Momentum Space Wave Functions. II. The Deuteron Ground State*

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(Received February 9, 1953)

Previously a procedure was given for carrying out numerically an iteration method to get approximate solutions for wave functions in momentum space and for eigenvalues. This procedure is applied to the ground state of the deuteron in nonrelativistic approximation, using central potentials only. Tables illustrating the rapidity of convergence of the iterations are given. Values for the momentum space wave functions corresponding to different potential shapes are tabulated, and simple analytic approximations are given.

1. INTRODUCTION

In a previous paper,[1] hereafter referred to as I, an iteration procedure suitable for numerical treatment was discussed with relation to solutions of the Schrödinger wave equation in momentum space. The present paper is a continuation of I; the iteration procedure is here applied to the ground state of the deuteron. We consider only central potentials and obtain nonrelativistic wave functions for several different potential shapes. Two to three iterations were carried out, both to determine the momentum distribution in the deuteron and to investigate the rapidity of convergence of the iteration. Details of the method and results are given in the following two sections. Some approximate analytic wave functions for the deuteron ground state, both in position space and in momentum space, are discussed in Sec. 4.

The notation used is the same as that of I, and frequent references to Table I of I will be made. We express all momenta in units of γ, where $\hbar\gamma^{-1}$ is the deuteron radius, viz.,

$$\gamma=(2ME_D)^{\frac{1}{2}}=45.69 \text{ Mev}/c =\hbar(4.317\times10^{-13}\text{ cm})^{-1}. \quad (1)$$

M is the reduced mass of the neutron-proton system and E_D is the deuteron binding energy (2.225 Mev). The experimental probable error in γ is about ±0.1 percent.

In Table I of I, a momentum μ, fixing the range of the position space potential $U_0(r)$, is defined for three forms of the dimensionless function $U_0(r)$, corresponding to Yukawa, exponential and Gaussian shapes. The corresponding momentum space potentials $[(2\pi)^{\frac{3}{2}}V_0(p)]$, which are the Fourier transforms of $U_0(r)$, are also given in I, Table I. We write the triplet neutron-proton potential in position space $U(r)$ in the form

$$U(r)=\lambda(\mu^2/2M)U_0(r), \quad (2)$$

where λ is a dimensionless parameter proportional to the "coupling constant." An analysis of low energy experiments on the neutron-proton system[2-4] gives a value for the triplet effective range of the $n-p$ potential of

$$\rho_t(0,-E_D)=1.703(1\pm0.02)\times10^{-13}\text{ cm}. \quad (3)$$

An effective range theory analysis,[3] using a compromise between Eq. (3) and data on photodisintegration of the deuteron,[5] then yields values for the momentum μ for the three potential shapes. We adopt the following values:

$$\gamma/\mu=0.313 \text{ (Yukawa)} \quad (4a)$$
$$\gamma/\mu=0.154 \text{ (exponential)} \quad (4b)$$
$$\gamma/\mu=0.349 \text{ (Gaussian)}. \quad (4c)$$

The probable error in (γ/μ), coming mainly from the experimental uncertainty in the effective range, is about ±2.5 percent in all three cases.

2. METHOD

The nonrelativistic[6] Schrödinger equation in momentum space for the spherically symmetric ground state of the deuteron has the form

$$(p^2+\gamma^2)\phi(p) =2\pi\lambda\mu^2\int_0^\infty k^2dk\phi(k)\int_{-1}^1 dxV_0(p^2+k^2-2pkx). \quad (5)$$

In Eq. (5), $(2\pi)^{\frac{3}{2}}V_0$ is one of the momentum space potentials defined in I, Table I, $\phi(p)$ is the momentum space wave function to be determined, and λ the "coupling constant," Eq. (2). We assume γ and μ to be known and consider λ as the eigenvalue of the problem.

It is convenient to define a modified function $\chi(p)$ by

$$\chi(p)=(p^2+\gamma^2)\phi(p). \quad (6)$$

As discussed in I, we write the nth iterated function

* Work supported in part by the U. S. Office of Naval Research.
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[4] H. A. Bethe, Phys. Rev. 76, 38 (1949); E. E. Salpeter, Phys. Rev. 82, 60 (1951); G. Snow, Phys. Rev. 87, 21 (1952).
[5] D. H. Wilkinson, Phys. Rev. 86, 373 (1952).
[6] We shall consider the nonrelativistic equation (5) for all momenta p. Actually $(20\gamma/c)$ is approximately equal to one nucleon mass, and Eq. (5) has, of course, no physical justification for $p\gtrsim20\gamma$.

TABLE I. Iterated values of $a_n(p)$ and of the coupling constant λ_{n+1} for a Yukawa potential.

p/μ	0	1	2	∞	λ_{n+1}
$a_0(p)$	1.858	1.858	1.858	1.858	2.359
$a_1(p)$	1.759	1.796	1.848	1.937	2.376_8
$a_2(p)$	1.758	1.797	1.854	1.943	2.376_6

TABLE II. Iterated values of $a_n(p)$ and of the coupling constant λ_{n+1} for an exponential potential.

p/μ	0	1	4	∞	λ_{n+1}
$a_0(p)$	1.380	1.380	1.380	1.380	2.070
$a_1(p)$	1.382	1.393	1.326	1.232	2.063_4
$a_2(p)$	1.379	1.395	1.322	1.229	2.062_0

TABLE III. Iterated values of $a_n(p)$ for $p/2\mu=0$ and 2, of $b_n(p)$ for $p/2\mu=4$ and ∞, and of the coupling constant λ_{n+1} for a Gaussian potential.

$p/2\mu$	0	2	4	∞	λ_{n+1}
$a_0(p), b_0(p)$	1.170	1.170	1.000	1.000	3.880
$a_1(p), b_1(p)$	1.165	1.209	0.819	1.015	3.864_3
$a_2(p), b_2(p)$	1.164	1.214	0.766	1.017_5	3.863_2
$a_3(p), b_3(p)$	1.163_5	1.215	0.757	1.018	3.862_5

$\chi_n(p)$ in the form

$$\chi_n(p)=[1+p^2/a_n^2(p)\mu^2]^{-1} \quad \text{(Yukawa)} \tag{7a}$$

$$\chi_n(p)=[1+p^2/a_n^2(p)\mu^2]^{-1} \quad \text{(exponential)} \tag{7b}$$

$$\chi_n(p)=\exp[-p^2/4a_n^2(p)\mu^2] \quad \text{(Gaussian).} \tag{7c}$$

The function $a_n(p)$, defined by Eqs. (7), is a slowly varying function of p and hence more suitable for numerical work than $\chi_n(p)$ itself. For the initial wave function of the iteration procedure, we take $a_0(p)$ to be a constant, a_0.

In I, Sec. 2, the iteration procedure was discussed in detail. Once $a_n(p)$ is known, the integral equation (5) furnishes a value λ_{n+1} for the coupling constant by requiring that $\chi_{n+1}(0)$ be unity. The integral equation then gives $\phi_{n+1}(p)$ and hence $a_{n+1}(p)$. In all three cases, an analytic expression for $a_1(0)$ as a function of a_0 could be found with relative ease. The requirement that $a_1(0)$ equal a_0 gave the following values for a_0: 1.74, 1.38, and 1.17 for the Yukawa, exponential, and Gaussian potentials, respectively. For the exponential and Gaussian potentials, these values of a_0 were actually used in the iteration. For the Yukawa potential, the ranges of integration involving large momenta were also found to be fairly important. For this reason, the value of a_0 for which $a_1(\infty)$ equals a_0 was also calculated. The mean between these two values for a_0, 1.86, was (rather arbitrarily) adopted for the iteration in the case of the Yukawa potential.

For the Yukawa and exponential potentials the first iteration can be carried out analytically, but the formulas for $\chi_1(p)$ are rather complicated. For all three potential shapes, $a_1(p)$ was evaluated for about ten values of p and a simple interpolation method used for intermediate values of p. Using these values of $a_1(p)$, a second iteration was then carried out giving values for λ_2 and $a_2(p)$ for a few values of p. At least for fairly small values of p, $a_1(p)$ and $a_2(p)$ differed from each other by only a small fraction. Finally, one single further integration, using the interpolated values for $a_2(p)$, gave λ_3. Values for λ_n and $a_n(p)$, for a few values of the momentum p, are given in Tables I, II, and III.

As may be seen from the tables, the rapidity of convergence of the iterated values of λ is very good in all three cases and of $a_n(p)$ fairly good for the Yukawa and exponential potentials. For the Gaussian potential the convergence of $a_n(p)$ is also fairly rapid for $p\lesssim\mu$ but becomes progressively worse for large values of p/μ. In fact, the correct asymptotic behavior of $\chi(p)$ for $p\rightarrow\infty$ is a much more slowly decreasing function of p than $\chi_0(p)$, Eq. (7c). Because of the rapid decrease of the Gaussian function, $\chi_{n+1}(p)$ depends mainly on the behavior of $\chi_n(k)$ for values of k in the neighborhood of p. For large values of p, therefore, the iterated functions $\chi_n(p)$ do not approach the correct function rapidly, once a function $\chi_0(p)$ with an incorrect asymptotic behavior has been used.

The correct asymptotic form of the wave function for the Gaussian potential can be obtained as follows. We write $\phi(p)$ in the form

$$\phi(p)\sim\exp[-pf(p)], \tag{8}$$

where $f(p)$ is a function to be determined. For $p\gg\mu$, the integral equation (5) takes the following asymptotic form:

$$p^3\exp[-pf(p)]$$
$$=(\lambda\mu/2\pi^{\frac{1}{2}})\int_0^\infty kdk\,\exp[-(p-k)^2/4\mu^2-kf(k)]. \tag{9}$$

The integrand has a sharp maximum near $k=[p-\frac{1}{2}f(p)]$; the integral can be evaluated approximately by using a Taylor expansion for the integrand about

TABLE IV. Extrapolated values for $a(p)$, $b(p)$ and of the coupling constant λ for three potential shapes.

	p/μ	0	0.5	1	2	3	4	5	∞	λ
Yukawa	$a(p)$	1.758	1.772	1.797	1.853	1.888	1.906	1.918	1.946	2.377
Exponential	$a(p)$	1.380	1.390	1.397	1.385	1.353	1.321	1.301	1.228	2.062
Gaussian	$a(2p)$	1.163	1.167	1.176	1.215					3.862
Gaussian	$b(2p)$				0.940	0.82	0.75	0.70	1.018	

TABLE V. Extrapolated values of $\chi(p)$ as a function of (p/γ) for three potential shapes.

p/γ	0	1	2	3	4	5	6	8	10	12	14
Yukawa	1.000	0.970	0.890	0.785	0.678	0.580	0.496	0.362	0.269	0.205	0.160
Exponential	1.000	0.976	0.908	0.811	0.700	0.589	0.485	0.317	0.203	0.130	0.085
Gaussian	1.000	0.978	0.914	0.818	0.700	0.575	0.453	0.252	0.123	0.052	0.020

this value of k and dropping higher powers of μ/p. The resulting equation for $f(p)$ has the solution

$$f(p)=2^{\frac{3}{2}}[\ln(bp)]^{\frac{1}{2}}, \tag{10}$$

where b is a known function of the coupling constant λ, independent of p.

The iteration procedure for the Gaussian potential was modified slightly by using Eq. (7c) only for $p<4\mu$, and using instead[7] for $p>4\mu$ the form

$$\phi_n(p)=A_n\exp\{-2^{\frac{3}{2}}p[\ln b_n(p)p]^{\frac{1}{2}}\}. \tag{11}$$

In Eq. (11), A_n is a constant determined by requiring the two formulas for $\phi_n(p)$ to agree for $p=4\mu$, and $b_n(p)$ is a function analogous to $a_n(p)$. The function $b_0(p)$ was taken to be a constant b_0 and was determined by requiring $b_1(\infty)$ to be equal to b_0. This modification of the iteration procedure improved the rapidity of convergence of $\phi_n(p)$ for $p\gg 4\mu$, but is unnecessary for small p and for evaluating λ.[8] Some iterated values for $b(p)$ are given in Table III.

3. RESULTS

In Tables I, II, and III, the iterated values of λ_n and of $a_n(p)$ are given for a few values of p, merely to demonstrate the rate of convergence. In Table IV we give the final estimates for λ and for $a(p)$ as a function of p/μ for each of the three potential shapes. For the values of γ/μ given in Eq. (4), the computational errors in the numbers given in Table IV are about ± 5 in the last figure given, or less. For values of p/μ not given in Table IV, a simple interpolation method can be used for $a(p)$. The probable errors in $a(p)$ and in λ due to the experimental uncertainty in the values used for γ/μ, Eq. (4), on the other hand, are slightly more than half a percent.

Using these values for $a(p)$ and Eqs. (4), (7), and (11), values for the modified function $\chi(p)$, Eq. (6), can be calculated. In Table V we give these values for $\chi(p)$ for the three potential shapes, with the momentum p expressed in units of the constant γ, Eq. (1). For the values of γ/μ given in Eq. (4) the probable errors, from computational sources alone, in $\chi(p)$ are of the order of magnitude of ± 0.002 for relatively large values of p. The experimental uncertainty in the values used for γ/μ introduces much larger errors for $\chi(p)$. This experimental probable error in $\chi(p)$ is of the order of magnitude of ± 0.010.

In Fig. 1, we give a rough plot of the wave function $\chi(p)$ as a function of p/γ for the three potential shapes. It will be seen from this graph that for $p\gtrsim 5\gamma\approx 230$ Mev/c, the momentum space wave functions (normalized to the same value at $p=0$) for the three potential shapes are very similar, although the Yukawa wave function decreases slightly more rapidly than the other two wave functions. For larger values of the momentum, however, the Yukawa wave function decreases much less rapidly with increasing momentum than the exponential wave function and the Gaussian wave function much more rapidly.

4. APPROXIMATE WAVE FUNCTIONS

It will be seen from Table IV that the functions $a(p)$ vary only slightly with p for all three potential shapes. One therefore obtains fairly accurate but simple approximations to the momentum space wave functions by replacing $a(p)$ in Eqs. (7) by a constant a, as was done for the initial wave function in our iteration method. Using Table IV, an appropriate value can be chosen for the constant a, if the momentum space wave function is required to be most accurate for a particular range of momenta. We also give below the values for a which give best agreement with the experimental effective range of the triplet neutron-proton system.

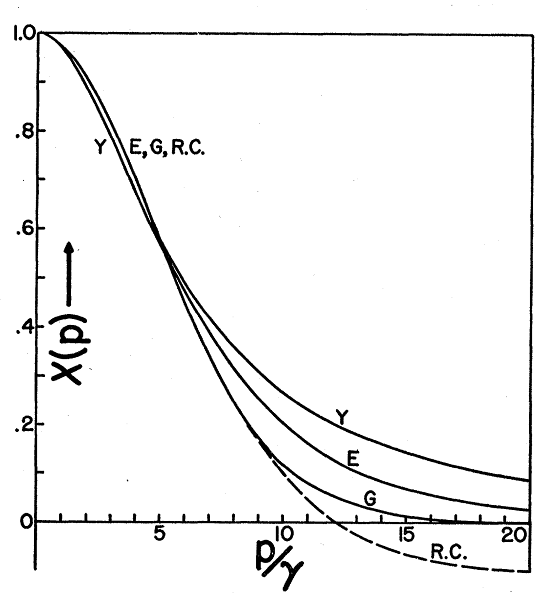

FIG. 1. The modified wave function in momentum space $\chi(p)$ for four different potential shapes.

[7] The logarithmic derivatives of the two forms for $\phi(p)$ are approximately equal for $p=4\mu$.

[8] In fact, the asymptotic behavior of $\phi(p)$ is only of academic interest, since the nonrelativistic approximation, used throughout this paper, breaks down for momenta p not much larger than 4μ.

For the Yukawa and exponential potentials, the Fourier transforms of these approximate wave functions are also simple functions. These approximate position space wave functions are (unnormalized)

$$\text{(Yukawa)}:\quad \psi_Y(r)=r^{-1}(e^{-\gamma r}-e^{-a\mu r});\qquad (12a)$$

$$\text{(exp)}:\quad \psi_e(r)=r^{-1}(e^{-\gamma r}-e^{-a\mu r})-Be^{-a\mu r},\qquad (12b)$$

where $B\equiv(a^2\mu^2-\gamma^2)/2a\mu$. The potentials in position space, for which these wave functions are exact solutions of the Schrödinger equation, are (except for constant multiplying factors)

$$\text{(Yukawa)}:\quad V(r)\propto e^{-a\mu r}/r\psi_Y(r);\qquad (13a)$$

$$\text{(exp)}:\quad V(r)\propto e^{-a\mu r}/\psi_e(r).\qquad (13b)$$

The values of the constant a which give the correct experimental effective ranges[2–5] are $a=1.84$, $(a\mu/\gamma=5.9)$ for the Yukawa potential and $a=1.38$, $(a\mu/\gamma=9.0)$ for the exponential potential.

The Fourier transform of the approximate momentum space wave function for the Gaussian potential[9] is not very simple. It is, however, similar to the wave function

$$\psi_G=r^{-1}[e^{-\gamma r}-\exp(-d^2\mu^2r^2-\alpha r)],\qquad (12c)$$

where

$$\alpha\equiv(2d^2\mu^2+\gamma^2)^{\frac{1}{2}}$$

and d is a constant of the order of magnitude of unity. The wave function, Eq. (12c) is the exact solution for a potential similar to the Gaussian potential, namely,

$$V(r)\propto(1+d^2\mu^2r/\alpha)\exp(-d^2\mu^2r^2-\alpha r)/\psi_G(r).\qquad (13c)$$

The value of the constant d, which gives the correct effective range is $d=0.98$, $(\alpha/\gamma=4.0)$.

[9] N. Svartholm, thesis, Lund (1945), unpublished.

Wave functions corresponding to an attractive position space potential with a "repulsive core"[10,11] may be of some interest for the deuteron problem. We consider a position space wave function of the form

$$\psi_{RC}(r)=r^{-1}\{e^{-\gamma r}-[1+(\alpha-\gamma)r]e^{-\alpha r}\},\qquad (12d)$$

corresponding to a position space potential of form

$$V(r)\propto[1-(\alpha+\gamma)r]/r\psi_{RC}(r).\qquad (13d)$$

This wave function is zero at the origin, the potential repulsive at short distances (with an r^{-2} singularity at the origin) and attractive at large distances (with asymptotic form $\exp[-(\alpha-\gamma)r]$). This potential thus bears some similarity to those employed by Jastrow[10] and Levy,[11] but the effect of its repulsive singularity is much less marked than that of an "infinite repulsive core."

The momentum space wave function corresponding to Eq. (12d) has the form

$$\phi_{RC}(p)\propto[\alpha(\alpha+2\gamma)-p^2]/(p^2+\gamma^2)(p^2+\alpha^2)^2,\qquad (14)$$

which changes sign as does the potential itself. The value of α which gives the correct effective range is approximately $11._2\gamma$. This wave function is illustrated by the dotted curve in Fig. 1, from which it will be seen that the function is similar to that for the Gaussian potential for small momenta but has a much larger "tail" of opposite sign for large momenta.

We are indebted to Mrs. D. Kellog, Mr. P. Kellog, and Mr. R. Margulies for extensive help with the computations and to Dr. McWeeney for pointing out related calculations.[12]

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