\mathbf{r})$ in terms of the complete set of normalized neutron-proton wave functions $\chi_n(\mathbf{r})$ so that $\Psi(\mathbf{R},\mathbf{r}) = \sum_n \chi_n(\mathbf{r})\Omega_n(\mathbf{R})$, then multiply from the left by $\chi_0^*(\mathbf{r})$, and finally integrate over r space, Eq. (1) becomes

$$\left[\frac{\hbar^2}{2m}(\nabla_R^2+k^2)-\frac{Ze^2}{R}\right]\Omega_0(\mathbf{R}) = Ze^2\int \chi_0^*(\mathbf{r})\left[\frac{1}{|\mathbf{R}+\frac{1}{2}\mathbf{r}|}-\frac{1}{R}\right]\Psi(\mathbf{R},\mathbf{r})(d\mathbf{r}), \quad (2)$$

where $\Omega_0(\mathbf{R})$ is the wave function for elastic scattering. Treating the right-hand side as a perturbation, we make the approximation $\Psi(\mathbf{R},\mathbf{r})=\chi_0'(\mathbf{r})\psi(\mathbf{R})$ where $\chi_0'(\mathbf{r})$ is the ground-state wave function of the deuteron as distorted by the electric field of the nucleus, so that approximately

$$\chi_0'(\mathbf{r})=\chi_0(\mathbf{r})-\sum_{n\neq 0}{}' [E_n+\epsilon]^{-1} \times\left(n\left|\frac{1}{|\mathbf{R}+\frac{1}{2}\mathbf{r}|}-\frac{1}{R}\right|0\right)\chi_n(\mathbf{r}), \quad (3)$$

where the summation is over both discrete and continuum states. We then have

$$\left[\frac{\hbar^2}{2m}(\nabla_R^2+k^2)-\frac{Ze^2}{R}\right]\Omega_0(\mathbf{R}) \approx Ze^2\left(0\left|\frac{1}{|\mathbf{R}+\frac{1}{2}\mathbf{r}|}-\frac{1}{R}\right|0\right)\psi(\mathbf{R}) - \sum_{n\neq 0}{}' [E_n+\epsilon]^{-1}\left|\left(n\left|\frac{1}{|\mathbf{R}+\frac{1}{2}\mathbf{r}|}-\frac{1}{R}\right|0\right)\right|^2\psi(\mathbf{R}). \quad (4)$$

In the French and Goldberger[3] treatment, just the first

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[1] Ramsey, Malenka, and Kruse, preceding paper [Phys. Rev. **91**, 1162 (1953)].

[2] N. F. Ramsey, Phys. Rev. **83**, 659 (1951); Breit, Hull, and Gluckstern, Phys. Rev. **87**, 74 (1952).

[3] J. B. French and M. L. Goldberger, Phys. Rev. **87**, 899 (1952).

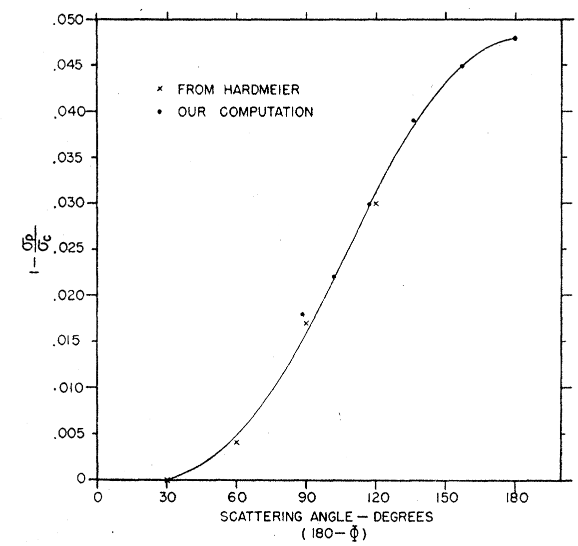

FIG. 1. The negative of the ratio of the polarizability to the Coulomb differential cross section plotted as a function of scattering angle for $\beta=0.2$.

term is obtained, since they introduce only the unperturbed deuteron wave function. In this paper, we are confining ourselves to the region where $Z\gg 1$ and $Ze^2/\hbar v\gg 1$; hence $\psi(\mathbf{R})$ is small for R less than the nuclear radius. On the other hand, the main contribution to the matrix elements on the right-hand side of Eq. (4) comes from the small values r since the wave function $\chi_0(\mathbf{r})$ vanishes exponentially for r greater than the deuteron radius. It is therefore a reasonable approximation to make a multipole expansion of the matrix arguments in terms of $R^{-1}(r/R)^l P_l(\cos\theta)$. From the angular integration, it then follows that the first term vanishes in agreement with the statement of French and Goldberger[3] for $R>\frac{1}{2}r$. If, in the second term, we keep only the lowest order term in r/R, Eq. (4) becomes approximately

$$[\tfrac{1}{2}(\hbar^2/m)(\nabla_R^2+k^2)-Ze^2R^{-1}]\Omega_0(\mathbf{R}) \approx -\tfrac{1}{2}\alpha(ZeR^{-2})^2\psi(\mathbf{R}), \quad (5)$$

where

$$\alpha=\tfrac{1}{2}e^2\sum_{n\neq 0}{}' [E_n+\epsilon]^{-1}|(n|r\cos\theta|0)|^2. \quad (6)$$

From our previous paper,[1] we identify the α of Eq. (6) as the deuteron polarizability, where $\frac{1}{2}\alpha\mathcal{E}^2=\frac{1}{2}\alpha(ZeR^{-2})^2$ is the polarization energy, so that for $Z\gg 1$ and $Ze^2/(\hbar v)\gg 1$ we can speak of the polarizability of the deuteron as causing the principal departure from Rutherford scattering.[4]

III. CLASSICAL SCATTERING

In a more approximate treatment of the scattering problem, the difficulties of solving Eq. (4) or (5) can be avoided by noting that the condition $zZe^2/(\hbar v)\gg 1$ is the criterion[5] for the validity of the classical Coulomb scattering. To be somewhat more general, in this section, we assume that the scattered particle has a charge z. We can then replace Eq. (5) with its classical counterpart,

$$E=\tfrac{1}{2}m\dot{R}^2+\tfrac{1}{2}mR^2\dot{\varphi}^2+zZe^2R^{-1}-\tfrac{1}{2}\alpha(ZeR^{-2})^2. \quad (7)$$

Let Φ be the supplement of the scattering angle, then

$$\Phi=2\int_0^{u_1}\frac{s\,du}{[1-zZe^2E^{-2}u+\frac{1}{2}\alpha(Ze)^2E^{-2}u^4-s^2u^2]^{\frac{1}{2}}}, \quad (8)$$

where $u=R^{-1}$, $u_1=$ inverse distance of closest approach, and s is the impact parameter using the results of classical scattering theory.[6] The differential cross section is

$$\sigma(\Phi)=\frac{s\,ds}{\sin\Phi\, d\Phi}, \quad (9)$$

and for pure Coulomb scattering

$$s_0=\tfrac{1}{2}(zZe^2/E)\tan(\tfrac{1}{2}\Phi_0). \quad (10)$$

Following the approach used by Hardmeier,[7] we say that for a given impact parameter s_0, the presence of the polarizability term produces a small decrease δ in the scattering angle, so that the supplement of the scattering angle can be written as

$$\Phi=\Phi_0+\delta. \quad (11)$$

The pure Coulomb impact parameter required for the deuteron to arrive at this angle is

$$s_1=\tfrac{1}{2}(zZe^2/E)\tan(\tfrac{1}{2}(\Phi_0+\delta))=\tfrac{1}{2}(zZe^2/E)\tan(\tfrac{1}{2}\Phi). \quad (12)$$

Then, it follows that the ratio of the Coulomb plus polarizability to the pure Coulomb differential cross section at $\Phi=\Phi_0+\delta$ is

$$\frac{\sigma_p}{\sigma_c}(\Phi)=\left(\frac{s_0ds_0}{\sin\Phi d\Phi}\right)\Big/\left(\frac{s_1ds_1}{\sin\Phi d\Phi}\right)=\left(\frac{ds_0{}^2}{d\Phi}\right)\Big/\left(\frac{ds_1{}^2}{d\Phi}\right)$$
$$=\frac{\tan(\frac{1}{2}\Phi_0)\sec^2(\frac{1}{2}\Phi_0)(d\Phi_0/d\Phi)}{\tan(\frac{1}{2}\Phi)\sec^2(\frac{1}{2}\Phi)}$$
$$=\frac{\sin(\frac{1}{2}\Phi_0)\cos^3\frac{1}{2}(\Phi_0+\delta)}{\sin(\frac{1}{2}(\Phi_0+\delta))\cos^3(\frac{1}{2}\Phi_0)}\left[1+\frac{d\delta}{d\Phi_0}\right]^{-1}. \quad (13)$$

From Eqs. (8) and (11) we can write δ as

$$\delta=2\int_0^{v_1}\frac{t\,dv}{[1-2v-t^2v^2+\beta v^4]^{\frac{1}{2}}}-2\int_0^{v_0}\frac{t\,dv}{[1-2v-t^2v^2]^{\frac{1}{2}}}, \quad (14)$$

[4] It should be noted, however, that in the neighborhood of the origin Eq. (5) breaks down, because although $\psi(\mathbf{R})$ becomes small as R goes to zero, it does remain finite.

[5] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), p. 124.

[6] H. Goldstein, *Classical Mechanics* (Addison-Wesley Press, Inc., Cambridge, 1950), p. 82.

[7] W. Hardmeier, Physik. Z. **28**, 181 (1925).

where we have made the substitution $v=\frac{1}{2}(zZe^2/E)u$ and are using the dimensionless constants

$$t=[2E/(zZe^2)]s=\tan(\tfrac{1}{2}\Phi_0), \quad (15)$$

$$\beta=\alpha 8(\hbar c/e^2)z^{-4}Z^{-2}(E/\hbar c)^3 = 2.68\times 10^{39}\alpha z^{-4}Z^{-2}E_{\text{Mev}}{}^3, \quad (16)$$

where α is in cm^3.

Hardmeier[7] has given a formula for δ at small scattering angles and has evaluated it explicitly for large scattering angles for $\beta=1, 2, 4, 6$. We have evaluated δ at large scattering angles for $\beta=0.2$ by numerically integrating the difference of the integrals appearing in Eq. (4) except near the ends of the region of integration. At the ends, the difference was approximated by expressions that could be integrated directly. $d\delta/d\Phi_0$ was then obtained by numerical differentiation. The results for σ_p/σ_c are given in Fig. 1 using also Hardmeier's[7] results for small scattering angles.

In the case of head on collisions, σ_p/σ_c reduces to $[1+(d\delta/d\Phi_0)]^{-2}$; also, as $t\rightarrow 0$, we have

$$\lim_{t\to 0}\frac{d\delta}{d\Phi_0}=\lim_{t\to 0}\frac{\delta}{\Phi_0}=\lim_{t\to 0}\frac{\delta}{2t}=\int_0^{v_1}\frac{dv}{[1-2v+\beta v^4]^{\frac{1}{2}}}-\int_0^{v_0}\frac{dv}{[1-2v]^{\frac{1}{2}}}. \quad (17)$$

If we evaluate this as a power series in β, we obtain for backward scattering

$$\frac{\sigma_p}{\sigma_c}(0)=1-0.229\beta-0.0354\beta^2. \quad (18)$$

For $\beta=0.2$, this gives us a 4.7 percent deviation from pure Coulomb scattering in the backward direction. For the deuteron $z=1$ and from reference 1, we have[8] $\alpha=0.56\times 10^{-39}$ cm^3. Then, applying Eqs. (16) and (18) to some examples where the nuclear effects should be comparatively small[2,9] and hence our treatment be valid, we find for 8-Mev deuterons scattered by $_{83}$Bi where $\beta=0.112$ a deviation of 2.7 percent in the backward direction, and for 10-Mev deuterons scattered by $_{92}$U where $\beta=0.177$ a deviation of 4.2 percent in the backward direction.

[8] The value of α is taken from Eqs. (13) and (21) of reference 1 by noting that for the scattering problem, we must average over the magnetic quantum number so that $\langle\alpha\rangle_{\text{Av}}=\alpha_{SS}+2\alpha_{DD}\approx\alpha_{SS}=0.56\times 10^{-39}$ cm^3.

[9] D. C. Peaslee, Phys. Rev. 74, 1001 (1948); C. J. Mullin and E. Guth, Phys. Rev. 82, 141 (1951).

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Exchange Scattering of an Electron by the Hydrogen Atom*

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The theory of scattering of electrons by hydrogen has been re-examined with the objective of identifying the matrix element for the exchange scattered amplitude from the same integral equation which provides direct scattered amplitudes. The theory of Mott and Massey is verified, but it is demonstrated that their result contains contributions from the incident wave which must be removed before computing exchange amplitudes. The result is a theoretical justification for the Oppenheimer (prior) matrix element.

THE analysis of exchange scattering was originally made by Oppenheimer,[1] and an entirely different treatment of this problem which has become standard was given by Mott and Massey.[2] The reason for the superiority of the latter method is that the general matrix element for the exchange scattered amplitude is identified, while Oppenheimer's solution is an approximate one from the outset so that the possibility for improved estimates is automatically ruled out. However, the method of Mott and Massey involves the assumption that the usual stationary state solution from which the direct scattered amplitude is derived has the asymptotic form

$$\operatorname*{Lim}_{r_2\to\infty}\psi(\mathbf{r}_1,\mathbf{r}_2)=\sum_\alpha\frac{e^{ik_\alpha r_2}}{r_2}\cdot g_\alpha\cdot\varphi_\alpha(\mathbf{r}_1), \quad (1)$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ refer to the primary and hydrogenic electrons, respectively. (This labeling will prevail throughout the present paper.) The φ's are Coulomb functions; the sum includes integration over continuum states. Mott and Massey point out that no proof of this assumption has been given but that one should be possible. It is the purpose of this paper to provide such

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[1] J. R. Oppenheimer, Phys. Rev. 32, 361 (1928).

[2] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, New York, 1949), second edition.