Elastic Scattering of Protons and Neutrons by Deuterons

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Phase shifts for *p-d* and *n-d* scattering are calculated in Born approximation for partial waves with $l \geq 1$. These are used as a starting point for a phase shift analysis of the *p-d* data in the energy range 0–10 Mev. For $l \geq 1$, the phase shifts resulting from the phase shift analysis agree with those calculated in Born approximation. The ^4S and ^2S phase shifts have a reasonable energy dependence; that is, the "$k \cot\delta$" plots are smooth functions of the energy and extrapolate to a set of scattering lengths near one of the known sets of *n-d* scattering lengths. It is concluded that the correct set of *n-d* scattering lengths is

$$a_4 = 6.2 \pm 0.2 \times 10^{-13} \text{ cm}, \quad a_2 = 0.8 \pm 0.3 \times 10^{-13} \text{ cm}.$$

Since this is in disagreement with some previous theoretical conjectures, the scattering lengths and S phase shifts in the energy region 0–10 Mev are calculated using a variational method with neglect of polarization (a theoretical estimate of the effect of polarization is made) and the results support the conclusion. *N-d* angular distributions are calculated and compared with experiments. The agreement of the theoretical results with the experimental ones provides a strong *a fortiori* justification of conclusions drawn from the theory about the importance of the internucleonic potentials in low energy *p-d* and *n-d* scattering. The scattering is nearly independent of the odd parity *n-p* potentials and of the forces between like particles. Furthermore, it is nearly independent of the shape of the ^3S and ^1S-*n-p* potentials. However, the ^2S scattering length is sensitive to the singlet even parity *n-n* potential, and is calculated as a function of the depth of this potential. It is insensitive to other *n-n* potentials.

I. INTRODUCTION

OUR primary purpose is to find out what can be learned about the internucleonic forces from the *p-d* and *n-d* scattering data. Of course, a complete solution of the three body problem is not possible, but the program described in the next section leads to a clarification of the importance of the forces between like particles and the odd parity *n-p* forces in the *p-d* and *n-d* scattering. Unfortunately, it will be shown that these forces influence the scattering to only a minor degree, and the scattering is determined by the ^3S and ^1S *n-p* force. Furthermore, the scattering is insensitive to the shape of the ^3S and ^1S *n-p* potentials.

An important result is obtained from a phase shift analysis of the *p-d* data in the energy range 0–10 Mev.[1–4] It is concluded that the correct set of *n-d* scattering lengths[5] is the set

$$a_4 = 6.2 \pm 0.2 \times 10^{-13} \text{ cm}, \quad a_2 = 0.8 \pm 0.3 \times 10^{-13} \text{ cm}. \tag{1}$$

Because this is in disagreement with some previous theoretical conjectures[6–8] even though in agreement with others,[9] we have recalculated the scattering lengths, and the results support the concluion. As a further test of the results of the phase shift analysis, we have calculated the S phase shifts in the energy region 0–10 Mev, and the calculation confirms the results of the phase shift analysis.

II. PROGRAM

The calculations are carried out for a general central nuclear potential with either a Yukawa or Gauss radial dependence. These potentials are adjusted in range and depth to fit the low energy *n-p*[10,11] and *p-p*[12] data and, in the case of the Yukawa potential, the high energy *n-p* data.[13] Tensor forces are not taken into account since the *n-p* data can be well described by an "equivalent central potential" and possible relatively weak tensor forces in *p-p* and *n-p* odd parity states are expected to play a small role in *n-d* or *p-d* scattering (Sec. VIII-A.2). Any effects attributable to inelastic scattering have been neglected throughout because the experimentally determined inelastic cross section[14] is much smaller than the elastic cross section at these energies (see also Sec. VIII-A.4).

The first step is to calculate the phase shifts for $l \geq 1$ using Born's approximation with the symmetry of the like particles taken into account (Secs. III, IV). The validity of this approximation is discussed in detail in Sec. VIII-A.1. It may be expected to yield reasonable results for the higher angular momentum states because of the large spatial extent of the deuteron and the presence of the centrifugal barrier. Further justification comes from the smallness of the calculated phase shifts.

The next step is to determine the ^4S and ^2S phase shifts so that calculated and experimental angular

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[1] Brown, Freier, Holmgren, Stratton, and Yarnell, Phys. Rev. **88**, 253 (1952).
[2] Sherr, Blair, Kratz, Bailey, and Taschek, Phys. Rev. **72**, 662 (1947).
[3] L. Rosen and J. C. Allred, Phys. Rev. **82**, 777 (1951).
[4] Allred, Armstrong, Bondelid, and Rosen, Phys. Rev. **88**, 433 (1952).
[5] D. G. Hurst and N. Z. Alcock, Can. J. Phys. **29**, 36 (1951); Wollan, Shull, and Koehler, Phys. Rev. **83**, 700 (1951). The latter paper contains references to other experimental work on the *n-d* scattering lengths.
[6] A. Troesch and M. Verde, Helv. Phys. Acta **24**, 39 (1951).
[7] M. M. Gordon, Phys. Rev. **80**, 111 (1950).
[8] F. G. Prohammer and T. A. Welton, Quarterly Report Oak Ridge National Laboratory, Oak Ridge, Tenn. ORNL-1005 (unpublished).
[9] L. Motz and J. Schwinger, Phys. Rev. **58**, 26 (1940).
[10] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).
[11] H. A. Bethe, Phys. Rev. **76**, 38 (1949).
[12] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950).
[13] R. S. Christian, Repts. Progr. Phys. **15**, 68 (1952).
[14] J. H. Coon and R. F. Taschek, Phys. Rev. **76**, 710 (1949).

distributions for p-d scattering in the energy range 0–10 Mev agree within quoted experimental error (Sec. V-A). Improvement in the fits is obtained by allowing the ^4P phase shift to change from its value calculated in Born approximation. When this is done, it is found that the change in the value of the ^4P phase shift is small and a smooth function of energy.

The ^4S and ^2S phase shifts found in this manner have a reasonable energy dependence. That is, the "$k \cot\delta$" plots extrapolate to zero energy to give a set of scattering lengths near one of the known sets of n-d scattering lengths. We find for the p-d scattering lengths

$$a_4 = 12.5 \pm 1.0 \times 10^{-13} \text{ cm}, \qquad (2)$$
$$a_2 = 1.4 \pm 1.0 \times 10^{-13} \text{ cm}.$$

Because of the connection between n-d and p-d scattering lengths (Sec. V-C), this means that the scattering lengths (l) are the correct n-d scattering lengths, in disagreement with some previous theoretical work.[6–8] A careful recalculation of the scattering lengths is undertaken in order to resolve this difficulty.

The calculation of the n-d scattering lengths is carried out using a variational method neglecting polarization of the deuteron (Sec. VI-A). The variational treatment is similar to that used by Massey and Buckingham[15] and Verde.[6,16] In its details, the work has some similarity to the work of Motz and Schwinger.[9] The neglect of polarization (discussed in Sec. VIII-A.3) is not expected to be serious in the case of the quartet state because the exclusion principle prevents the three particles from being close together simultaneously at low energies. Because the phase shift analysis shows that the doublet state makes only a small contribution to the low energy scattering, errors attributable to neglect of polarization are of little consequence for the angular distribution calculations. The results of the calculation,

$$a_4 = 5.9 \times 10^{-13} \text{ cm}, \quad a_2 = 1.5 \times 10^{-13} \text{ cm}, \qquad (3)$$

check the scattering lengths (1) closely for the quartet state and are even in fair agreement for the doublet state. However, they definitely rule out the other set allowed by the experiments:

$$a_4 = 2.4 \times 10^{-13} \text{ cm}, \quad a_2 = 8.3 \times 10^{-13} \text{ cm}. \qquad (4)$$

The quartet and doublet n-d S phase shifts at energies up to 10 Mev are calculated using a somewhat more crude approximation than that employed for the the scattering length calculation (Sec. VI-B). The energy variation of the "$k \cot\delta$" plots is in fair agreement with that found from the phase shift analysis.

n-d angular distributions are calculated (Sec. VII-A) using the theoretical corrections to the p-d phase shifts necessitated by the absence of the Coulomb forces (Sec. IV-D). These are compared with the experimental data.

In Sec. VII-B the problem of deducing information about the n-n forces from the n-d scattering data is discussed. It is shown that the angular distributions are not sensitive to the n-n forces. The ^2S scattering length is, however, and is calculated as a function of the depth of the singlet even parity n-n potential. It is shown to be insensitive to other n-n potentials. The ^4S scattering length depends only on the triplet odd parity n-n potential, to which it is very insensitive.

III. THE BORN APPROXIMATION

Others have described the formulation of Born's approximation for n-d[17,18] and p-d[19] scattering.

There are eight two body potentials $^3V_{np}^+$, $^3V_{np}^-$, $^1V_{np}^+$, $^1V_{np}^-$, $^1V_{pp}^+$, $^3V_{pp}^-$, $^1V_{nn}^+$, $^3V_{nn}^-$. The superscript to the left of $^3V_{np}^+$, for example, specifies that $^3V_{np}^+$ is the potential for a triplet state, while the superscript to the right specifies that it is the potential for a state of even parity. The subscript specifies that $^3V_{np}^+$ is the potential between a neutron and a proton. Thus $^3V_{np}^+$ is the potential for the deuteron ground state. We suppose that all of these potentials have the same radial dependence, so that, for example, $^1V_{np}^+(r) = {}^1V_{np}^+U(r)$, the V's becoming pure numbers. $U(r)$ is taken to be the potential for the ground state of the deuteron, so that $^3V_{np}^+=1$. $U(r)$ is understood to be multiplied by $4M/3\hbar^2$ which reduces it dimensionally to the reciprocal of the square of a length. Let

$$J_1(\theta) = \int \exp(-i\mathbf{k}'\cdot\mathbf{q})\phi(|\mathbf{r}|)U(|\mathbf{q}+\tfrac{1}{2}\mathbf{r}|) \times \exp(i\mathbf{k}\cdot\mathbf{q})\phi(|\mathbf{r}|)d\mathbf{r}d\mathbf{q},$$

$$J_2(\theta) = \int \exp(-i\mathbf{k}'\cdot\mathbf{q})\phi(|\mathbf{r}|)U(|\mathbf{q}+\tfrac{1}{2}\mathbf{r}|) \times \exp[-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{q}-\tfrac{3}{4}\mathbf{r})]\phi(|\mathbf{q}+\tfrac{1}{2}\mathbf{r}|)d\mathbf{r}d\mathbf{q}, \qquad (5)$$

$$J_3(\theta) = \int \exp(-i\mathbf{k}'\cdot\mathbf{q})\phi(|\mathbf{r}|)U(|\mathbf{q}+\tfrac{1}{2}\mathbf{r}|) \times \exp[-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{q}+\tfrac{3}{4}\mathbf{r})]\phi(|\mathbf{q}-\tfrac{1}{2}\mathbf{r}|)d\mathbf{r}d\mathbf{q},$$

where

$$\mathbf{r} = \mathbf{r}_3 - \mathbf{r}_2, \quad \mathbf{q} = -\mathbf{r}_1 + \tfrac{1}{2}(\mathbf{r}_2 + \mathbf{r}_3), \qquad (6)$$

are the center-of-mass coordinates; $\mathbf{k}$, given by

$$k^2 = \frac{8}{9}\frac{M}{\hbar^2}E(\text{lab}), \qquad (7)$$

where E is the energy of the incident nucleon in the

[15] R. A. Buckingham and H. S. W. Massey, Proc. Roy. Soc. (London) **A179**, 123 (1941).
[16] M. Verde, Helv. Phys. Acta **22**, 339 (1949).
[17] M. Verde, Helv. Phys. Acta **22**, 339 (1949).
[18] T.-Y. Wu and J. Ashkin, Phys. Rev. **73**, 986 (1948). In the last of their Eqs. (29), the coefficient of λ^2 is 1 instead of $\frac{1}{2}$.
[19] J. L. Gammel, "Elastic scattering of neutrons and protons by deuterons," Cornell thesis, 1950 (unpublished).

laboratory, is the wave number vector of the incident particle, and $\mathbf{k}'$ is the wave number vector of the scattered particle. Because the scattering is elastic, $|\mathbf{k}|=|\mathbf{k}'|$. The angle between $\mathbf{k}$ and $\mathbf{k}'$ is the scattering angle in the center-of-mass system, θ. $\phi(r)$ is the wave function for the ground state of the deuteron.

For *n-d* scattering, the scattering amplitude is given by

$$4\pi f(\theta)=\alpha J_1(\theta)+\beta J_2(\theta)+\gamma J_3(\theta), \qquad (8)$$

where

$$\begin{aligned}\alpha&=\tfrac{1}{2}\,{}^3V_{np}{}^{+}+\tfrac{1}{2}\,{}^3V_{np}{}^{-}+{}^3V_{nn}{}^{-},\\ \beta&=-{}^3V_{np}{}^{+},\\ \gamma&=\tfrac{1}{2}\,{}^3V_{np}{}^{+}-\tfrac{1}{2}\,{}^3V_{np}{}^{-}-{}^3V_{nn}{}^{-},\end{aligned} \qquad (9)$$

for $S=\frac{3}{2}$; and

$$\begin{aligned}\alpha&=\tfrac{1}{8}\,{}^3V_{np}{}^{+}+\tfrac{1}{8}\,{}^3V_{np}{}^{-}+\tfrac{3}{8}\,{}^1V_{np}{}^{+}+\tfrac{3}{8}\,{}^1V_{np}{}^{-}\\ &\qquad+\tfrac{3}{4}\,{}^1V_{nn}{}^{+}+\tfrac{1}{4}\,{}^3V_{nn}{}^{-},\\ \beta&=+\tfrac{1}{2}\,{}^3V_{np}{}^{+},\\ \gamma&=\tfrac{1}{8}\,{}^3V_{np}{}^{+}-\tfrac{1}{8}\,{}^3V_{np}{}^{-}+\tfrac{3}{8}\,{}^1V_{np}{}^{+}-\tfrac{3}{8}\,{}^1V_{np}{}^{-}\\ &\qquad+\tfrac{3}{4}\,{}^1V_{nn}{}^{+}-\tfrac{1}{4}\,{}^3V_{nn}{}^{-},\end{aligned} \qquad (10)$$

for $S=\frac{1}{2}$.

Corrections to Eq. (8) required by the presence of the Coulomb force in *p-d* scattering are discussed in Sec. IV-D.

The phase shifts are computed from

$$kf(\theta)=\textstyle\sum_l(2l+1)\delta_l P_l(\cos\theta), \qquad (11)$$

and these phase shifts are substituted in the rigorous formula for the angular distribution, which for *p-d* scattering is

$$\begin{aligned}k^2\sigma(\theta)=&\frac{2}{3}\left|\frac{\eta}{1-\cos\theta}\exp\left(i\eta\ln\frac{2}{1-\cos\theta}\right)+\frac{1}{2i}\sum_l(2l+1)\right.\\ &\left.\times[\exp(2i\,{}^4\delta_l)-1]\exp(i\phi_l)P_l(\cos\theta)\right|^2\\ &+\frac{1}{3}\left|\frac{\eta}{1-\cos\theta}\exp\left(i\eta\ln\frac{2}{1-\cos\theta}\right)+\frac{1}{2i}\sum_l(2l+1)\right.\\ &\left.\times[\exp(2i\,{}^2\delta_l)-1]\exp(i\phi_l)P_l(\cos\theta)\right|^2, \end{aligned}\qquad (12)$$

where the superscript on the δ_l's specify the spin state for which they are the phase shifts, and

$$\eta=e^2/\hbar v, \quad \phi=\text{phase}\Gamma(l+1+i\eta). \qquad (13)$$

Reasons for using the exact expression (12) rather than $f^2(\theta)$ corrected for S phase shifts are discussed in Sec. VIII-A.5.

Each of the above integrals forming the scattering amplitude can be given a simple physical interpretation[20] in the sense of validity of Born's approximation, since only a single collision takes place. $J_1(\theta)$ represents potential scattering in which the incident particle scatters off one of the two particles in the deuteron. $J_3(\theta)$ represents a type of scattering or direct "knock out" in which the incident particle exchanges directly with one of the particles in the deuteron so that the "struck" particle is ejected from the deuteron and the original incident particle remains to form the deuteron. (This exchange is attributable in part to the exclusion principle and in part to explicit exchange forces.) $J_2(\theta)$ is a manifestation of the exclusion principle and represents pickup by the incident particle of a dissimilar particle in the deuteron to form a new deuteron, the remaining particle becoming the scattered particle. It can, therefore, be expected that $J_1(\theta)$ is peaked in the forward direction. $J_3(\theta)$ should also be peaked in the forward direction since in order to reconstitute a deuteron the incident particle must transfer most of its momentum to the struck particle. [Actually $J_3(\theta)$ is almost spherically symmetrical.] $J_2(\theta)$ is peaked in the backward direction since following the pickup process the "new" deuteron is most probably traveling in the direction of the incident nucleon.

[20] The physical significance of these integrals has been discussed by others. See the papers of reference 29.

IV. COMPUTATION OF PHASE SHIFTS

The integrals $J_1(\theta)$, $J_2(\theta)$, and $J_3(\theta)$ are calculated using Gauss and Yukawa radial forms for $U(r)$ with ranges adjusted to fit the low energy data.

A. Zero Range Approximation

In order to gain some further insight into the nature and magnitude of the terms in Eq. (5), we first consider the "zero range" approximation. Actually, this need not be thought of as strrictly a zero range approximation; it can be thought of as an approximation in which it is assumed that the other terms in the integrands in Eq. (5) vary slowly compared to the potential. This would be the case if the range of $U(r)$ were very small compared to the deuteron radius and the wavelength of the incident particle. With the potential

$$U(r)=U_0\delta(|\mathbf{r}|), \qquad (14)$$

where $\delta(|\mathbf{r}|)=1$ for $r<r_0$ and $\delta(|\mathbf{r}|)=0$ for $r>r_0$, and U_0 is a potential depth such that $U_0r_0{}^2$ is a constant whose value is $2\pi/3$, we find

$$\begin{aligned}J_1(\theta)&=\frac{4\pi}{3}(2r_0)^3U_0\int\exp(-i\mathbf{k}'\cdot\mathbf{q})\phi^2(|2\mathbf{q}|)\\ &\qquad\times\exp(i\mathbf{k}\cdot\mathbf{q})d\mathbf{q},\\ J_2(\theta)&=\frac{4\pi}{3}(2r_0)^3U_0\langle\phi(r_0)\rangle_{\text{Av}}\int\exp(-i\mathbf{k}'\cdot\mathbf{q})\\ &\qquad\times\phi(|2\mathbf{q}|)\exp(-2i\mathbf{k}\cdot\mathbf{q})dq,\\ J_3(\theta)&=\frac{4\pi}{3}(2r_0)^3U_0\int\exp(-i\mathbf{k}'\cdot\mathbf{q})\phi^2(|2\mathbf{q}|)\\ &\qquad\times\exp(i\mathbf{k}\cdot\mathbf{q})d\mathbf{q},\end{aligned} \qquad (15)$$

$\langle\phi(r_0)\rangle_{\text{Av}}$ is the average value of the deuteron wave function for $r<r_0$. In this approximation we may evidently use the zero range deuteron wave function

$$\phi(r)=(\alpha/2\pi)^{\frac{1}{2}}e^{-\alpha r}/r, \tag{16}$$

inside the integrands as we are only interested in its low Fourier components. The results are

$$J_1(\theta)=\left(\frac{8\pi}{3}\right)^2\frac{\alpha r_0}{k\sin(\theta/2)}\tan^{-1}\left(\frac{k\sin(\theta/2)}{2\alpha}\right),$$

$$J_2(\theta)=\left(\frac{8\pi}{3}\right)^2\left[r_0\langle\phi(r_0)\rangle_{\text{Av}}\left(\frac{2\pi}{\alpha}\right)^{\frac{1}{2}}\right]\frac{2\alpha}{4\alpha^2+k^2(5+4\cos\theta)}$$

$$=\left(\frac{8\pi}{3}\right)^2\frac{\pi\alpha}{4\alpha^2+k^2(5+4\cos\theta)},$$

$$J_3(\theta)=J_1(\theta),$$

$\langle\phi(r_0)\rangle_{\text{Av}}$ was calculated for square wells, and in evaluating $J_2(\theta)$,

$$\lim_{r_0\to0}\langle\phi(r_0)\rangle_{\text{Av}}=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\frac{\pi}{2r_0}$$

is used.

We note that $J_1(\theta)$ is peaked in the forward direction as is $J_3(\theta)$ and $J_2(\theta)$ is peaked in the backward direction in agreement with our physical arguments of the last section.

Also we note that $J_1(\theta)$ and $J_3(\theta)$ are of order $(r_0\alpha)J_2(\theta)$. Detailed analysis shows they also have less angular variation than $J_2(\theta)$, and the contributions of $J_1(\theta)$ and $J_3(\theta)$ to the phase shifts fall off more rapidly with increasing l than the contribution of $J_2(\theta)$.

In calculating $J_1(\theta)$ and $J_2(\theta)$ for other radial forms of the potential, it is convenient to make the transformation

$$\mathbf{q}+\tfrac{1}{2}\mathbf{r}=\mathbf{w},\quad \mathbf{r}=\mathbf{z}.$$

Then

$$J_1(\theta)=\int\exp(-i(\mathbf{k}-\mathbf{k}')\cdot\tfrac{1}{2}\mathbf{z})\phi^2(|\mathbf{z}|)d\mathbf{z}\times\int\exp(i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{w})U(|\mathbf{w}|)d\mathbf{w},$$

$$J_2(\theta)=\int\exp(-i(\mathbf{k}+\tfrac{1}{2}\mathbf{k}')\cdot\mathbf{z})\phi(|\mathbf{z}|)d\mathbf{z}\times\int\exp(-i(\mathbf{k}'+\tfrac{1}{2}\mathbf{k})\cdot\mathbf{w})\phi(|\mathbf{w}|)U(|\mathbf{w}|)d\mathbf{w}.$$

From these forms we note several things. First, as the energy increases, the angular variation of $J_1(\theta)$ increases more rapidly than the angular variation of $J_2(\theta)$ primarily attributable to the part of $J_1(\theta)$ containing $\phi^2(|\mathbf{z}|)$ in its integrand. Thus at very high energy the elastic scattering becomes potential scattering. Second, $J_1(\theta)$ is sensitive to the shape of the potential. At low energies, it is larger for potentials which have a long tail. Third, the potential may be eliminated from $J_2(\theta)$ by making use of the two-body equations of motion

$$U(w)\phi(w)=(\nabla^2-\alpha^2)\phi(w).$$

Then $J_2(\theta)$ is of the form

$$J_2(\theta)=[(\tfrac{1}{2}\mathbf{k}+\mathbf{k}')^2+\alpha^2]\int\exp(i(\mathbf{k}+\tfrac{1}{2}\mathbf{k}')\cdot\mathbf{z})\phi(z)d\mathbf{z}\times\int\exp(-i(\mathbf{k}'+\tfrac{1}{2}\mathbf{k})\cdot\mathbf{w})\phi(w)d\mathbf{w}. \tag{17}$$

At low energies, $J_2(\theta)$ is practically independent of the shape or range of the potential.

$J_3(\theta)$ cannot be separated into the product of two integrals nor can the potential be eliminated. We have seen that in "zero range" approximation it is $J_1(\theta)$. However, this is an overestimate for finite range. A detailed examination of the expansion of $J_3(\theta)$ in spherical harmonics shows that its contributions to the higher phase shifts are less than those of $J_1(\theta)$ because there is no line up of the potential and a deuteron wave function or two deuteron wave functions, which makes it necessary to perform an average over $P_l(\cos\psi)$, where ψ is the angle between $\mathbf{q}$ and $\frac{1}{2}\mathbf{q}+\frac{3}{4}\mathbf{r}$ which varies from 0° to $3\alpha r_0/2$ (about 30°). The effect is more pronounced the longer the tail of the potential.

B. Gauss Radial Form

We use

$$U(r)=U_0\exp(-(r/\lambda)^2),\quad U_0=86.4\text{ Mev},\quad \lambda=1.332\times10^{-13}\text{ cm}. \tag{18}$$

In this calculation, the unit of length was taken to be 1.332×10^{-13} cm, so that

$$k^2=0.03806E.$$

It is necessary to represent the deuteron wave function by the sum of three Gauss terms:

$$\phi(x)=0.02133\exp(-0.03x^2)+0.08582\exp(-0.16x^2)+0.18115\exp(-0.76x^2), \tag{19}$$

which approximates the actual deuteron wave function for the potential (17) to within 3 percent to three times the deuteron radius. Wu and Ashkin[18] have given formulas for $J_1(\theta)$, $J_2(\theta)$, and $J_3(\theta)$ in this case.

Each of the integrals is of the form

$$(k/4\pi)J(\theta)=k\textstyle\sum_i a_i\exp(-b_ik^2)\exp(-c_ik^2\cos\theta), \tag{20}$$

where the a's, b's, and c's are given in Table I.

A partial wave analysis of $kJ_1(\theta)$, $kJ_2(\theta)$, and $kJ_3(\theta)$ is made as follows:

$$(k/4\pi)J_i(\theta)=\textstyle\sum_l(2l+1)\delta_i^lP_l(\cos\theta). \tag{21}$$

TABLE I. Constant for Gauss potential.

	J_1			J_2			J_3		
i	a_i	b_i	c_i	a_i	b_i	c_i	a_i	b_i	c_i
1	0.37966	2.5834	−2.5834	1.0197	10.720	8.5760	0.36854	2.3659	−1.8062
2	0.53992	1.1579	−1.1579	3.4332	10.686	8.5489	0.51846	0.95370	−0.39784
3	0.13499	0.65822	−0.65822	3.8774	10.594	8.4752	0.12840	0.45752	0.089197
4	0.49912	0.89061	−0.89061	1.1217	2.2225	1.7780	0.44144	0.65104	−0.13021
5	0.43223	0.63585	−0.63585	1.2668	2.1307	1.7045	0.35617	0.39603	0.10081
6	0.21063	0.58224	−0.58224	0.25829	0.58874	0.47099	0.13153	0.28604	0.12157
7				0.33314	2.2566	1.8053			
8				0.05854	0.64716	0.53405			
9				0.22869	0.68058	0.54528			

$J_1(\theta)$ and $J_3(\theta)$ do not contribute to the phase shifts for $l \geq 3$.

$\delta_1{}^l$, $\delta_2{}^l$, and $\delta_3{}^l$ are presented graphically in Fig. 1.

C. Yukawa Radial Form

We use

$$U(r)=U_0\frac{\exp(-r/\lambda)}{r/\lambda},$$
$$U_0=68.0 \text{ Mev},\quad \lambda=1.18\times10^{-13}\text{ cm}. \tag{22}$$

In this calculation, the unit of length was taken to be 10^{-13} cm, so that

$$k^2=0.0214E.$$

For the deuteron wave function we use

$$\phi(r)=\left[\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha+\beta)^2}\right]^{\frac{1}{2}}\frac{\exp(-\alpha r)-\exp(-\beta r)}{r}, \tag{23}$$

where[21]

$$\alpha=0.2316,\quad \beta=1.268. \tag{24}$$

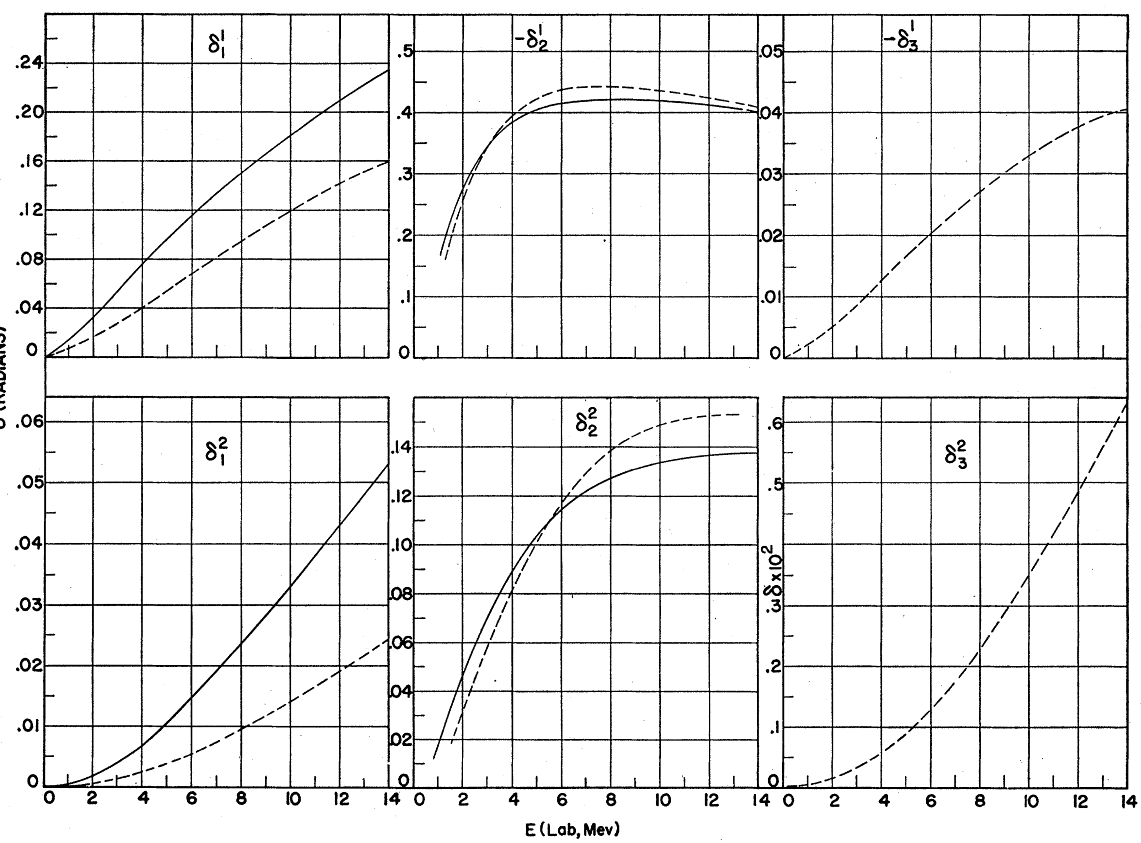

FIG. 1. Plots of $\delta_i{}^l$ for the Gauss (dotted curves) and Yukawa (solid curves) potentials as a function of energy.

[21] G. F. Chew, Phys. Rev. **74**, 809 (1948). We have used Chew's value $\beta=5.5\alpha$. More recently, R. L. Gluckstern and H. A. Bethe, Phys. Rev. **81**, 761 (1951), have given $\beta=7.0\alpha$. This makes little difference (see their footnote 32).

We find for $J_1(\theta)$:

$$\frac{k}{8\pi}J_1(\theta)=\frac{2k}{K}\left[\tan^{-1}\frac{K}{4\alpha}-2\tan^{-1}\frac{K}{2(\alpha+\beta)}+\tan^{-1}\frac{K}{4\beta}\right]\frac{g}{\nu^2+K^2}, \quad (25)$$

where

$$\nu=\frac{1}{\lambda},\quad K=2k\sin\frac{\theta}{2},\quad g=\frac{4M}{3\hbar^2}U_0\lambda\frac{\alpha\beta(\alpha+\beta)}{(\alpha-\beta)^2}=1.056, \quad (26)$$

and for $J_2(\theta)$:

$$\frac{k}{8\pi}J_2(\theta)=\frac{k}{K'}\left[\frac{1}{K'^2+\beta^2}-\frac{1}{K'^2+\alpha^2}\right]\times\left[\tan^{-1}\frac{K'}{\nu+\alpha}-\tan^{-1}\frac{K'}{\nu+\beta}\right]g, \quad (27)$$

where

$$K'=k(\cos\theta+5/4)^{\frac{1}{2}}. \quad (28)$$

A partial wave analysis is also made of these. Again $J_1(\theta)$ does not contribute to the phase shifts for $l\geq 3$. The calculations with a Gauss potential show that $J_3(\theta)$ is nearly spherically symmetrical and does not contribute appreciably to the phase shifts for $l\geq 1$, and exploratory numerical calculations of $J_3(\theta)$ showed that with a Yukawa potential it gives even smaller contributions to the phase shifts for $l\geq 1$. Rather than evaluate $J_3(\theta)$ numerically, we have assumed that it is also spherically symmetrical for a Yukawa potential.

$\delta_1{}^l$ and $\delta_2{}^l$ are presented graphically in Fig. 1.

D. Effect of the Coulomb Force in *p-d* Scattering

The *n-n* potentials are replaced by *p-p* potentials, of course. There are two additional integrals

$$C_1(\theta)=\frac{4M}{3\hbar^2}e^2\int \exp(-i\mathbf{k}'\cdot\mathbf{q})\phi(|\mathbf{r}|)\left(\frac{1}{|\mathbf{q}|}-\frac{1}{|\mathbf{q}+\frac{1}{2}\mathbf{r}|}\right)\exp(i\mathbf{k}\cdot\mathbf{q})\phi(|\mathbf{r}|)d\mathbf{r}d\mathbf{q}, \quad (29)$$

$$C_3(\theta)=\frac{4M}{3\hbar^2}e^2\int \exp(-i\mathbf{k}'\cdot\mathbf{q})\phi(|\mathbf{r}|)\left(\frac{1}{|\mathbf{q}|}-\frac{1}{|\mathbf{q}+\frac{1}{2}\mathbf{r}|}\right)\times\exp(-i\mathbf{k}\cdot(\tfrac{1}{2}\mathbf{q}+\tfrac{3}{4}\mathbf{r})\phi(|\mathbf{q}-\tfrac{1}{2}\mathbf{r}|)d\mathbf{r}d\mathbf{q}. \quad (30)$$

These allow for the fact that the deuteron's charge is not concentrated at its center, but distributed throughout it. The scattering amplitude is given by

$$4\pi f(\theta)=P[\alpha J_1(\theta)+\beta J_2(\theta)+\gamma J_3(\theta)+C_1(\theta)+\epsilon C_3(\theta)], \quad (31)$$

where

$$\epsilon=+1,\ S=\tfrac{3}{2},\quad \epsilon=-1,\ S=\tfrac{1}{2}, \quad (32)$$

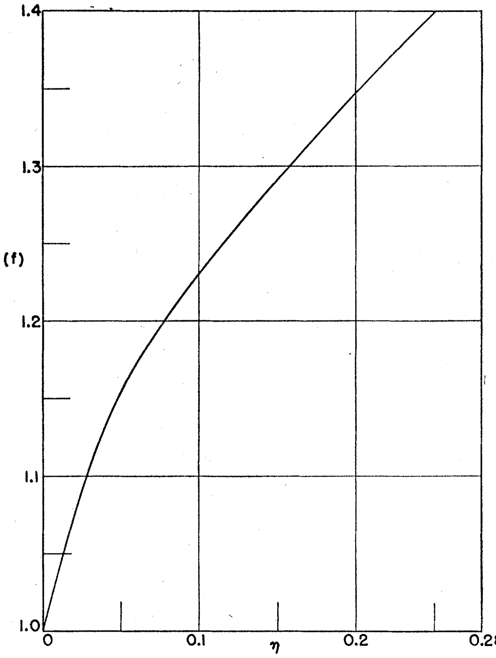

FIG. 2. f, a correction factor to the usual Coulomb penetration factor, defined by Eq. (34) *versus* η.

and P is a penetration factor given by

$$P=fC_0{}^2,\quad C_0{}^2=\frac{2\pi\eta}{\exp(2\pi\eta)-1}. \quad (33)$$

$C_0{}^2$ is the penetration factor one might use offhand. However, the regular Coulomb function $F(\rho)/\rho$ is greater than $C_0{}^2j_l(\rho)$.[22] Therefore, $C_0{}^2$ over estimates the effect of the Coulomb force in reducing the *p-d* phase shifts from the *n-d* phase shifts. To correct for this, we have defined the penetration factor P as follows

$$P=\frac{\frac{1}{k^2}\int_0^\infty \exp(-2\alpha q)F_l(kq)F_l(2kq)(dq/q)}{\int_0^\infty \exp(-2\alpha q)j_l(kq)j_l(2kq)qdq}=fC_0{}^2. \quad (34)$$

We have used the zero range approximation for this simply because it is the only case for which the integrals with Coulomb functions can be handled. f furns out to be independent of l. It is plotted as a function η in Fig. 2.

Actually, we have used only a mean value of f in calculating the Born approximation *p-d* phase shifts; namely

$$\bar{f}=1.273. \quad (35)$$

The integrals $C_1(\theta)$ and $C_3(\theta)$ are found to contribute nothing to the phase shifts for $l\geq 1$.

[22] Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. 23, 147 (1951).

TABLE II. Born approximation phase shifts for p-d scattering (radians).

				For $l \geq 3$, ${}^2\delta_l = -\frac{1}{2}\,{}^4\delta_l$				
E (lab, Mev)	${}^4\delta_1$	${}^4\delta_2$	${}^4\delta_3$	${}^4\delta_4$	${}^4\delta_5$	${}^4\delta_6$	${}^2\delta_1$	${}^2\delta_2$
9.66	0.549	−0.127	0.0478	−0.0154	0.00618	−0.00246	−0.058	0.103
5.18	0.466	−0.101	0.0336	−0.0094			−0.116	0.064
3.49	0.380	−0.074	0.0204	−0.0049			−0.118	0.043
3.00	0.345	−0.064					−0.113	0.036
2.53	0.307	−0.053	0.0120	−0.0025			−0.110	0.029
2.08	0.264	−0.042					−0.096	0.023
1.61	0.212	−0.029					−0.082	0.016
1.51	0.200	−0.027	0.0043	−0.0006			−0.078	0.014
1.495	0.196	−0.026					−0.077	0.014
1.355		−0.022						0.011
1.23	0.162	−0.019					−0.065	0.010
1.105		−0.0164						0.0082
0.985		−0.0130						0.0065
0.825		−0.0099						0.0049
0.73		−0.0082						0.0041
0.6		−0.0062						0.0031
0.48		−0.0047						0.0023

V. ANALYSIS OF p-d EXPERIMENTS

A. Phase Shift Analysis

According to Eqs. (8), (11), and (21), the phase shifts are given by

$$\delta_l = \alpha\delta_1{}^l + \beta\delta_2{}^l + \gamma\delta_3{}^l, \tag{36}$$

where α, β, and γ are given by (9) for $S=\frac{3}{2}$ and by (10) for $S=\frac{1}{2}$.

In calculating the phase shifts, we have taken ${}^3V_{np}{}^+=1$, ${}^1V_{np}{}^+=0.69$ in agreement with the low energy n-p data. We have taken the odd parity potentials to be zero in agreement with the high energy n-p data. We have assumed that n-n, p-p, and n-p potentials are equal in equivalent states. Because of the small size of $\delta_1{}^l$ and $\delta_3{}^l$ compared with $\delta_2{}^l$ for $l \geq 1$, and because $\delta_2{}^l$ depends only on ${}^3V_{np}{}^+$, it does not matter much (not at all for $l \geq 3$) what we take for the odd parity forces or what we assume about the n-n and p-p potentials, a point returned to in Sec. VII-B. The phase shifts are tabulated in Table II.

Using these phase shifts as a starting point, a phase shift analysis of the p-d data in the energy range 0–10 Mev[1–4] is made using the rigorous expression (12). The method used in making the phase shift analysis is the same as that used by Dodder and Gammel in analyzing the p-He^4 elastic scattering data.[23] The phase shift analyses were performed on IBM card-programmed electronic calculators. The decks were prepared so that if fits could not be obtained with the S phase shifts alone, the P phase shifts could be changed from their values calculated in Born approximation and the data analyzed for them; the same is true of the D phase shifts.

The results of the phase shift analysis are given in Table III. Some improvement is obtained by allowing the ${}^4P$ phase shift to change from its value calculated in Born approximation. The ${}^4P$ phase shift found if this is done is compared with that calculated in Born approximation in Fig. 3. The ${}^2P$ phase shift and phase shifts for $l \geq 2$ agree with those calculated in Born approximation. In analyzing for the S phase shifts or the S and ${}^4P$ phase shifts, the ${}^2P$ phase shifts and phase shifts for $l \geq 2$ used are those calculated with the Yukawa shaped potential. No significance is to be attached to this since those calculated with the Gauss shaped potential would have served as well.

Calculated and experimental angular distributions are compared in Fig. 4. The effect of omitting the contribution of partial waves with $l \geq 3$ is illustrated in several figures, making it clear that the higher phase shifts are certainly necessary in the analysis.

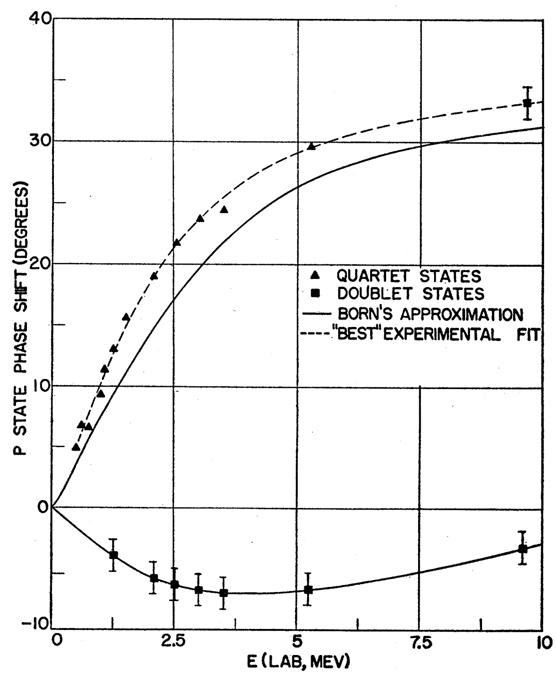

FIG. 3. Comparison of calculated and "experimental" P phase shifts.

[23] D. C. Dodder and J. L. Gammel, Phys. Rev. 88, 520 (1952).

TABLE III. Results of phase shift analysis.

E	^4S	^2S	^4P	rms	^4P on Born approx. ^4S	^2S	rms
9.66	73.34°	79.05°	31.78°	3.2 percent	72.64°	78.65°	4.72 percent
5.18	−78.93°	−39.84°	29.73°	3.56 percent	−78.23°	−40.45°	5.13 percent
3.49	−66.94°	−22.2°	24.5°	6.70 percent			
3.00	−66.27°	−19.3°	23.77°	2.63 percent			
2.53	−61.01°	−17.1°	21.78°	2.0 percent			
2.08	−54.89°	−14.1°	18.93°	1.9 percent			
1.61	−52.48°		14.67°	0.9 percent			
1.51	−46.92°	−10.5°	15.76°	3.2 percent			
1.495	−49.27°		15.60°	1.46 percent			
1.355	−45.38°		14.33°	1.73 percent			
1.23	−44°		13.0°	2.1 percent			
1.105	−42.40°		11.54°	1.9 percent			
0.985	−38.50°		9.50°	1.67 percent			
0.73	−32.09°		6.69°	3.75 percent			
0.6	−27.5°		6.83°	2.58 percent			
0.48	−23.19°		5.08°	1.6 percent			

The question of how well the data determines a given phase shift is an important one. The *p-d* data at 5.2 and 9.7 Mev were carefully examined to determine how well the ^4S and ^2S phase shifts are determined with the phase shifts for $l\geq 1$ given values calculated in Born approximation. Figures 5(a) and (b) show plots of the sum of the squares of the percent deviations from the experimental points ($\sum$) as a function of the ^2S phase shift for several values of the ^4S phase shift. These make it possible to visualize a surface

$$\sum=\sum({}^4\delta_0, {}^2\delta_0).$$

It can be seen that at 5.2 Mev. there are two solutions for the ^2S phase shift. The solution with $^2\delta_0=57°$ can be ruled out after the calculations of Sec. VI-B make it certain that the "k cotδ" plot for the ^2S state has a certain energy dependence. At 9.7 Mev, these two solutions blend, which makes for a very great uncertainty in the value of the ^2S phase shift. The ^4S phase shifts are reasonably well determined, however. All this has its origin in the fact that the doublet state makes only a small contribution to the elastic scattering. Unfortunately, only the ^2S phase shift is sensitive to the *n-n* forces in *n-d* scattering, and it is sensitive only to the singlet even parity *n-n* potential. As this analysis shows, the ^2S phase shift is hard to determine from the experimental data. In Sec. VII-B, further illustrations of the fact that wide changes in the ^2S phase shift do not affect the angular distribution are presented.

Uncertainties in quantities (phase shifts, "k cot δ") deduced from experiment are indicated on the graphs. These are based on studies similar to those described in the previous paragraph.

B. *p-d* Scattering Length and Effective Ranges

Plots of

$$\text{“}k\cot\delta\text{''}=C_0{}^2k\cot\delta+h(\eta)/R,\quad R=\tfrac{3}{4}(\hbar^2/Me^2)=21.6\times10^{-13}\text{ cm},\qquad(37)$$

for the ^4S and ^2S states are shown in Fig. 6. The scattering lengths are the set (2), and the effective range for the ^4S state is

$$\rho_4=1.99\pm0.07\times10^{-13}\text{ cm}.\qquad(38)$$

For $S=\frac{1}{2}$, the scattering length is very short so that an effective range theory is not very useful. The large uncertainties in the ^2S phase shifts make it unadvisable to quote a value of ρ_2 deduced from experiment.

C. Connection between *p-d* and *n-d* Scattering Lengths

Blatt and Jackson[12] give the following connection between the *p-p* and *n-p* scattering lengths

$$\frac{1}{a_p}=\frac{1}{a_N}+\frac{1}{R}\left[\ln\frac{r_0}{R}+0.330\right].\qquad(39)$$

They use for the wave function in the "internal" region

$$u=\sin(\pi r/2r_0).$$

Our wave functions (Sec. VI-B, Fig. 9) have maxima at about 2.0×10^{-13} cm. Taking this as an estimate of r_0, we find

$$1/a_p-1/a_N=-0.095.$$

For the ^4S state, with our value of $^4a_p=12.5\times10^{-13}$ cm, this gives

$$^4a_N=5.7\times10^{-13}\text{ cm}.$$

For the ^2S state, with our value $^2a_p=1.4$, it gives

$$^2a_N=1.2\times10^{-13}\text{ cm}.$$

Of course, the value for the doublet state means almost nothing because we cannot get an accurate value of ^2a_p from the experiments. Had we used a slightly larger value of r_0, the value for the quartet state would have been in better agreement with (1).

However, the set (4) is definitely ruled out, and the *p-d* scattering lengths indicate that the correct set of *n-d* scattering lengths is the set (1).

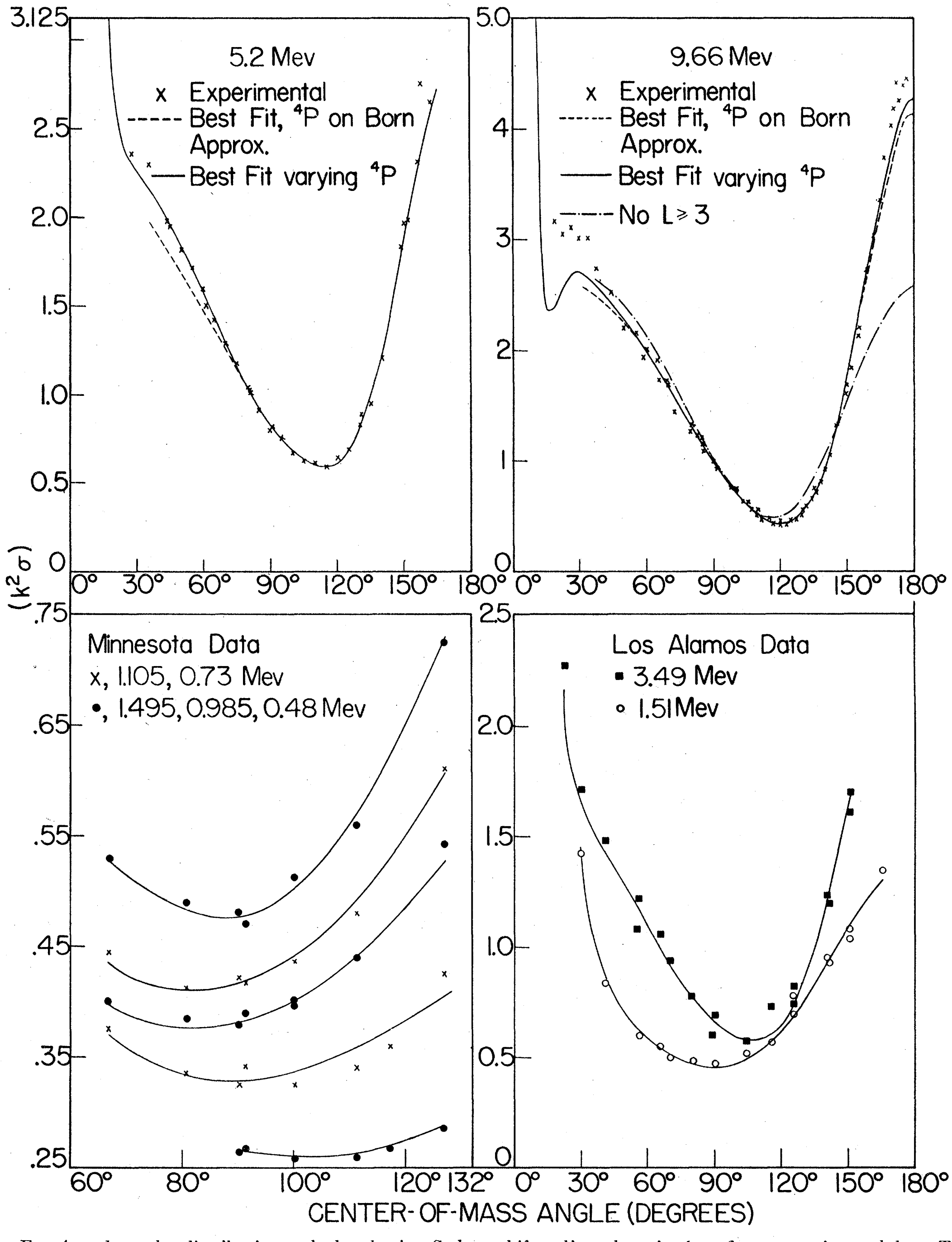

FIG. 4. p–d angular distributions calculated using S phase shifts adjusted to give best fits to experimental data. The The ordinate is the dimensionless quantity $k^2\sigma(\theta)$.

VI. S-STATE CALCULATIONS

A. Scattering Lengths, Effective Ranges

To support the conclusion of the last paragraph, we have calculated the n-d scattering lengths.

For $S=\frac{3}{2}$, assuming for purposes of illustration that the nuclear potentials are spin independent but that the odd parity forces are zero, the equations of motion are

$$\begin{aligned} T\psi' &= -\tfrac{1}{2}U^s\psi' - \tfrac{1}{2}(U''\psi' - U'\psi''), \\ T\psi'' &= -\tfrac{1}{2}U^s\psi'' - \tfrac{1}{2}(U'\psi' - U''\psi''). \end{aligned} \tag{40}$$

This is written in Verde's notation.[16] This can be written

$$T\begin{pmatrix}\psi' \\ \psi''\end{pmatrix} = (U)\begin{pmatrix}\psi' \\ \psi''\end{pmatrix}, \tag{41}$$

where (U) is a matrix. Verde[16] gave the variational principle

$$\delta\int(\psi',\psi'')[T-(U)]\binom{\psi'}{\psi''}d\tau_1 d\tau_2 d\tau_3=0. \quad (42)$$

In the no polarization approximation, ψ' and ψ'' are written

$$\begin{aligned}\psi'&=\tfrac{1}{2}\sqrt{3}\{\phi(12)f(3)-\phi(13)f(2)\},\\ \psi''&=-\phi(23)f(1)+\tfrac{1}{2}\phi(12)f(3)+\tfrac{1}{2}\phi(13)f(2),\end{aligned} \quad (43)$$

where ϕ is the wave function of the ground state of the deuteron and f is a function to be determined. The coordinate are labeled such that a pair of numbers indicate a vector extending from the first particle of the pair to the second number while a single number indicates a vector extending from that particle to the center of the other two remaining particles.

Substituting Eq. (43) into Eq. (42), eliminating integrals with $f(1)$ by permuting 1 and 2 (the integrals are invariant with respect to these permutations), varying $f(2)$ and $f(3)$ independently, and equating the coefficients of $\delta f(2)$ and $\delta f(3)$ to zero leads to a differential equation for $f(3)$:

$$\begin{aligned}(\nabla_3^2+k^2)f(3)-\int\phi(12)(\nabla_2^2+k^2)f(2)\phi(13)d\tau_1d\tau_2\\ =\int\phi(12)U(12)\phi(13)f(2)d\tau_1d\tau_2\\ -\tfrac{1}{2}\int\phi(12)U(13)\phi(12)f(3)d\tau_1d\tau_2\\ -\tfrac{1}{2}\int\phi(12)U(23)\phi(13)f(2)d\tau_1d\tau_2.\end{aligned} \quad (44)$$

The second term on the left and the first term on the right may be put in a different form by making use of the fact that $\phi(13)$ is a solution to the deuteron problem. We may then replace Eq. (44) by

$$\begin{aligned}(\nabla_3^2+k^2)\left[f(3)-\int f(2)\phi(13)\phi(12)d\tau_1d\tau_2\right]\\ =\int\phi(12)U(13)\phi(13)f(2)d\tau_1d\tau_2\\ -\tfrac{1}{2}\int\phi(12)U(13)\phi(12)f(3)d\tau_1d\tau_2\\ -\tfrac{1}{2}\int\phi(12)U(23)\phi(13)d\tau_1d\tau_2.\end{aligned} \quad (44')$$

Equations (44) and (44′) lead to identically the same results if ϕ is an exact solution of the deuteron problem. The exact time dependent equations of motion satisfy the reciprocity theorem;[24] that is, the transition probability for a process and its inverse (obtained by reversal of time) are equal. To insure this property for any choice of a trial function in which ϕ is not an exact solution, we must use a fifty-fifty mixture of Eqs. (44) and (44′).

Use of the zero energy Green's function $1/4\pi|\mathbf{r}_3-\mathbf{r}_3'|=1/4\pi R_3$ in (44′) leads to

$$\begin{aligned}f(3')=1+\int f(2)\phi(13')\phi(12)d\tau_1d\tau_2\\ +\int\frac{1}{4\pi R_3}\phi(12)U(13)\phi(13)f(2)d\tau_1d\tau_2d\tau_3\\ -\tfrac{1}{2}\int\frac{1}{4\pi R_3}\phi(12)U(13)\phi(12)f(3)d\tau_1d\tau_2d\tau_3\\ -\tfrac{1}{2}\int\frac{1}{4\pi R_3}\phi(12)U(23)\phi(13)f(2)d\tau_1d\tau_2d\tau_3.\end{aligned} \quad (45)$$

The first integral, which we call the "orthogonality" integral arises from the Pauli principle and vanishes in Born approximation. (Because of this term it appears

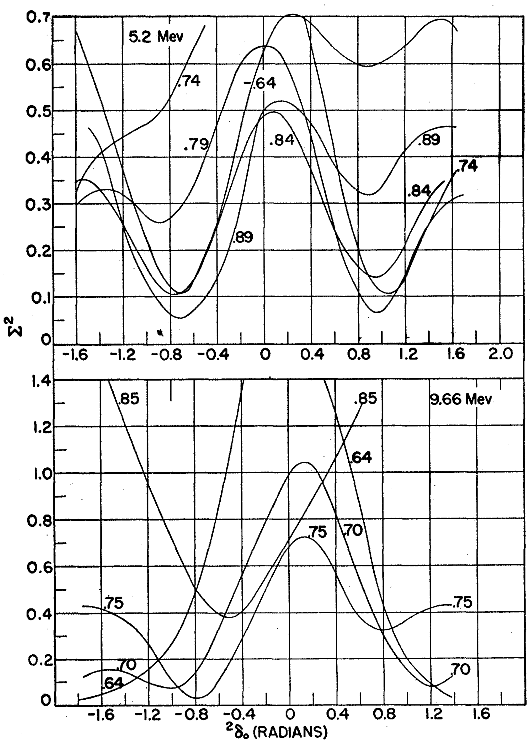

FIG. 5. Plots of the sum of the squares of the percent deviations of calculated points from experimental points *versus* the ^2S phase shift for several values of the ^4S phase shift (the curves are labeled with one-half of the value of the ^4S phase shift). At 5.2 Mev, 13 points are used; at 9.66 Mev, 15 points are used.

[24] B. A. Lippman and J. Schwinger, Phys. Rev. **79**, 469 (1950).

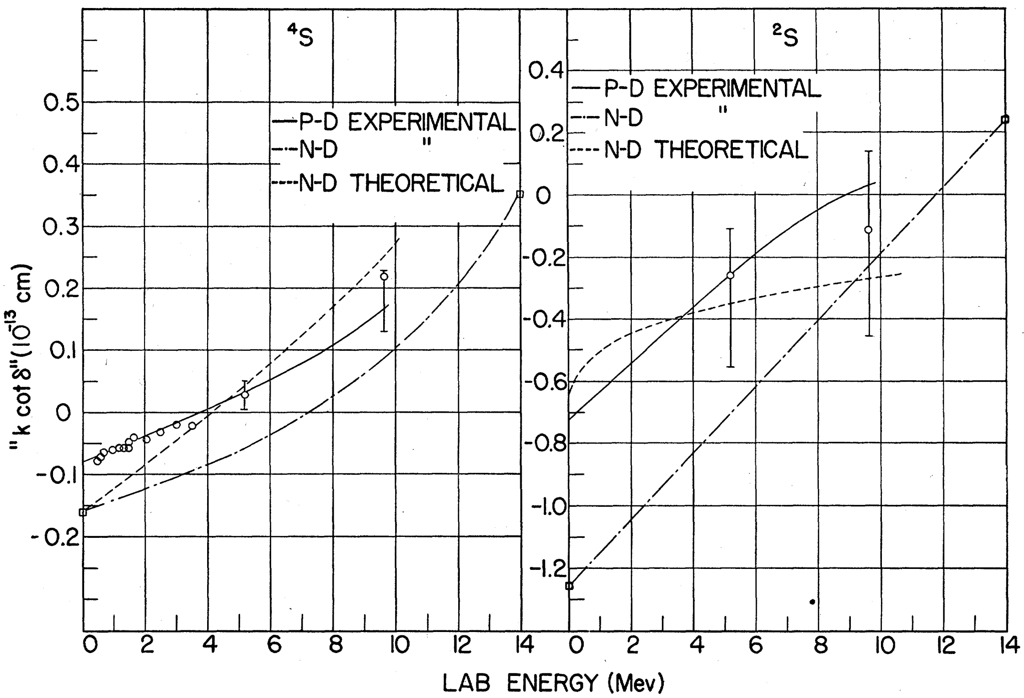

FIG. 6. Plots of "$k\cot\delta$" defined by Eq. (37) for the ^4S and ^2S phase shifts.

as though the interaction would be different from zero even if the force between the incoming particle and the deuteron were zero. However, the interaction between the deuteron and the incoming particle cannot be zero because of the identity of the two like particles unless the interaction of the two particles forming the deuteron is also zero.) When Born's approximation is made and the second, third, and fourth integrals examined asymptotically in the variable 3, they are of the form $J_2(\theta)$, $J_1(\theta)$, and $J_3(\theta)$, respectively, and are labeled accordingly.

Calculations were performed with three forms of J_2; namely, (1) that resulting from Eq. (44), (2) that resulting from a fifty-fifty mixture of Eqs. (44) and (44′), and (3) that resulting from Eq. (44′). In all these calculations, only the form of O in Eq. (44′) or (45) was used. This means that the scattering lengths resulting with the first and second forms of J_2 are not stationary as is the scattering length resulting with the third form. However, if ψ' and ψ'' (Eq. (43)) were exact solutions of the wave equations, $\phi(12)f(3)$, $\phi(13)$-$f(2)$, and $\phi(23)f(1)$ would also be exact (unsymmetrized) solutions of the wave equation, and both forms of O used with any combination of J_2's would lead to the same results. Thus a comparison of the results of the calculations with the forms (1) and (2) of J_2 and the results of the calculation with from (3) of J_2 reflects in some manner the adequacy of the no polarization approximation.

Motz and Schwinger had equations similar to Eq. (45), but their derivation is not based on a variational principle and did not contain the orthogonality integral.

For the rest, the procedure is the same as that of Motz and Schwinger. For this calculation we use the potential (17) and the wave function (19). The dummy variable and angle variables are integrated out leading to an equation of the form

$$qf(q)=q+\int K(q,q')q'f(q')dq', \tag{46}$$

where

$$K(q,q')=-\beta O(q,q')+\alpha J_1(q,q')+\beta J_2(q,q')+\gamma J_3(q,q'). \tag{47}$$

Here the dependence on the two body potentials has been put back in. Contour plots of the kernels $O(q,q')$, $J_1(q,q')$, both forms of $J_2(q,q')$ and $J_3(q,q')$ are shown in Fig. 7. Equation (46) was solved by replacing it by set of linear algebraic equations. Twenty points were used for this. Contributions to the integral from four times the deuteron radius were required and retained. The sets of linear equations were solved on the MANIAC.

The results of the calculation are summarized in Table IV. Graphs of $u(q)=qf(q)$ are shown in Fig. 8.

Effective ranges can be derived from $u(q)$. Verde[25] and Breit[26] have given formulas for the effective ranges.

[25] M. Verde, Atti accad. nazl. Lincei **8**, 228 (1950). (See also reference 6.)

[26] G. Breit, Revs. Modern Phys. **23**, 228 (1951).

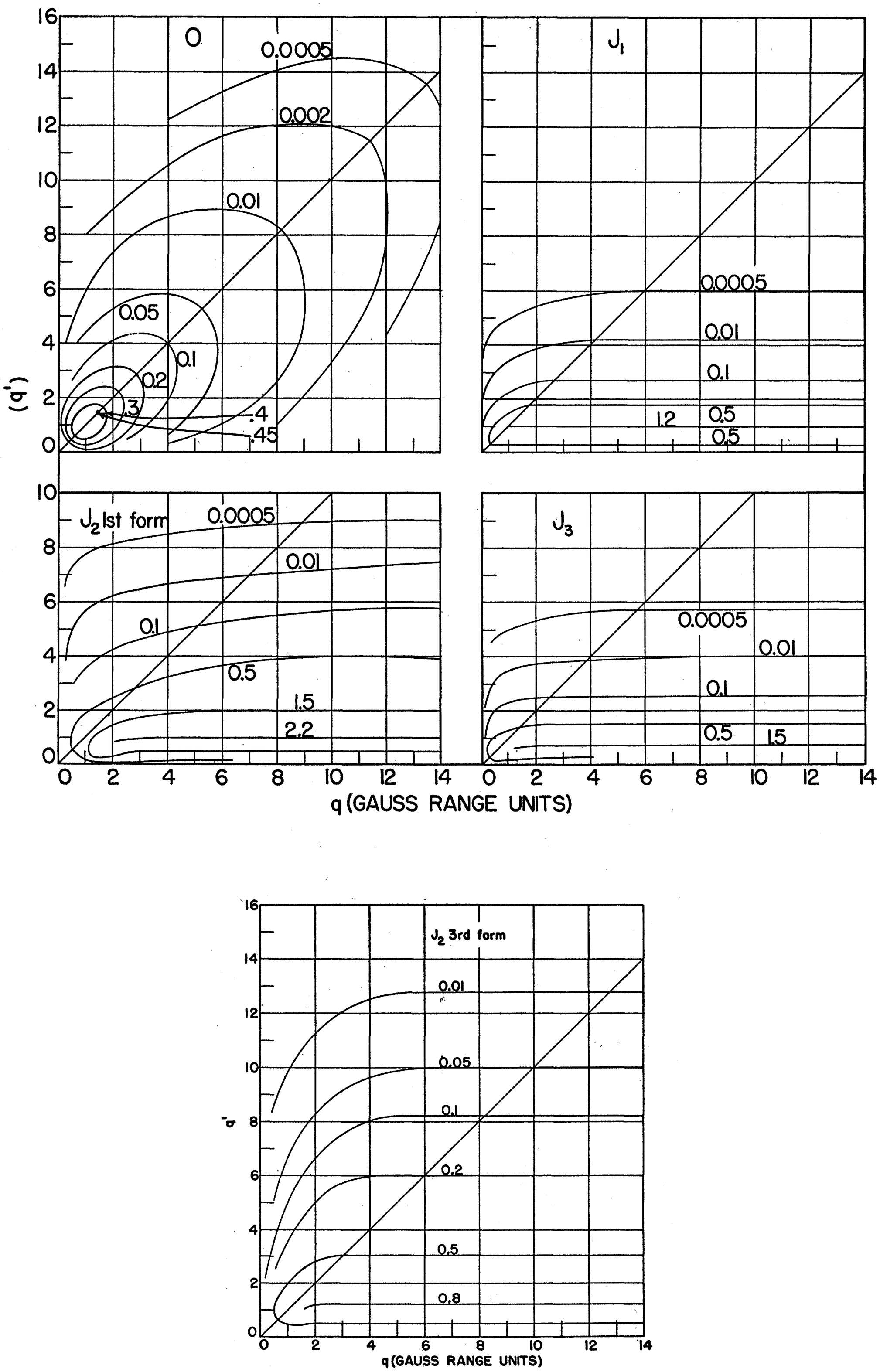

FIG. 7. Contour plots of the zero energy kernels q and q' are plotted in Gauss range units (1.332×10^{-13} cm). The value of the kernel in Eq. (46) is found from Eqs. (9), (10), and (47) with ${}^3V_{np}{}^+=1$.

TABLE IV. Results of scattering length calculations.

Case	$^3V_{np}^+$	$^1V_{np}^+$	$^3V_{np}^-$	$^1V_{np}^-$	$^1V_{nn}^+$	$^3V_{nn}^-$	Form of J_2	a_4 ($\times 10^{-13}$ cm)	a_2
1	1	0.69	0	0	0.69	0	(1) As in Eq. (44) (2) Fifty-fifty mixture (3) As in Eq. (44′)	6.398 5.871 5.334	1.651 1.544 1.520
2	1	0.69	0	0	0.552	0	(1) (2) (3)	6.398 5.871 5.334	2.507
3	1	0.69	0	0	0.345	0	(1) (2) (3)	6.398 5.871 5.334	4.280
4	1	0.69	0	0	0	0	(1) (2) (3)	6.398 5.871 5.334	11.75 9.774
5	1	0.69	1	0.69	0.69	1.0	(1) (2)	6.396 5.903	1.412

These reduce in the no polarization approximation to

$$\rho_4=\frac{2r_0^3}{a_4^2}\left\{\int[(-q+a_4)^2-u^2(q)]dq+\int O(q,q')u(q)u(q')dqdq'\right\}, \quad (48)$$

for $S=\frac{3}{2}$, and

$$\rho_2=\frac{2r_0^3}{a_2^2}\left\{\int[(-q+a_2)^2-u^2(q)]dq-\tfrac{1}{2}\int O(q,q')u(q)u(q')dqdq'\right\}, \quad (49)$$

for $S=\frac{1}{2}$. The effective ranges are calculated with the wave functions resulting with the form (1) (see Table IV) for J_2 and the potentials of cases 1 and 3 of Table IV. The results are

$$\rho_4=2.9\times 10^{-13}\text{ cm},\quad \rho_2=45.1\times 10^{-13}\text{ cm}. \quad (50)$$

However, we wish to stress that since these are not variationally determined quantities the error in using the no polarization approximation may be considerably larger than in the case of the scattering lengths.

B. *S* Phase Shifts 0–10 Mev

A very approximate calculation of the S-phases in the 0–10 Mev energy range was considered necessary from a theoretical standpoint because of the uncertainty in the experimental determination of the ^2S phase shifts. Most situations investigated so far in potential scattering have a bound state lying close to zero energy and the "$k\cot\delta$" plot extrapolates reasonably well into the negative energy region using the scattering length and zero energy effective range to give the binding energy approximately (as it does for the ^3S state in the two body problem). A square well which gives the binding energy of the extra neutron in the triton and the short n-d ^2S scattering length has a negative effective range. To eliminate this possibility, it was thought that a very elaborate calculation was not necessary since there is little likelihood that an equivalent potential could be found for the three body problem. We consider the finite energy extension of the no polarization approximation Eq. (45)

$$\begin{aligned}f(q)=\frac{\sin(kq)}{kq}&+\left(\frac{4}{3}\right)^3\int\phi\left(\left|\frac{2}{3}\mathbf{q}+\frac{4}{3}\mathbf{q}'\right|\right)\phi\left(\left|\frac{2}{3}\mathbf{q}'+\frac{4}{3}\mathbf{q}\right|\right)f(q')d\mathbf{q}'+\left(\frac{4}{3}\right)^3\int\int\frac{\exp(ik|\mathbf{q}-\mathbf{q}''|)}{4\pi|\mathbf{q}-\mathbf{q}''|}\\&\times\phi\left(\left|\frac{2}{3}\mathbf{q}'+\frac{4}{3}\mathbf{q}''\right|\right)\frac{1}{2}\left[U\left(\left|\frac{4}{3}\mathbf{q}'+\frac{2}{3}\mathbf{q}''\right|\right)+U\left(\left|\frac{4}{3}\mathbf{q}''+\frac{2}{3}\mathbf{q}'\right|\right)\right]\phi\left(\left|\frac{4}{3}\mathbf{q}'+\frac{2}{3}\mathbf{q}''\right|\right)f(q')d\mathbf{q}'d\mathbf{q}''\\&-\frac{1}{2}\int\int\frac{\exp(ik|\mathbf{q}-\mathbf{q}'|)}{4\pi|\mathbf{q}-\mathbf{q}'|}\phi^2(|\mathbf{q}''|)U(|\mathbf{q}'+\tfrac{1}{2}\mathbf{q}''|)f(q')d\mathbf{q}'d\mathbf{q}''\\&-\frac{1}{2}\left(\frac{4}{3}\right)^3\int\int\frac{\exp(ik|\mathbf{q}-\mathbf{q}''|)}{4\pi|\mathbf{q}-\mathbf{q}''|}\phi\left(\left|\frac{4}{3}\mathbf{q}'+\frac{2}{3}\mathbf{q}''\right|\right)U\left(\left|\frac{2}{3}\mathbf{q}'-\frac{2}{3}\mathbf{q}''\right|\right)\phi\left(\left|\frac{2}{3}\mathbf{q}'+\frac{4}{3}\mathbf{q}''\right|\right)f(q')d\mathbf{q}'d\mathbf{q}''. \quad (51)\end{aligned}$$

Employing the "zero range" approximation allows us to write

$$f(q)=\frac{\sin(kq)}{kq}+\left(\frac{4}{3}\right)^3\int\phi\left(\left|\frac{4}{3}\mathbf{q}+\frac{2}{3}\mathbf{q}'\right|\right)\phi\left(\left|\frac{4}{3}\mathbf{q}'+\frac{2}{3}\mathbf{q}\right|\right)f(q')d\mathbf{q}'$$

$$+\frac{1}{2}\left\{(2r_0)^3\left(\frac{4\pi}{3}\right)U_0\langle\phi(r_0)\rangle_{\text{Av}}\int\frac{\exp(ik|\mathbf{q}+2\mathbf{q}'|)}{4\pi|\mathbf{q}+2\mathbf{q}'|}\phi(2q')f(q')d\mathbf{q}'\right.$$

$$\left.+(r_0)^3\left(\frac{4\pi}{3}\right)U_0\langle\phi(r_0)\rangle_{\text{Av}}\int\frac{\exp(ik|\mathbf{q}+\frac{1}{2}\mathbf{q}'|)}{4\pi|\mathbf{q}+\frac{1}{2}\mathbf{q}'|}\phi(q')f(q')d\mathbf{q}'\right\}$$

$$-\tfrac{1}{2}(2r_0)^3\left(\frac{4\pi}{3}\right)U_0\int\frac{\exp(ik|\mathbf{q}-\mathbf{q}'|)}{4\pi|\mathbf{q}-\mathbf{q}'|}\phi^2(2q')f(q')d\mathbf{q}'-\tfrac{1}{2}(2r_0)^3\left(\frac{4\pi}{3}\right)U_0\int\frac{\exp(ik|\mathbf{q}-\mathbf{q}'|)}{4\pi|\mathbf{q}-\mathbf{q}'|}\phi^2(2q')f(q')d\mathbf{q}'. \quad (52)$$

Thus in "zero range" approximation the net effect is simply to change the "Green's function factor" from

$$\frac{1}{4\pi|\mathbf{q}-\mathbf{q}'|}\rightarrow\frac{\exp(ik|\mathbf{q}-\mathbf{q}'|)}{4\pi|\mathbf{q}-\mathbf{q}'|}, \quad (53)$$

for $J_1(q,q')$ and $J_3(q,q')$ type terms, and

$$\frac{1}{4\pi|\mathbf{q}+2\mathbf{q}'|}\rightarrow\frac{\exp(ik|\mathbf{q}+2\mathbf{q}'|)}{4\pi|\mathbf{q}+2\mathbf{q}'|}, \quad (54)$$

for the first form of $J_2(q,q')$, and by

$$\frac{1}{4\pi|\mathbf{q}+\frac{1}{2}\mathbf{q}'|}\rightarrow\frac{\exp(ik|\mathbf{q}+\frac{1}{2}\mathbf{q}'|)}{4\pi|\mathbf{q}+\frac{1}{2}\mathbf{q}'|}, \quad (55)$$

for the third form of $J_2(q,q')$. $O(q,q')$ remains unchanged. (It should be noted that only the spherically symmetric term in the expansion of the source function is necessary; however, to simplify the writing we have not indicated this. It is also interesting to note that these "zero range" forms of the "Green's function factor" for $J_2(q,q')$ are physically intelligible and represent a pick up process even when Born's approximation is not valid. The reason that $\frac{1}{2}\mathbf{q}'$ and $2\mathbf{q}'$ occur instead of $\mathbf{q}'$ when integrating the wave function in the neighborhood of the source function is that the neutron that we were originally following was picked up by the proton in the deuteron and afterward the wave function describes the motion of the "freed" neutron which will not originate at the point of collision. In the same manner we see it is justified to say that $J_3(q,q')$ represents a direct knock out" process.) We thus divided our zero energy kernels $K(q,q')$ of the previous section by the zero energy "Green's function factor." We believe this procedure eliminates most of the error arising from the fact that the range of the forces is not very small compared to the deuteron radius. The method should be satisfactory as long as $kr_0<1$. As 9.7 Mev, $kr_0=0.6$.

With these kernels we proceed as before, replacing the integral equations by sets of 20×20 linear systems. The wave functions are shown in Fig. 9, and the "$k\cot\delta$" plots are compared with those deduced from experiment in Fig. 6. The theoretical "effective ranges" determined in this way are in good agreement with those calculated in the preceding section. The results conclusively rule out the possibility mentioned at the beginning of this section. They also rule out the solution for the S phase shifts at 5.2 Mev which has $^2\delta_0=57°$ (see Sec. V-A, Fig. 5).

VII. *n-d* SCATTERING

A. *n-d* Angular Distributions

A phase shift analysis of the 14-Mev *n-d* data[27] is made. The results are summarized in Table V. Phase shifts for $l\geq2$ agrees with those calculated in Born approximation. The phase shift analysis makes it

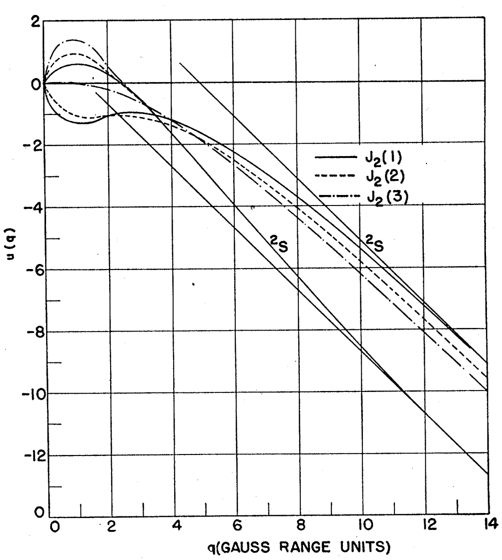

FIG. 8. Graphs of the zero energy radial wave functions $u(q)$ calculated with the "no polarization" approximation. Curves are labeled by $J_2(1)$, $J_2(3)$, and $J_2(2)$ depending upon whether the form of $J_2(q,q')$ has the form it has in Eq. (44), the Motz-Schwinger form, or a fifty-fifty mixture of the two, respectively. (Note: the ^{2}S on the right in the figure should read ^{4}S.)

[27] Allred, Armstrong, and Rosen (preceding paper), Phys. Rev. **91**, 90 (1953).

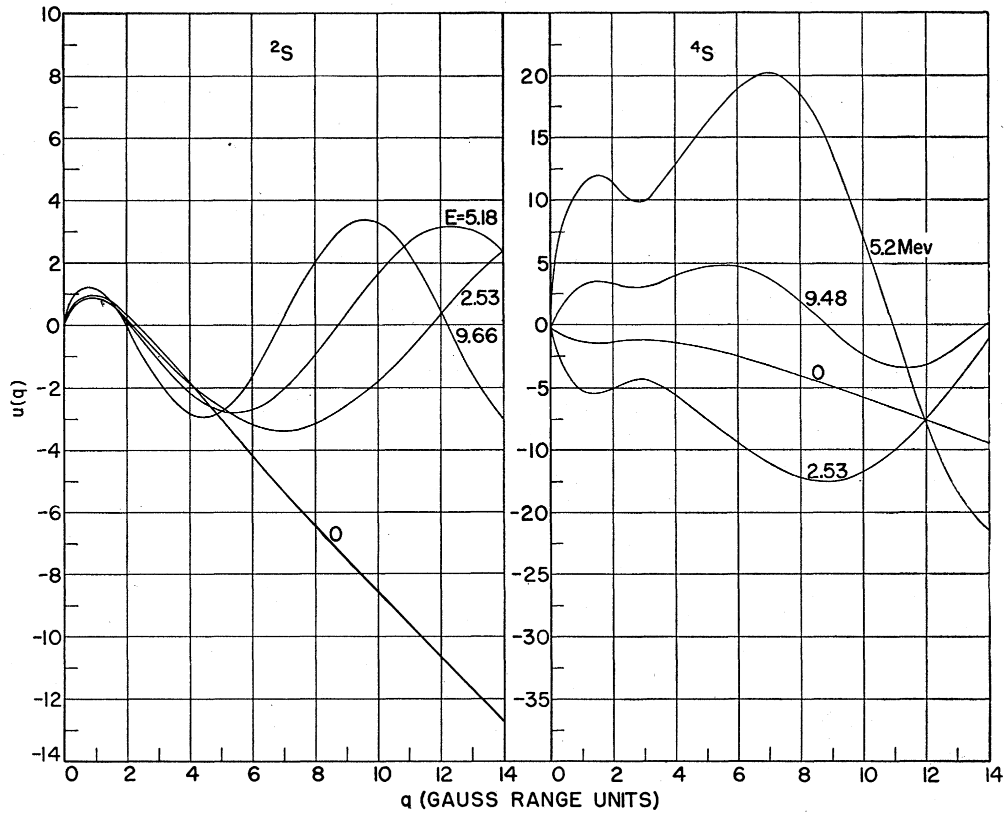

FIG. 9. Graphs of the radial S wave function $u(q)$ calculated with the "no polarization" approximation at several energies. The fifty-fifty mixture form of $J_2(q, q')$ is used in this calculation. The normalization is such that for large q, $u(q)=[\sin(kq)+\tan\delta\cos(kq)]/k$.

possible to pin down the 14-Mev points on the n-d "k cotδ" plots as shown in Fig. 6. The n-d "k cotδ" plots are completed by drawing them to intersect the zero energy axis at the points determined by the known n-d scattering lengths.[1]

This makes it possible to find the S phase shifts for any intermediate energy. Phase shifts for $l\geq 1$ are taken from the Born approximation. Angular distributions computed at 14, 5.5, and 3.27 Mev are shown in Fig. 10 and compared with the experimental data.[28,29] The agreement is good except for the low angle measurements at $E=3.27$ Mev. This suggests that this data is not reliable because of a hydrogen contamination in the target.

TABLE V. Phase shift analysis of 14-Mev n-d data.

	Phase shift analysis	Born approximation
^4S	57.53°	
^2S	66.23°	
^4P	33.23°	29.91°
^2P	0.0±1.7°	0°
^4D	−3.217°	−3.217°
^2D	3.37°	3.37°
rms	26 points, 9.3 percent 23 points, 6.2 percent	

[28] Hamouda, Halter, and Scherrer, Phys. Rev. **79**, 539 (1950); I. Hamouda and G. de Montmollin, Phys. Rev. **83**, 1277 (1951).

[29] E. Wantuch, Phys. Rev. **86**, 679 (1952).

B. Sensitivity of *n-d* Scattering to *n-n* Forces

1. *Angular Distribution*

In Fig. 11, the effects of various potentials on angular distributions are shown. Effects of these potentials on partial waves with $l\geq 1$ could be estimated using Eqs. (9) and (10) because it has been found that the Born approximation is satisfactory for them. (When an increase in the ^4P phase shift was necessary to achieve a fit with experiment, it is assumed that all of the increase occurs in $J_2(\theta)$ in making these estimates; this is justified in Sec. VIII-A.1.) While there is less basis for estimating the effect of the n-n potentials on the S phase shifts it is shown in Fig. 11 that large changes in the ^2S phase shift do not affect the angular distributions. At 15 Mev, appreciable changes do occur near the minimum, but it is just at this point that the experiments are most inaccurate. The ^4S phase shift does not depend on $^1V_{nn}^{+}$ and is weakly dependent on $^3V_{nn}^{-}$ because $^3V_{nn}^{-}$ occurs with different signs in the coefficients of $J_1(\theta)$ and $J_3(\theta)$ (see the following Sec. VII-B.2).

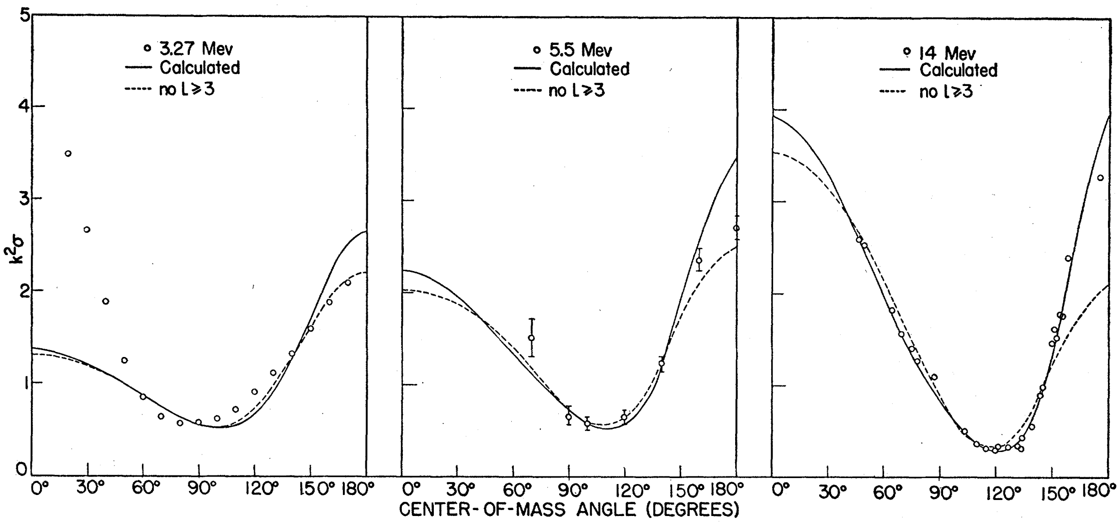

FIG. 10. Calculated and experimental n-d angular distribution at several energies.

2. *Scattering Lengths*

Table IV shows that the scattering lengths do not depend on the n-n potentials. The exception is the sensitive dependence on ${}^1V_{nn}{}^+$ of the ${}^2S$ scattering length. From the data of Table IV. a graph of the ${}^2S$ scattering length as a function of ${}^1V_{nn}{}^+$ is prepared and shown in Fig. 12.

Why is the n-d scattering insensitive to the n-n forces? The answer is contained in Eqs. (9) and (10). For states with $l\geq 1$, $J_2(\theta)$ is large compared to $J_1(\theta)$ and $J_3(\theta)$, but the coefficient of $J_2(\theta)$ depends only on ${}^3V_{np}{}^+$. For S states, $J_1(q, q')$ and $J_3(q, q')$ are about equal, so that terms with odd parity potentials cancel, since these occur with opposite signs in the coefficients of $J_1(q, q')$ and $J_3(q, q')$. The ${}^4S$ phase shifts do not depend on the even parity n-n potential, and the ${}^2S$ state contributes so little to the scattering that it hardly matters what the ${}^2S$-phase shifts are.

Were our theory stronger, we could deduce a value for ${}^1V_{nn}{}^+$ from Fig. 12. At first glance, Fig. 12 might suggest that ${}^1V_{nn}{}^+$ is small and the second of the scattering lengths (4) is permissible, but it must be re-

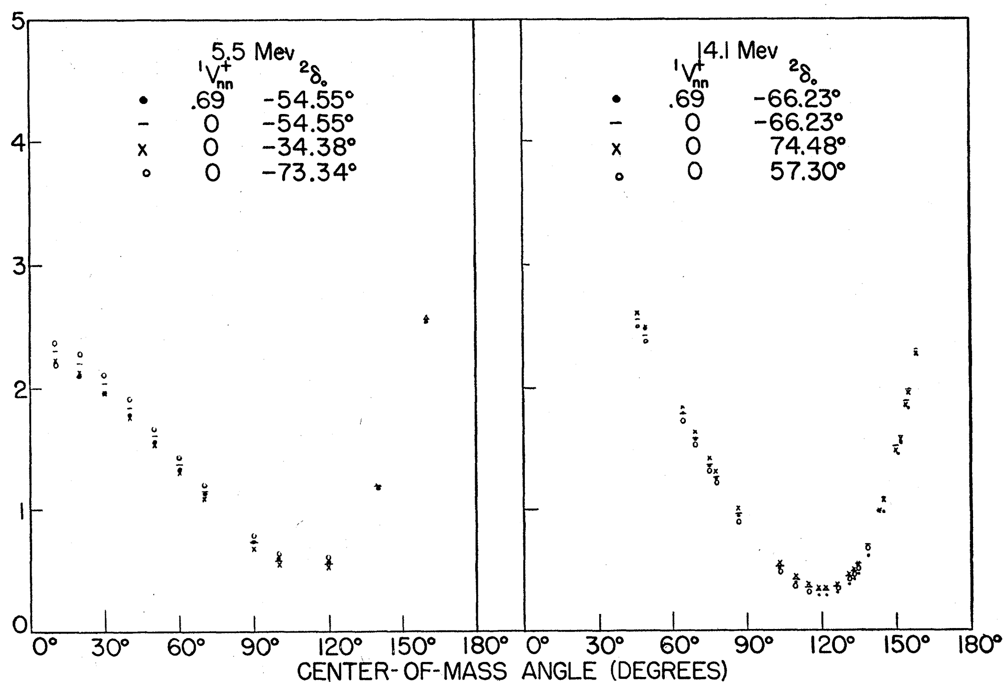

FIG. 11. Effect of the n-n potential on the n-d distributions. ${}^1V_{nn}{}^+=0.69$ is the value obtained assuming ${}^1V_{nn}{}^+={}^1V_{np}{}^+={}^1V_{pp}{}^+$. Also the effect of varying the ${}^2S$ phase shift is shown.

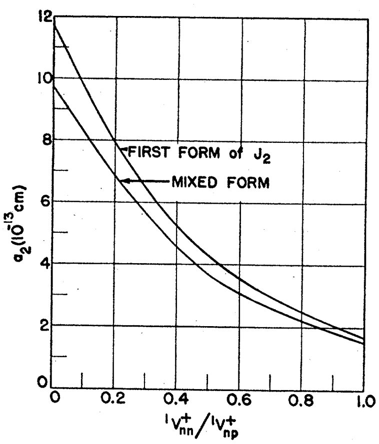

FIG. 12. The ${}^2S$ scattering length *versus* the ratio ${}^1V_{nn}^+/{}^1V_{np}^+$ for the first and second (mixed) form of $J_2(q, q')$.

membered that the first of the scattering lengths (4) does not follow from any combination of potentials allowed by the low energy two body data. Figure 12 strongly suggests that ${}^1V_{nn}^+={}^1V_{np}^+$. It could not be much larger than ${}^1V_{np}^+$, otherwise there would exist a bound singlet state of the "di-neutron."

It is clearly indicated that a comparison of the binding energies of H^3 and He^3 should make it possible to set an accurate limit on ${}^1V_{nn}^+/{}^1V_{np}^+$.

VIII. THEORETICAL CONSIDERATIONS

A. Validity of Approximations

To discuss the various approximations made let us consider the quartet states using ordinary forces (no charge, spin, or parity dependence of the potentials). We assume the wave function for the n-d system may be expanded in the complete set of deuteron function labeled $\phi_\kappa(\mathbf{r})$ for the continuum states, so that

$$\psi(\mathbf{r},\mathbf{q})=\phi(r)f(\mathbf{q})+\int\phi_\kappa(\mathbf{r})f_\kappa(\mathbf{q})d\kappa \equiv\phi(r)f(\mathbf{q})+\chi(\mathbf{r},\mathbf{q}). \quad (56)$$

Substituting the antisymmetrical combination

$$\psi_A(\mathbf{r},\mathbf{q})=\psi(\mathbf{r},\mathbf{q})-\psi(\mathbf{q}+\tfrac{1}{2}\mathbf{r},\tfrac{3}{4}\mathbf{r}-\tfrac{1}{2}\mathbf{q}) =\psi(\mathbf{r},\mathbf{q})-\psi(\mathfrak{r},\mathfrak{q}), \quad (57)$$

into the variational principle (42) we find the following coupled integral equations for $f(\mathbf{q})$ and $\chi_A(\mathbf{r},\mathbf{q})$

$$f(\mathbf{q})=\exp(i\mathbf{k}\cdot\mathbf{q})+\int\phi(r)[\phi(\mathfrak{r})f(\mathfrak{q})+\chi_A(\mathfrak{r},\mathfrak{q})]d\mathbf{r} +\int\frac{\exp(ik|\mathbf{q}-\mathbf{q}'|)}{4\pi|\mathbf{q}-\mathbf{q}'|} \times\phi(r')[U(|\mathbf{q}'+\tfrac{1}{2}\mathbf{r}'|)+U(|\mathbf{q}'-\tfrac{1}{2}\mathbf{r}'|)] +[\phi(r')f(\mathbf{q}')-\phi(\mathfrak{r}')f(\mathfrak{q}')+\chi_A(\mathfrak{r}',\mathfrak{q}')]d\mathbf{r}'d\mathbf{q}', \quad (58)$$

$$\chi_A(\mathbf{r},\mathbf{q})=\int K(rr';qq')[U(|\mathbf{q}'+\tfrac{1}{2}\mathbf{r}'|)+U(|\mathbf{q}'-\tfrac{1}{2}\mathbf{r}'|)] \times[\phi(r')f(\mathbf{q}')-\phi(\mathfrak{r}')f(\mathfrak{q}')+\chi_A(\mathfrak{r}',\mathfrak{q}')]d\mathbf{r}'d\mathbf{q}',$$

where

$$K(rr';qq') =\int_{\kappa^2<k^2-\alpha^2}d\kappa\phi_\kappa(\mathbf{r})\frac{\exp(i(k^2-\alpha^2-\kappa^2)^{\frac{1}{2}}|\mathbf{q}-\mathbf{q}')}{4\pi|\mathbf{q}-\mathbf{q}'|}\phi_\kappa(\mathbf{r}') +\int_{\kappa^2>k^2-\alpha^2}d\kappa\phi_\kappa(\mathbf{r})\frac{\exp(-(\kappa^2-k^2+\alpha^2)^{\frac{1}{2}}|\mathbf{q}-\mathbf{q}'|)}{4\pi|\mathbf{q}-\mathbf{q}'|}\phi_\kappa(\mathbf{r}'), \quad (59)$$

and

$$\chi_A(\mathbf{r},\mathbf{q})=\chi(\mathbf{r},\mathbf{q})-\chi(\mathfrak{r},\mathfrak{q}). \quad (60)$$

$\chi_A(\mathbf{r},\mathbf{q})$ gives the effect of including excited states of the deuteron in the expansion and therefore contains the effects of polarization (not caused by the exclusion principle) and inelastic scattering.

Let us consider then the effect of neglecting $\chi_A(\mathbf{r},\mathbf{q})$, (the so-called no polarization approximation). We may then make a partial wave analysis of $f(\mathbf{q})$ in the usual way

$$f(\mathbf{q})=\sum(i)^l(2l+1)\exp(i\delta_l)f_l(q)P_l(\mathbf{k},\mathbf{q}), \quad (61)$$

where

$$f_l(q)\rightarrow\sin(kq-\tfrac{1}{2}l\pi+\delta_l)/kq, \quad (62)$$

so that the resulting integral equation for $f_l(q)$ becomes

$$f_l(q)=j_l(kq)+(-1)^l\left(\frac{4}{3}\right)^3\int\phi\left(\left|\frac{2}{3}\mathbf{q}-\frac{4}{3}\mathbf{q}'\right|\right)\phi\left(\left|\frac{4}{3}\mathbf{q}-\frac{2}{3}\mathbf{q}'\right|\right)P_l(\cos\mathbf{q}\mathbf{q}')f_l(q')d\mathbf{q}'$$
$$+\left(\frac{4}{3}\right)^3\int G_l(kq;kq'')\phi\left(\left|\frac{2}{3}\mathbf{q}'+\frac{4}{3}\mathbf{q}''\right|\right)\frac{1}{2}\left[U\left(\left|\frac{4}{3}\mathbf{q}'+\frac{2}{3}\mathbf{q}''\right|\right)+U\left(\left|\frac{4}{3}\mathbf{q}''+\frac{2}{3}\mathbf{q}'\right|\right)\right]$$
$$\times\phi\left(\left|\frac{4}{3}\mathbf{q}'+\frac{2}{3}\mathbf{q}''\right|\right)P_l(\cos\mathbf{q}',\mathbf{q}'')f_l(q')d\mathbf{q}'d\mathbf{q}''-\frac{1}{2}\int G_l(kq;kq')\phi^2(q'')U(|\mathbf{q}'+\tfrac{1}{2}\mathbf{q}''|)f_l(q')d\mathbf{q}'d\mathbf{q}''$$
$$\times(-1)^l\frac{1}{2}\left(\frac{4}{3}\right)^3\int G_l(kq;kq'')\phi\left(\left|\frac{4}{3}\mathbf{q}'+\frac{2}{3}\mathbf{q}''\right|\right)U\left(\left|\frac{2}{3}\mathbf{q}'-\frac{2}{3}\mathbf{q}''\right|\right)\phi\left(\left|\frac{2}{3}\mathbf{q}'+\frac{4}{3}\mathbf{q}''\right|\right)$$
$$\times P_l(\cos\mathbf{q}',\mathbf{q}'')f_l(q')d\mathbf{q}'d\mathbf{q}'', \quad (63)$$

where

$$G_l(kq;kq')=kj_l(kq_<)j_{-l}(kq_>),\tag{64}$$

where $j_l(kq)$ and $j_{-l}(kq)$ are the regular and irregular spherical Bessel functions of order l, respectively, and $q_<$ means the lesser of q and q'. In the above a fifty-fifty mixture of the possible forms of $J_2(q,q')$ is used. We have adopted the half exchange force and changed the normalization of $f_l(q)$ from that of Eq. (62). In the case of the S state these equations have been solved numerically. For the higher partial waves only the first Born approximation has been computed; that is, wherever $f_l(q)$ occurs inside the integrals it has been replaced by $j_l(kq)$.

1. *Validity of First Born Approximation for* $l\geq 1$

We shall compute the first-order distortion of the wave function and the corresponding second Born approximation to the phase shift. Fortunately, the distortion arising from $J_1(q,q')$ and $J_3(q,q')$ is small and may be neglected compared to that arising from $J_2(q,q')$ and $O(q,q')$. This is most easily seen from the "zero range" approximation in which the integrands in $J_1(q,q')$ (exclusive of the Green's functions) are of the nature of weak long-range potentials (exactly for $J_1(q,q')$) for which it is known that the Born approximation is quite good. The first-order distortion of the incident plane wave caused by terms which give rise to δ^l and the "orthogonality" integral have been calculated numerically. In the case of the orthogonality term this has been done using the Gauss potential (17) and the deuteron wave function (19). The "zero range" approximation has been used in the calculation of the distortion caused by the $J_2(q,q')$ type term. This is not believed to lead to any appreciable error since the results of the first Born approximation for the zero range, Yukawa, and Gauss potentials were nearly the same (see Fig. 1). The first-order distorted ^4P wave function at 5 Mev is shown in Fig. 13. The corrected ^4P phase shift is 0.38 radian as compared to the first Born approximation value of 0.41 radian. The correction to the ^2P phase shift is $+0.016$ radian. This shows that the use of Born's approximation in the no polarization approximation leads to very little additional error. The distortion of the incident wave is even smaller for D and higher partial waves. Further justification comes from the calculation of Buckingham and Massey[15] who numerically integrated the no polarization equation (cast in a somewhat different form) and found results for the phase shifts agreeing closely with those of the Born approximation.

2. *Neglect of Tensor Forces*

Let us consider next the effect of tensor forces which are known to be necessary in the two body problem. When tensor forces are included, integrals of the type $J_1(\theta)$, $J_2(\theta)$, and $J_3(\theta)$ still occur.[18] Those of the type $J_1(\theta)$ and $J_3(\theta)$ are small compared to those of the type $J_2(\theta)$, so that the most important effects of tensor forces arise from integrals of the type $J_2(\theta)$. Fortunately, however, it is possible to eliminate the two body potentials in the Born approximation for $J_2(\theta)$. The result has been given previously in Eq. (17). In the presence of tensor forces the deuteron function is

$$\phi(r)+\frac{1}{\sqrt{2}}P_2(\boldsymbol{\sigma},\mathbf{r})\omega(r),\tag{65}$$

so that

$$J_2(\theta)=\frac{4}{3}(P^2+\alpha^2)\left\{\int\phi(r)j_0(Pr)r^2dr+\frac{1}{\sqrt{2}}P_2(\sigma,\mathbf{P})\int\omega(r)j_2(Pr)r^2dr\right\}\times\left\{\int\phi(r)j_0(Qr)r^2dr+\frac{1}{\sqrt{2}}P_2(\sigma,\mathbf{Q})\int\omega(r)j_2(Qr)r^2dr\right\},\tag{66}$$

where

$$\mathbf{P}=\tfrac{1}{2}\mathbf{k}+\mathbf{k}',\quad \mathbf{Q}=\mathbf{k}+\tfrac{1}{2}\mathbf{k}'.$$

From the above expression it is clear why $J_2(\theta)$ is not dependent upon the tensor character of the two body potential. First the S state part of the deuteron wave function is very little changed even in the presence of a strong tensor force. Secondly, the additional term in Eq. (66) which depends upon the D state part of the deuteron ground-state wave function are small because the D state part of the deuteron wave function is at most 20 percent of the amplitude of the S state part even for extremely strong tensor interactions. Its effect is further diminished because the Fourier-Bessel coefficients of order two rather than zero occur together with $\omega(r)$ in Eq. (66).

This argument is weakened for the S state calculations because $J_1(q,q')$ and $J_3(q,q')$ are not small and

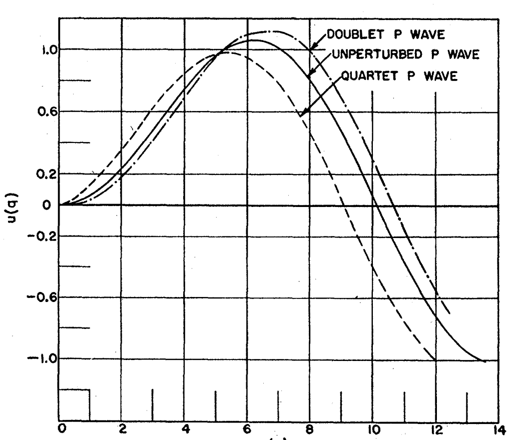

FIG. 13. The first-order distortion of the P-wave radial wave function $u(q)$ at 5 Mev.

the potential cannot be eliminated. However, $J_2(q, q')$ is still the most important kernel in determining the wave function for the quartet state. Since the potential may again be eliminated from this kernel, the effect of the tensor force in causing transitions to the ^2S state and altering the scattering length must be small. The effect on the ^2S state is probably more pronounced, but large errors in the ^2S phase shifts can be tolerated in calculating the angular distributions. However, neglect of tensor forces adds to the doubt about the accuracy of the ^2S scattering length calculation and makes it impossible to draw quantitative conclusions from Fig. 12.

3. *Effect of Polarization*

The effect of polarization has been estimated for zero energy using Eq. (58). The first approximation to $\chi_A(\mathbf{r}, \mathbf{q})$ is calculated from the second equation using the no polarization result. Then this is substituted into the first equation to find a correction to the scattering length. Again use is made of the "zero range" approximation in that all terms in which the combination $\phi(|\mathbf{q}+\frac{1}{2}\mathbf{r}|)U(|\mathbf{q}+\frac{1}{2}\mathbf{r}|)$ or $\phi_\mathbf{k}(\mathbf{q}+\frac{1}{2}\mathbf{r})U(|\mathbf{q}+\frac{1}{2}\mathbf{r}|)$ does not occur in the integrands are neglected. All other terms are expected to be smaller by the first or second power of the ratio of the range of forces to the deuteron radius. The correction due to terms of the type retained is

$$\frac{\Delta f(q)}{q}=\frac{1}{16\pi^2}\int\frac{1}{|\mathbf{q}-\mathbf{q}''|}\phi(r)V(|\mathbf{q}''+\tfrac{1}{2}\mathbf{r}|)\phi_\mathbf{k}(\mathbf{q}+\tfrac{1}{2}\mathbf{r})$$
$$\times\frac{\exp(-(k^2+\alpha^2)^{\frac{1}{2}}|\frac{1}{2}\mathbf{q}''-\frac{3}{4}\mathbf{r}+\mathbf{q}'|)}{|\frac{1}{2}\mathbf{q}''-\frac{3}{4}\mathbf{r}+\mathbf{q}'|}$$
$$\times\phi_\mathbf{k}(\mathbf{r})V(|\mathbf{q}'+\tfrac{1}{2}\mathbf{r}'|)\phi(|\mathbf{q}'+\tfrac{1}{2}\mathbf{r}'|)$$
$$\times\frac{f(|\frac{1}{2}\mathbf{q}'-\frac{3}{4}\mathbf{r}'|)}{|\frac{1}{2}\mathbf{q}'-\frac{3}{4}\mathbf{r}'|}d\mathbf{q}'d\mathbf{q}''d\mathbf{r}d\mathbf{r}'d\mathbf{k}. \quad (67)$$

For large q,

$$1/|\mathbf{q}-\mathbf{q}''|=1/q. \quad (68)$$

Using the "zero range" approximation, the correction to the scattering length is

$$\Delta a=\frac{1}{16\pi^2}\int d\mathbf{q}''d\mathbf{q}'d\mathbf{k}\phi_\mathbf{k}(0)V_0{}^2\phi(0)\left[\frac{4\pi}{3}(2r_0)^3\right]^2$$
$$\times\phi(2q'')\frac{\exp(-(k^2+\alpha^2)^{\frac{1}{2}}|2\mathbf{q}''+\mathbf{q}'|)}{|2\mathbf{q}''+\mathbf{q}'|}\phi_\mathbf{k}(2\mathbf{q}')\frac{f(2q')}{2q'}. \quad (69)$$

Let

$$2\mathbf{q}''+\mathbf{q}'=\mathbf{Q},\quad \mathbf{q}'=\mathbf{q}'. \quad (70)$$

The Jacobian of this transformation is $\frac{1}{8}$. Then

$$\Delta a=\frac{1}{16\pi^2}\frac{1}{8}\int d\mathbf{k}\phi_\mathbf{k}(0)V_0{}^2\phi(0)\left[\frac{4\pi}{3}(2r_0)^3\right]^2\int d\mathbf{q}'$$
$$\times\int d\mathbf{Q}\phi(|\mathbf{Q}-\mathbf{q}'|)\frac{\exp(-(k^2+\alpha^2)^{\frac{1}{2}}|\mathbf{Q}|)}{|\mathbf{Q}|}$$
$$\times\phi_\mathbf{k}(2\mathbf{q}')\frac{f(2q')}{|2\mathbf{q}'|}. \quad (71)$$

Now, with (16) for the deuteron wave function and $\beta=(\alpha^2+k^2)^{\frac{1}{2}}$, the integral over $\mathbf{Q}$ is

$$\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\int d\mathbf{Q}\frac{\exp(-\alpha|\mathbf{Q}-\mathbf{q}'|)}{|\mathbf{Q}-\mathbf{q}'|}\frac{\exp(-\beta Q)}{Q}$$
$$=\frac{4\pi}{2q'}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\left\{\exp(-\alpha q')\int_0^{q'}\sinh\alpha Q\exp(-\beta Q)dQ\right.$$
$$\left.+\sinh\alpha q'\int_{q'}^{\infty}\exp(-(\alpha+\beta)Q)dQ\right\},$$
$$=\frac{4\pi}{2q'\alpha}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\left(\frac{2\alpha}{k^2}\right)\exp(-\alpha q')$$
$$\times(1-\exp(-(\beta-\alpha))q'),$$
$$\approx\frac{4\pi}{2\alpha}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\exp(-\alpha q'),\quad k<\alpha, \quad (72)$$

or

$$\approx\frac{4\pi}{k^2q'}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\exp(-\alpha q'),\quad k>\alpha,$$

so that

$$\Delta a=\frac{1}{16\pi^2}\frac{1}{8}\int_0^{\alpha}d\mathbf{k}\phi_\mathbf{k}(0)V_0{}^2\phi(0)\left[\frac{4\pi}{3}(2r_0)^3\right]^2$$
$$\times\frac{4\pi}{2\alpha}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\int d\mathbf{q}'\phi_\mathbf{k}(2\mathbf{q}')\frac{f(2q')}{2q'}\exp(-\alpha q')$$
$$+\frac{1}{16\pi^2}\frac{1}{8}\int_{\alpha}^{\infty}d\mathbf{k}\phi_\mathbf{k}(0)V_0{}^2\phi(0)\left[\frac{4\pi}{3}(2r_0)^3\right]^2$$
$$\times\int d\mathbf{q}'\frac{4\pi}{k^2q'}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\phi_\mathbf{k}(2\mathbf{q}')\frac{f(2q')}{2q'}\exp(-\alpha q'). \quad (73)$$

We take

$$\phi_{\mathbf{k}}(\mathbf{r})=\frac{1}{(2\pi)^{\frac{3}{2}}}\left(\exp(i\mathbf{k}\cdot\mathbf{r})+\frac{\exp(2i\delta_0)^{-1}}{2i}\frac{\exp(ikr)}{kr}\right), \quad (74)$$

so that

$$\lim_{r_0\to 0} r_0\phi_{\mathbf{k}}(r_0)=\frac{1}{(2\pi)^{\frac{3}{2}}}\exp(i\delta_0)\frac{\sin\delta_0}{k}. \quad (75)$$

Then

$$\lim_{r_0\to 0} r_0\phi_{\mathbf{k}}(r_0)=\frac{1}{2\pi\alpha^{\frac{3}{2}}}\lim_{r_0\to 0} r_0\phi(r_0), \quad k<\alpha, \quad (76)$$

since in zero range approximation

$$\sin\delta_0=-k(\alpha^2+k^2)^{-\frac{1}{2}}. \quad (77)$$

For $k<\alpha$, we take

$$\phi_{\mathbf{k}}(2\mathbf{q}')=\frac{1}{2\pi\alpha^{\frac{3}{2}}}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\frac{\exp(-2\alpha q')}{2\alpha q'}, \quad (78)$$

that is, we assume that the shape of the wave function for a low excited state of the deuteron is the same as the shape of the ground-state wave function out to the deuteron radius for $k<\alpha$.

The second integral with $k>\alpha$ cuts off for several reasons. It can be seen from Eq. (75) that $\phi_{\mathbf{k}}(0)$ decreases as $1/k$. The zero range approximation is not very reasonable for large k because the continuum wave function oscillates inside the potential once $kr_0>1$ which of course cuts off the integral. In the following we drop out the second integral.

Let

$$B=\frac{4\pi}{3}(2r_0)^3\frac{4}{3}\frac{M}{\hbar^2}U_0\phi(0). \quad (79)$$

Substituting Eq. (78) and Eq. (76) into Eq. (73), the integral over k gives just $(4\pi/3)\alpha^3$. Then if $\alpha q'=x$,

$$\Delta a=\frac{B^2}{384\pi^2}\int_0^\infty dx f\left(\frac{2x}{\alpha}\right)\exp(-3x). \quad (80)$$

In "zero range" approximation, the part of the scattering length arising from terms of the type $J_2(q, q')$ is

$$a=\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\frac{B}{4}\int_0^\infty dx f\left(\frac{2x}{\alpha}\right)\exp(-2x). \quad (81)$$

The integral which occurs in (81) is less than the integral in (82) because of the exponential factors. However, taking as an estimate of the effect of polarization just the ratio of coefficients

$$\frac{\Delta a}{a}=\frac{4B}{384\pi^2}\left(\frac{2\pi}{\alpha}\right)^{\frac{1}{2}}\approx 0.19. \quad (82)$$

Thus it seems that polarization could change the ^4S scattering length calculations by about 20 percent. The third form of $J_2(q, q')$ gives a ^4S scattering length which is about 15 percent too low. One does not believe the estimate for the ^2S state because the scattering length is so short (were the scattering length calculated in "zero range" approximation zero, the percentage correction would be infinite). Probably the absolute correction is about the same for both states, about 1×10^{-13} cm, so that our ^2S calculations are untrustworthy.

4. *Inelastic Scattering*

Our calculated total elastic cross section at 14.1 Mev is 0.62 barn as compared to an experimental total cross section of 0.79 barn. This makes the inelastic cross section about 0.17 barn. This is larger than the value 0.05 barn quoted by Coon and Taschek.[14]

5. *Use of the Exact Scattering Formula*

We wish to emphasize that it is necessary to use the rigorous scattering formula in the calculation of the angular distributions even though the higher phase shifts are calculated in Born approximation. The imaginary part of the scattering amplitude [Eq. (12)] is not small compared to the real part because the S phase shifts are large. Thus, for example, even if the D phase shifts are small, S-D interference may lead to considerable angular variation in the cross section arising from the imaginary part of the scattering amplitude.

Use of the rigorous Eq. (12) guarantees that the scattering matrix is unitary so that at least one does not get theoretical nonsense.

It is worth noticing that a variational expression for the higher phase shifts can be derived similar to that derived for the S phase shifts, where now the form of $f_l(q)$ is not varied but the trial function is the appropriate plane wave component times a deuteron wave function. Then instead of Eq. (36) we find

$$\tan\delta_l=\alpha\delta_1{}^l+\beta\delta_2{}^l+\gamma\delta_3{}^l,$$

where $\delta_i{}^l$ is given as before by Eq. (21). Because the $\delta_i{}^l$'s are small, Eq. (36) and the equation above give numerically equivalent results. The point is that if we use the rigorous scattering formula, we have a variational calculation of the angular distribution.

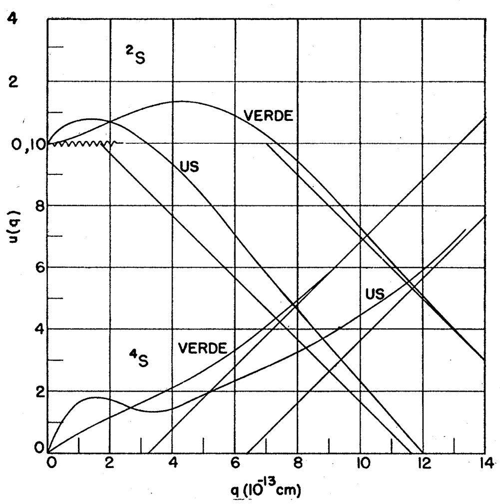

FIG. 14. Comparison of the zero energy radial wave functions determined by Verde (reference 25) and the present authors. (Note the difference in the zero of the ordinate for the quartet and doublet states.)

B. Comparison with and Criticism of Previous Work

It has been thought that since states of higher angular momentum appear at lower energies in *p-d* and *n-d* scattering than in *n-p* or *p-p* scattering the exchange nature of nuclear forces would be determined at lower *p-d* and *n-d* energies. This argument received some support from the calculations of Buckingham and Massey. At that time, (1941), nuclear ranges were generally taken somewhat longer than recent determinations have shown are necessary. Furthermore, nothing was known about the exchange nature of the forces and they assumed large forces in the odd parity states. It is known from *n-p* and *p-p* scattering that actually the odd parity forces are quite weak. Thus their two assumptions combined to give great theoretical promise for the determination of exchange forces in the 5- to 10-Mev region. The calculations presented here with more realistic internucleonic forces suggest that the scattering is determined primarily by the even parity *n-p* potentials. Possibly the only advantage to be gained in *n-d* or *p-d* scattering stems from the increase by a factor 1.33 of the energy in the center of mass system. This advantage is more than offset by the greater theoretical complexity of the problem.

Critchfield[30] has made a phase shift analysis of the data of reference 2. He assumes that the ^4S and ^2S phase shifts are equal and finds some justification for this in the work of Buckingham and Massey. However, it is known that the ^4S and ^2S *n-d* scattering lengths are unequal, so that it is unlikely that the ^4S and ^2S phase shifts are equal. With a different approach to the phase shift analysis, we were able to avoid this assumption. The data at these energies admit of many solutions for the phase shifts and it is necessary to start with some idea of what they should be in fitting the data. We have used the scattering lengths as a guide in finding the S state phase shifts and the Born approximation as a guide in finding the phase shifts for partial waves with $l\geq 1$.

Other workers[31] studying the 5-Mev data on *n-d* scattering have used Critchfield's method and tried to draw conclusions from the work of Buckingham and Massey about the exchange character of nuclear forces. Latter and Latter[32] avoided the assumption that the ^4S and ^2S phase shifts are equal. Although the neglect of partial waves with $l\geq 3$ is not justified at 5 Mev (see figures) perhaps the most severe criticism of both these works is that data at only one energy is examined, and it is impossible to correlate the results with the known *n-d* scattering lengths.

Previous calculations of the scattering lengths exist. Gordon[7] replaced the *n-d* interaction by an effective central potential. This could be done if $O(q,q')$, $J_2(q,q')$, and $J_3(q,q')$ had the form of $J_1(q,q')$; that is, if for a given q', they were linear in q to $q=q'$ and constant thereafter. However, they do not have this form.

Prohammer and Welton[8] assumes that the "$k\cot\delta$" plot for the doublet state in *n-d* scattering should extrapolate to the binding energy of the triton, just as the *n-p* "$k\cot\delta$" for the triplet state extrapolates to the binding energy of the deuteron in *n-p* scattering. However, since we know that the potential Eq. (17) gives the scattering lengths Eq. (3) and the binding energy of the triton at least approximately, we can conclude only that this is not the case for the ^2S state in the three body problem.

The wave functions found by Troesch and Verde[6] are compared with ours in Fig. 14. Our calculation is equivalent to theirs except we have a continuum of variational parameters where they had only a few.

IX. CONCLUSIONS

The excellent fits obtained of both *n-d* and *p-d* angular distributions attest to the reasonableness of our calculations. The close agreement between phase shifts calculated using Born's approximation in states with $l\geq 1$ and a "no polarization" approximation in the S states with those found from the phase shift analysis provide strong *a posterior* justification for the approximations. The neglect of deuteron polarization has been shown on theoretical ground to give little error in the case of the ^4S state, and because of its unimportance in scattering, to give no serious discrepancy for the ^2S state. In higher angular momentum states polarization is of even less importance and it has been shown that

[30] C. L. Critchfield, Phys. Rev. **73**, 1 (1948).

[31] M. M. Gordon and W. D. Barfield, Phys. Rev. **86**, 679 (1952).
[32] A. L. Latter and R. Latter, Phys. Rev. **86**, 727 (1952).

for the *P* states the distortion of the incident waves may be neglected, allowing one to use the Born approximation. Since the major physical process that determines the phase shifts in the higher angular momentum states at low energies has been shown to be the pick up by one of the particles in the deuteron of the incident particle to form a "new" deuteron, the two body force of primary importance is the triplet even parity *n-p* force. It has been further shown that only very slight differences occur in the phase shifts depending upon the range, shape, or tensor nature of this force, it being sufficient to bind the deuteron. Thus tensor forces may be replaced by central forces in the higher states. In the *S* states, the neglect of tensor forces is not so justified; however, even here the pickup process accounts for a major portion of the scattering.

The calculated phase shifts are hardly dependent upon the nature of the odd parity two body interaction for any reasonable choice of the parameters.

The ^2S phase shifts are sensitive to the even parity singlet interactions. However, since this is the state in which the calculations are most susceptible to error due to the "no polarization" approximation in the calculations and the difficulty in determining the phase shifts from experimental data, the limits on the singlet *n-n* potential in terms of the singlet *n-p* potential can not be accurately determined. It is extremely improbable though that the singlet *n-n* potential depth is less than 0.7 that of the singlet *n-p* potential depth if they both have the same radial dependence. It is quite likely that very accurate limits on the *n-n* singlet depth could be determined by a comparison of the binding energies of H^3 and He^3.

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