rent density of 8.7×10^6 amperes/cm^2 at the cross section XX of the emitter profile in Fig. 7(B). That current density is about equal to the value required to raise the tip of the emitter 7(B) to its melting point in one microsecond in view of the calculations presented in Part II of this paper. The heat of fusion is small, and no large amount of material was vaporized during arc judged from the electron micrographs of Fig. 7. The evidence indicates that the resistive mechanism was adequate to account for the heat required by the observed emitter deformation during arc. Apparently energy was not supplied by other mechanisms at a rate large compared with that of the resistive mechanism.

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The Field Emission Initiated Vacuum Arc. II. The Resistively Heated Emitter*

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Electrical breakdown between clean metal electrodes in high vacuum was observed when the field current density at the single crystal tungsten cathode exceeded a critical value of the order of 10^8 amperes/cm^2. At current densities just below the critical value, an electron emission process was observed which apparently involved both high temperature and high electric field. Calculations are presented for the emitter temperature increase due to the resistive mechanism for both the steady state and the transient solution. Emitter geometries used for the calculations approximated those obtained from electron micrographs of several emitters. The calculations show that the resistive heating was sufficient to melt the emitter at the critical current density, assuming the accepted value of the physical constants for the polycrystalline metal.

IN Part I of this paper evidence has been presented showing that the interruption of the microsecond field emission from the tungsten emitter by the occurrence of a vacuum arc is dependent principally upon current density J under conditions of clean emitter surfaces and excellent vacuum. It also has been pointed out that the experimentally observed values of the critical current density $\bar{J}_x$ for arc initiation lie in the range $10^7<\bar{J}_x<10^8$ amperes/cm^2 for microsecond operation, with the suggestion that heating of the emitter by a current density dependent mechanism may be the initiating factor of the breakdown. The purpose of this part of the paper is to present an analysis of the heat flow problem when an emitter of idealized geometry approximating those used in actual operation is heated resistively. The current density dependent mechanism proposed by Nottingham[1] is briefly considered. A comparison will be made between values of the current density for which an arbitrary large temperature increase is predicted by resistive heating and experimental values of $\bar{J}_x$.

A mathematical analysis of the resistive generation of heat and its simultaneous dissipation by conduction, using physical constants for the polycrystalline metal, was made possible when electron micrographs had revealed the geometry of the emitter,[2] whose shape in the present experiments was a cone whose half-angle was in the range 2.75° to 15.5°, with a hemispherical tip of radius between 1.5×10^{-5} and 1.5×10^{-4} cm. Although this geometry does not lend itself directly to any simple coordinate system, it may be closely approximated by a portion of a cone bounded by concentric spherical surfaces, orthogonal to the cone, for which ordinary spherical coordinates are suitable. Figure 1 shows a comparison between the idealized geometry and that of several typical emitters. The point $u=m$ is chosen to correspond to the position of maximum current density, which may be expected at the "neck" of constricted emitters [see Sec. XX of Fig. 4(D), Part I], or in the case of emitters without constriction, at a distance from the vertex of the emitter about equal to the radius of its hemispherical tip.

The flow lines for both electric current and heat are assumed to follow the radial coordinate curves of the system. Heat radiation is supposed negligible. Consideration of the resistive generation of heat, together with the usual laws of heat conduction, leads to the differential equation

$$u^4\partial^2T/\partial u^2+2u^3\partial T/\partial u-\alpha^2u^4\partial T/\partial t=b. \qquad (1)$$

Here u is the distance in cm from the vertex of the cone, T is temperature in degrees centigrade, t is time in seconds; $\alpha^2=c\delta/\kappa$, where c is specific heat, δ is density, and κ is thermal conductivity; b is defined by

* Support for this work was extended by the U. S. Air Force through the Microwave Laboratory of the University of California, and by the U. S. Office of Naval Research.
[1] W. B. Nottingham, Phys. Rev. 59, 906 (1941).
[2] Dyke, Trolan, Dolan, and Barnes, J. Appl. Phys. 24, 570 (1953).

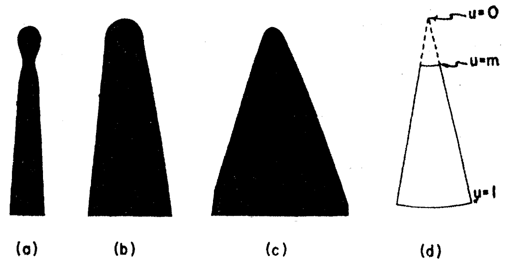

FIG. 1. A comparison between the geometry of typical field emitters (a, b, c) and the idealized conical geometry (d) used in heat flow calculations.

the equation

$$b=\frac{-I^2\rho}{4\pi^2(4.18\kappa)(1-\cos v_0)^2}, \tag{2}$$

with I indicating electric current, ρ the resistivity, and v_0 the interior half-angle of the cone. Equation (1) overlooks the variation of the physical constants with temperature which if included would unreasonably complicate the solution. The use of values for intermediate temperatures is a satisfactory alternative.

If the term involving t in Eq. (1) is omitted, a steady-state solution $s(u)$, useful in the general case, is obtained,

$$s(u)=b/u^2+c_1/u+c_2, \tag{3}$$

where c_1 and c_2 are constants of integration. For the general equation (1), the method of separation of variables may be used. A solution of the form

$$T=U(u)\theta(t)+s(u) \tag{4}$$

is assumed. $\theta(t)$ has the usual exponential form appearing in Eq. (8) below and the function $U(u)$ must satisfy the equation

$$\frac{d^2U}{du^2}+\frac{2}{u}\frac{dU}{du}+k^2U=0, \tag{5}$$

k being arbitrary. Boundary conditions are based on the assumptions of zero temperature at a relatively great distance $u=l$ from the apex of the emitter, and no heat flow through the apex where $u=m$, that is,

$$U(l)=0, \quad dU/du]_{u=m}=0. \tag{6}$$

A related problem treated in Churchill[3] suggests a solution of Eq. (5) satisfying the first of conditions (6), namely,

$$U=(a/u)\sin(k(l-u)),$$

where a is arbitrary. To satisfy the second of conditions (6) requires the relation

$$\tan(k(l-m))=-mk, \tag{7}$$

an equation with infinitely many solutions k_n. The functions $\sin(k_n(l-u))$ are orthogonal for all values of k_n so that they may serve as a basis for the series expansion needed below.

The solution (4) now takes the form

$$T=\frac{1}{u}\sum_{n=1}^{\infty}\exp(-k_n^2t/\alpha^2)A_n\sin(k_n(l-u))+s(u). \tag{8}$$

Clearly T approaches the steady-state value $s(u)$ when $t\rightarrow\infty$, and the coefficients A_n must be so chosen that $T=0$ when $t=0$, i.e., the function $-us(u)$ must be expanded in a series of the orthogonal functions $\sin(k_n(l-u))$. A process analogous to that used in the usual Fourier series development leads to an evaluation of the coefficients in the form

$$A_n=\frac{1}{M_n}\int_m^l xs(x)\sin(k_n(l-x))dx, \tag{9}$$

where

$$M_n=\int_m^l \sin^2(k_n(l-x))dx$$
$$=\frac{l-m}{2}-\frac{1}{4k_n}\sin(2k_n(l-m)). \tag{10}$$

The procedure for writing the series (8) is now straightforward. The roots k_n of Eq. (7) are obtained by successive approximations, and thereafter the values M_n, A_n, and the exponential factors of (8) are computed. If the experimental current is drawn for short intervals of time, many terms are necessary to get the desired convergence of the series (80 terms in the present case of microsecond pulses). Computation of steady-state temperatures on the other hand is relatively simple. The physical constants for the tungsten emitter used are those tabulated by Worthing and Halliday.[4] For resistivity ρ and specific heat c, which vary considerably with temperature, intermediate values $\rho=50\times10^{-6}$ ohm cm and $c=0.045$ cal/g°C have been arbitrarily chosen for the present examples; a further comment on a possible dependence of resistivity on current density appears below. The value of l is taken as one millimeter, which approximates the usual length of the emitters used, and is large compared with the radius at the apex. The value of m varies with cone angle in such a way as to make the cross-section radius where $u=m$ equal to the true emitter radius. The value of the cone angle, within broad limits exceeding those encountered in practice, makes relatively little difference to the results in terms of permissible current density, not more than a factor of 3 between half-angles of 5° and 20°. In the calculation of Eq. (11) an arbitrary angle of $\arctan\frac{1}{5}$ (approximately 11°) is assumed.

[3] R. V. Churchill, *Fourier Series and Boundary Value Problems* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 113–114.

[4] A. G. Worthing and D. Halliday, *Heat* (John Wiley and Sons, Inc., New York, 1948), p. 496.

The maximum steady-state temperature, which occurs at $u=m$, may be expressed by the relation

$$T_{\max}=9.5\times10^{-4}J^2r^2 \text{ deg C}, \tag{11}$$

where J is current density in amperes/cm^2, and r is emitter radius in cm. If $T_{\max}$ is to be held at less than 1000°C, an arbitrary value used for illustration, it is clear that the product J^2r^2 must not exceed 10^6. Thus in the range of radii from 10^{-4} to 10^{-5} cm, J for direct current operation may reach corresponding values of 10^7 to 10^8 amperes/cm^2 under the conditions here described. It should be noted that while the permissible J varies inversely as the radius, the total current involves emitting area and so varies directly as radius.

The temperature rise-time indicated by the calculation of the series in Eq. (8) is such that the steady state is closely approached in 10^{-5} sec, and about one-fourth of the temperature increase is to be expected in a microsecond. Since T is proportional to J^2, microsecond operation should permit about twice as large a current density as can be sustained in the steady state, and shorter pulses offer the possibility of attaining still higher levels.

The analysis for a cylindrical emitter is useful in comparison with the foregoing because it represents a limiting case for very small cone angles. In cylindrical coordinates the equation corresponding to Eq. (1) is

$$\partial^2T/\partial u^2-\alpha^2\partial T/\partial t=-a, \tag{12}$$

where T, t, and α have the same significance as before; u is the length coordinate along the cylindrical wire, and a is defined by the relation

$$a=I^2\rho/4.18\pi^2\kappa v_0^4, \tag{13}$$

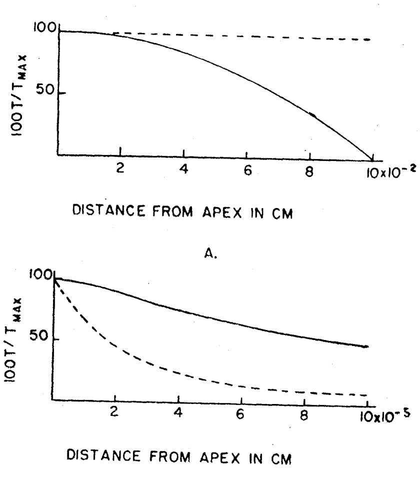

FIG. 2. A comparison between steady-state temperature gradients (solid curves) in the idealized field emitters of (A) cylindrical and (B) conical geometries, the latter assuming an emitter tip radius of 10^{-5} cm. Dotted curves are corresponding gradients if heat conduction is neglected, adjusted to arbitrary maxima.

v_0 being the radius of the wire. The steady-state solution is

$$T=-\tfrac{1}{2}au^2+c_1u+c_2. \tag{14}$$

If the origin is chosen so that $u=0$ at the emitting end of the wire, and $u=l$ at the cool end, the boundary conditions are: (1) $T=0$ when $u=l$, (2) $dT/du=0$ when $u=0$. By use of these conditions, Eq. (14) becomes

$$T=\tfrac{1}{2}a(l^2-u^2). \tag{15}$$

If the emitter is one millimeter long as in the conical case, the maximum value of T found where $u=0$ is given by the relation,

$$T_{\max}=2.45\times10^{-7}J^2 \text{ deg C}, \tag{16}$$

which may be compared with Eq. (11) for the cone. The temperature in the present case is independent of radius.

Equation (16) immediately gives a limiting value of J of about 10^5 amperes/cm^2 for operation at 1000°C in the steady state, which is a factor of several hundred less than the level permitted by conical emitters of typical radius. Although the temperature in conical emitters is not very sensitive to changes in cone angle within the stated limits, the change is very rapid as the angle approaches zero, and it may be shown that Eq. (16) is the limit of Eq. (11) under such conditions.

A comparison of temperature gradients for the cylindrical and conical cases in the steady state is instructive. Figure 2 shows such a comparison, with the dotted curves indicating the corresponding gradients if heat conduction is neglected. The latter curves reach no steady state and are arbitrarily adjusted to the same maxima. It is observed that while the temperature in the conical case decreases to one-half of its maximum in a distance of ten emitter radii from the apex, the same fraction is reached for the cylinder at a distance of several thousand radii.

Details of the transient case for a cylindrical wire will not be included. The procedure is straightforward, except that the boundary conditions do not permit expansion of the steady-state function in ordinary Fourier series; however, a series of terms in the orthogonal functions $\cos(2n-1)\pi u/2l$ is permissible and the solution is given explicitly by the expression

$$T=\frac{a}{2}(l^2-u^2)-\frac{16al^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\times\exp[-(2n-1)^2Ct]\cos\left(\frac{(2n-1)\pi u}{2l}\right), \tag{17}$$

where C represents the quantity $\pi^2/4l^2\alpha^2$.

In experimental work[5] direct current operation of emitters has been observed at current densities in excess of 10^6 amperes/cm^2 as compared with predicted values for the conical case of approximately 10^7 am-

[5] W. P. Dyke and J. K. Trolan, Phys. Rev. **89**, 799 (1953).

TABLE I. A comparison between observed experimental current densities $\bar{J}_x$ required to initiate arc for several emitters (Column B) and current densities J_R (Column C) for which the calculated temperature reaches 3000°C in the corresponding pulse times.

A Emitter number	B $\bar{J}_x$ (amp/cm²)	C J_R (amp/cm²)	D V_{max} (kv)	E Radius of the emitter tip (cm)	F Half-angle of the emitter cone (degrees)	G Pulse length (μsec)	H[a] Tube type, electrode spacing (cm), anode material
O-38	6×10^7	7.0×10^7	9.2	2.4×10^{-5}	5	1	S, 4.5, ABW
X-62[b]	4×10^7	2.4×10^7	8.8		10	$\frac{1}{2}$	S, 8.5, ABW
X-62-A	3×10^7	2.5×10^7	60.1	1.5×10^{-4}	10	1	S, 8.5, ABW
Q-1	4×10^7	7.1×10^7	4.9	1.5×10^{-5}	3	1	PTP, 1, Moly.
Q-29	7×10^7	7.4×10^7	14.2	3.2×10^{-5}	6	1	S, 8.5, ABW
O-54	1×10^8	2.7×10^8	16.1	2.0×10^{-5}	16	1	S, 4.5, ABW
2-X-4	5×10^7	5.8×10^7	13.3	3.8×10^{-5}	6	1	S, 4.5, ABW

[a] Experimental tube type S is shown in Fig. 1; type PTP, i.e., point-to-plane, is shown in reference 5; ABW, aluminum-backed willemite.
[b] Electron micrographs of this emitter were not available; hence $\bar{J}_x$ was calculated from its electrical behavior with the aid of Eq. (1) of Part I. In this case, $\bar{J}_x$ was known within a factor of 3. For the other emitters, electron micrographs were available and $\bar{J}_x$ was known within ±10 percent.

peres/cm². Larger experimental current densities may yet be obtained for the steady state since the limitation imposed in reference 5 was not due to the cathode.

Table I exhibits the comparison of the maximum current densities J_R predicted by the resistive process (Column C) with the values of the critical current density $\bar{J}_x$ observed experimentally during the microsecond operation for several emitters (Column B). Methods for calculating the value of $\bar{J}_x$ were presented in Part I. J_R was arbitrarily defined as that current density required to raise the emitter temperature to 3000°K in a time equal to the duration of the experimental current pulse. For all emitters except X-62 the tip radius and cone angle were obtained from electron micrographs, and the resulting values of the experimental current densities are correct within 20 percent. The critical current density showed approximately the expected dependence on radius and cone angle. It will be noted that electrical breakdown depended on current density. It was independent of voltage, total current, anode material, and gap spacing.

With regard to the assumed values of the physical constants for tungsten, it may be pointed out that recent work by Ignateva and Kalashnikov[6] suggests that for impulse operation at values of J above 10^6 amperes/cm², the resistivity of some metals may be increased by a considerable factor. The reference notes an increase by a factor of 2 in resistivity of tungsten at the largest current density reported, i.e., 4×10^6 amperes/cm². Current densities two orders of magnitude larger were observed in the present work, and the increase of resistivity with current density, if any, must be less than an order of magnitude for $J<10^8$ amperes/cm² unless it is later shown that the physical constants for the polycrystalline metal are inapplicable to the small single crystals used herein.

The heating process suggested by Nottingham,[1] a quantitative analysis of which has not been made, is based on the suggestion that emitted electrons must be replaced in the metal at their respective energy levels by electrons supplied at the top Fermi level. The energy lost in the replacement process was assumed to heat the emitter. The analysis of this mechanism requires knowledge of (1) the energy distribution of the emitted electrons under conditions of simultaneously high field and high temperature, and (2) the spatial distribution of the heating within the emitter. The former is presently under study and he lattter is in question.[7] The effect of this heating mechanism would be small, according to preliminary analysis, unless it can be shown that the average energy per electron given to the metal is about 1 ev and that this energy is released within a distance from the emitter tip which is small compared to the emitter radius.

The resistive heating process provides adequate emitter temperature increase to support the suppositions in Part I which were advanced to account for the observed ring and tilt prior to arc formation. Thus, no other source is required for the generation of the heat which presumably initiated the observed electrical breakdown.

The experimental data for Table I were supplied by the authors of Part I. The continued interest of Professor J. E. Henderson of the Physics Department, University of Washington, is appreciated.

[6] L. A. Ignateva and S. G. Kalashnikov, Zhur. Eksptl. i Teort. Fiz. 22, 385 (1952).

[7] G. M. Fleming and J. E. Henderson, Phys. Rev. 59, 907 (1941).