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Inner Bremsstrahlung in μ-Meson Decay*

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The transition probability for the decay of a μ-meson resulting in the simultaneous ejection of an electron and a photon is computed. Its dependence on the energies of the two emitted particles and on the angle between their direction of emission is given for the general case of an arbitrary mixture of three basic Fermi-type interactions for the μ-decay process.

I. INTRODUCTION

INNER bremsstrahlung, i.e., photon production accompanying charged particle transformations, has been studied in the past in connection with nuclear β-decay,[1–3] K capture,[4,5] charged meson production in high energy nucleon collisions,[6,7] and the decay of charged π-mesons.[8,9] The occurrence of inner bremsstrahlung in μ-decay has also been considered,[10] but the calculations have been made with the now abandoned one-neutrino model of μ-decay. It is the purpose of the present paper to derive the transition probability for inner bremsstrahlung in μ-decay on the basis of the presently accepted model.[11]

The results may be of more than theoretical interest. The interaction Hamiltonian responsible for μ-decay can be chosen in a variety of ways, and, in the general case, can be expressed as a sum of certain basic interaction types. The coupling constants appearing in such a sum are to be determined by experiment. It is known, however,[11] that even a complete knowledge of the energy spectrum of the decay electrons can yield at most two relations between these coupling constants. For this reason, one is naturally led to search for other phenomena involving the basic μ-decay that depend on the coupling constants, in order to obtain further relations between the latter from experiment. Inner bremsstrahlung is such a phenomenon. This investigation shows that, indeed, a sufficiently accurate measurement of certain features of the phenomenon would yield additional new information on the coupling constants, although the effect is small and its detection would require the production of large μ-meson densities.

II. GENERAL FORM OF THE INTERACTION

The interaction Hamiltonian density for the μ-decay is constructed out of the quantized spinor field components Ψ, ψ, and φ of the μ-mesons, electrons, and neutrinos, respectively. The presently adopted model requires the emission of two neutrinos in the basic decay process, consequently, the Hamiltonian is quadratic in φ, while it is linear in both Ψ and ψ. The procedure for forming Lorentz-invariant Hamiltonians from four spinors and their Hermitian conjugates is well known from the theory of nuclear β-decay. Let us write symbolically $\Gamma_S=1$, $\Gamma_V=i\gamma_\mu$,[12] $\Gamma_T=i\gamma_{\mu\nu}=\frac{1}{2}i(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$, $\Gamma_A=\gamma_\mu\gamma_5$, where $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$, and finally $\Gamma_P=\gamma_5$. Furthermore, let χ_1 and χ_2 be any two spinors. It is known that, if we define the spinor conjugates as[13]

$$\bar{\chi}=\chi^*A \tag{1}$$

with A defined by

$$A\gamma_\mu A^{-1}=-\gamma_\mu{}^\dagger, \quad A=A^\dagger, \tag{2}$$

the five expressions $(\bar{\chi}_1\Gamma_t\chi_2)(t=S,\,V,\,T,\,A,\,P)$ have the transformation properties of a scalar, vector, antisymmetric tensor of second rank, axial vector, and pseudoscalar, respectively, under Lorentz transformations excluding the time reflections. Thus, from four spinors one forms the five invariants,

$$(\bar{\chi}_1\Gamma_t\chi_2)(\bar{\chi}_3\Gamma_t\chi_4), \tag{3}$$

where summation convention with respect to the suppressed tensor indices is understood. It is convenient to consider, together with any spinor χ, its charge conjugate,

$$\chi^c=C^{-1}\chi^*, \tag{4}$$

where C is defined by

$$C\gamma_\mu C^{-1}=\gamma_\mu{}^*, \quad C^{-1}=C^*. \tag{5}$$

For quantized field operators charge conjugation means

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11 L. Michel, Proc. Phys. Soc. (London) **A63**, 514 (1950); Phys. Rev. **86**, 814 (1952). A calculation of the effect has already been presented by A. Abragam and J. Horowitz, J. phys. et radium **12**, 952 (1951), but they limit themselves to the case of pure scalar interaction for the spinor particles. Our interest is precisely the dependence on the coupling type.

12 γ_μ are four Dirac matrices satisfying $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}$, where $g_{11}=g_{22}=g_{33}=-g_{00}=1$, $g_{\mu\nu}=0$ for $\mu\neq\nu$.
13 It will be understood that, if χ is a quantized field operator, χ^* denotes the hermitian conjugate operator. Matrix multiplication with respect to spinor indices is indicated unambiguously by the order of factors. † denotes Hermitian conjugation with respect to spinor indices.

simply the replacement of Jordan-Wigner operators of particles by those of their antiparticles. Under Lorentz transformations, χ^c behaves the same way as χ; consequently, one can replace each spinor in (3) by its charge conjugate without destroying relativistic invariance. Further arbitrariness is involved in the order in which the four spinors appear in (3).

All this is well known, of course. Perhaps it has not been sufficiently emphasized in the past, however, that the number of physically different interaction Hamiltonians is much smaller than the above considerations would seem to indicate. In particular, the classes of interactions designated by various authors as "charge retention," "charge exchange," etc.[14] do not really represent physically different theories. The reason for this is to be found in the following two theorems:

Theorem 1

$$(\bar{\chi}_1\Gamma_t\chi_2)=-\zeta_t(\overline{\chi_2{}^c}\Gamma_t\chi_1{}^c), \quad (6)$$

where

$$\zeta_S=\zeta_A=\zeta_P=1,$$
$$\zeta_V=\zeta_T=-1.$$

Theorem 2

$$(\bar{\chi}_1\Gamma_t\chi_2)(\bar{\chi}_3\Gamma_t\chi_4)=\sum_{t'}F_{tt'}(\bar{\chi}_3\Gamma_{t'}\chi_2)(\bar{\chi}_1\Gamma_{t'}\chi_4), \quad (7)$$

where $F_{tt'}$ is a square matrix whose rows and columns are labeled by the five interaction types S, V, T, A, and P in this order:

$$F_{tt'}=\frac{1}{8}\begin{pmatrix} 2 & -2 & 1 & -2 & 2 \\ -8 & -4 & 0 & 4 & 8 \\ 24 & 0 & -4 & 0 & 24 \\ -8 & 4 & 0 & -4 & 8 \\ 2 & 2 & 1 & 2 & 2 \end{pmatrix}.$$

Theorem 1 follows easily from the two relations,

$$A\Gamma_t A^{-1}=\Gamma_t{}^\dagger \quad (8)$$

and

$$C\Gamma_t C^{-1}=\zeta_t\Gamma_t{}^*, \quad (9)$$

together with the antisymmetry property of the matrix A^*C,[15]

$$A^*C=-(A^*C)^T=-C^TA. \quad (10)$$

Equation (10) is verified most easily with the frequently used standard representation of the Dirac matrices, but it is not dependent on it. Theorem 1 has been used in the past in various contexts.[11,16] Theorem 2 is also known and was first given by Fierz.[17]

The immediate consequence of these theorems is that the order of the four spinors in an expression like (3) is purely conventional and can have no physical significance, provided linear combinations of all five interaction types are admitted. It will become evident in the following that there is a particular order which is more convenient than the others. We shall adopt this one and write for the Hamiltonian density for μ-decay:

$$H_{(\mu)}=\sum_t g_t(\bar{\Psi}\Gamma_t\psi)(\bar{\varphi}\Gamma_t\varphi)+\text{Herm. conj.}, \quad (11)$$

where g_t are five coupling constants of the dimension of the square of a length.[18] We may point out immediately that replacing Ψ by Ψ^c, for instance, will also not alter the physical content of the interaction Hamiltonian (11), for this transformation leaves unchanged its matrix elements between states specified physically in terms of occupation numbers, momenta, spins, and charges of particles involved. Somewhat less trivial is a similar question raised in connection with the neutrino field operators. We are concerned with replacing the factor $(\bar{\varphi}\Gamma_t\varphi)$ by $(\bar{\varphi}\Gamma_t\varphi^c)$ and want to inquire whether this change can have any physical significance. The matrix elements of $(\bar{\varphi}\Gamma_t\varphi^c)$ corresponding to the emission of two neutrinos are antisymmetric in the quantum numbers characterizing the neutrino states. On the other hand, $(\bar{\varphi}\Gamma_t\varphi)$ can only emit a neutrino and an antineutrino, so that no requirement of antisymmetry arises from the exclusion principle. Note, however, that the antisymmetric matrix element from the Hamiltonian $(\bar{\varphi}\Gamma_t\varphi^c)$ is

$$(\bar{\nu}_1\Gamma_t\nu_2{}^c)-(\bar{\nu}_2\Gamma_t\nu_1{}^c)=(1+\zeta_t)(\bar{\nu}_1\Gamma_t\nu_2{}^c), \quad (12)$$

where we have used Theorem 1. But in the last line, we have precisely the matrix element of the Hamiltonian $(\bar{\varphi}\Gamma_t\varphi)$ for the emission of two neutrinos, multiplied by a numerical factor $1+\zeta_t$. This establishes the equivalence of the two theories and also shows that (11) is indeed the most general form. We may also remark that, if the Majorana theory[19] of the neutrino is adopted, there is no choice in the matter to start with, because then $\varphi=\varphi^c$ is true as an operator identity.

From a practical standpoint a further restriction is possible. Consider, for instance, the relationship of the scalar and pseudoscalar interactions. Since γ_5 anticommutes with all four Dirac matrices γ_μ, ψ, and $\gamma_5\psi$ satisfy Dirac equations with opposite sign of the electron rest mass. Thus, the relationship between matrix elements is simply that of reversal of the sign of the rest mass. But, in view of the largeness of the μ-meson rest mass compared to the electron rest mass, momenta of all particles participating in a reaction will be large compared to the electron rest mass over an overwhelming portion of their momentum space. Thus, the difference between the scalar and the pseudoscalar cases can be neglected for all but the smallest momenta. A similar argument applies to the vector and axial

[14] This terminology was introduced by J. Tiomno and J. A. Wheeler, Revs. Modern Phys. **21**, 144 (1949).
[15] The superscript T denotes transposition with respect to spinor indices.
[16] An interesting application has been found by S. R. De Groot and H. A. Tolhoek, Physica **16**, 456 (1950).
[17] M. Fierz, Physik **104**, 553 (1937).
[18] We shall use units such that $\hbar=c=1$, thus energies and momenta are measured in wave numbers.
[19] See, for instance, W. Pauli, Revs. Modern Phys. **13**, 203 (1941).

vector interactions. We shall thus retain only the three interactions (scalar, vector, and tensor) and, consistently with this approximation, neglect all terms that are multiplied by the electron rest mass.

III. ELECTRON SPECTRUM FROM μ-DECAY

We shall recall here the computation of the energy spectrum of electrons resulting from μ-decay, because it illustrates the method that will be used in the inner bremsstrahlung calculation without the additional complications of that case. In particular, a short method of integrating over neutrino momenta had to be devised to prevent calculations in the latter case from becoming prohibitively lengthy. It was found that this problem can be dealt with rather satisfactorily by making full use of relativistic covariance properties.

The transition probability per unit time for the decay of a μ-meson into an electron of given momentum $\mathbf{p}$ and two neutrinos is given by

$$\mathcal{P}_{(\mu)}d^3p=(2\pi)^{-5}\int d^3k\int d^3k' \times\tfrac{1}{2}\sum_{\text{spins}}|(f|R|i)|^2\delta(P-p-k-k')d^3p. \quad (13)$$

Here k and k' are neutrino momenta and the argument of the δ-function contains the energy-momentum four-vectors P, p, k, and k' of the initial μ-meson, the electron, and the two neutrinos. The transition amplitude $(f|R|i)$ is obtained from the Hamiltonian (11) by first-order perturbation theory in the usual way. We take note of the fact that this matrix element is actually a product of two factors, one involving the μ-meson and electron momenta and one involving the neutrino momenta only. It is evident from the form of the Hamiltonian that these two factors have the respective transformation properties of a scalar, a vector, and an antisymmetric tensor of second rank and are contracted over corresponding tensor indices. We may now square the neutrino factor, carry out the spin summation, and integrate over neutrino momenta, regardless of the other factor which contains dependence on quantum numbers of the observed particles only. The advantage of this procedure is that the same integral occurs in the problem of inner bremsstrahlung or, for that matter, in the expression for the probability for any process which involves the basic interaction (11) in a linear fashion (this condition can always be regarded as satisfied on account of the smallness of the coupling constants g_t). We write then[20]

$$\mathcal{P}_{(\mu)}=\frac{1}{2(2\pi)^5\epsilon E}\sum_{tt'}g_tg_{t'}{}^*\tfrac{1}{4}\,\mathrm{Tr}\{(p\gamma)\Gamma_t(P\gamma)\Gamma_{t'}\}J_{t,t'}, \quad (14)$$

where $\epsilon=(\mathbf{p}^2+m^2)^{\frac{1}{2}}$, $E=(\mathbf{P}^2+M^2)^{\frac{1}{2}}$, m and M being the rest masses of electron and μ-meson, respectively. The contribution of the neutrinos is included in the tensors

$$J_{t,t'}=\tfrac{1}{4}\,\mathrm{Tr}\{\gamma^\mu\Gamma_t\gamma^\nu\Gamma_{t'}\}I_{\mu\nu}, \quad (15)$$

where

$$I_{\mu\nu}=\int\int d^3kd^3k'\delta(P-p-k-k')k_\mu k_\nu/\kappa\kappa'. \quad (16)$$

κ and κ' are neutrino energies: $\kappa=|\mathbf{k}|$, $\kappa'=|\mathbf{k}'|$. In (14), terms proportional to m have been neglected already. The integral (16) can be evaluated efficiently by noting that it depends only on the vector

$$G=P-p, \quad (17)$$

that it is of the dimension of a momentum squared, and that it is a symmetric tensor under Lorentz transformations. These conditions suffice to fix its form as $I_{\mu\nu}=AG_\mu G_\nu+Bg_{\mu\nu}(GG)$, where A and B are dimensionless numerical coefficients. One can easily find them for a special tensor component. The result is

$$I_{\mu\nu}=\tfrac{1}{6}\pi[2G_\mu G_\nu+g_{\mu\nu}(GG)]. \quad (18)$$

We next observe the important fact that, for $t\neq t'$,

$$\mathrm{Tr}\{\gamma^\mu\Gamma_t\gamma^\nu\Gamma_{t'}\}=-\mathrm{Tr}\{\gamma^\nu\Gamma_t\gamma^\mu\Gamma_{t'}\}. \quad (19)$$

This has the consequence that the double sum in (14) reduces to a single sum and that the transition probability for a general linear combination of interaction types appears as the linear sum of transition probabilities computed from pure interactions with the squares of the respective coupling constants as coefficients. This simple result holds only if the order of field operators is chosen as in (11). Equation (19) actually holds for t as well as t' running through all five types. The evaluation of (15) is immediate, and we list the results below:

$$\begin{aligned}
J_{S,S}&=J=\pi(GG),\\
J_{V,V}&=J_{\sigma,\rho}=\tfrac{2}{3}\pi[(GG)g_{\sigma\rho}-G_\sigma G_\rho],\\
J_{T,T}&=J_{\sigma\rho,\kappa\lambda}=\tfrac{2}{3}\pi[G_\rho G_\kappa g_{\sigma\lambda}-G_\sigma G_\kappa g_{\rho\lambda}+G_\sigma G_\lambda g_{\rho\kappa}\\
&\quad -G_\rho G_\lambda g_{\sigma\kappa}]+\tfrac{1}{3}\pi(GG)[g_{\sigma\kappa}g_{\rho\lambda}-g_{\sigma\lambda}g_{\rho\kappa}].
\end{aligned} \quad (20)$$

Here we have again written out the tensor indices for the sake of clarity. One can even go one step further. The tensors (20) are always contracted with expressions of the form

$$\tfrac{1}{4}\mathrm{Tr}\{Q\Gamma_tQ'\Gamma_{t'}\}, \quad (21)$$

where Q and Q' are two Dirac matrix products depending on momentum vectors. It is easy to verify that the following results obtain:

$$\tfrac{1}{4}\,\mathrm{Tr}\{Q\Gamma_SQ'\Gamma_S\}J_{S,S}=\pi(GG)\tfrac{1}{4}\,\mathrm{Tr}\{QQ'\}, \quad (22)$$

$$\tfrac{1}{4}\,\mathrm{Tr}\{Q\Gamma_VQ'\Gamma_V\}J_{V,V}=\tfrac{2}{3}\pi[\tfrac{1}{4}\,\mathrm{Tr}\{Q(G\gamma)Q'(G\gamma)\} -(GG)\tfrac{1}{4}\,\mathrm{Tr}\{Q\gamma^\mu Q'\gamma_\mu\}], \quad (23)$$

$$\tfrac{1}{4}\,\mathrm{Tr}\{Q\Gamma_TQ'\Gamma_T\}J_{T,T}=(8\pi/3)[\tfrac{1}{4}\,\mathrm{Tr}\{Q(G\gamma)\gamma_\mu Q'(G\gamma)\gamma^\mu\} -(GG)\tfrac{1}{4}\,\mathrm{Tr}\{QQ'\}]-\tfrac{2}{3}\pi(GG)\tfrac{1}{4}\,\mathrm{Tr}\{Q\gamma_{\mu\nu}Q'\gamma^{\mu\nu}\}. \quad (24)$$

[20] The Lorentz invariant product of two vectors a and b will be denoted by $(ab)=a_\mu b^\mu=\mathbf{a}\cdot\mathbf{b}-a^0b^0$.

The last term in (24) can frequently be omitted, since it vanishes whenever Q or Q' contains an odd number of γ-factors.

The foregoing results can be now applied to evaluate (14). The result will be expressed in terms of the invariants (pP) and $(PP)=-M^2$. It is customary to write it in the rest system of the initial μ-meson. One obtains the following:

$$\mathcal{P}_{(\mu)}=\frac{\epsilon_m}{(2\pi)^4}\left[(g_S^2+2g_V^2+8g_T^2)(\epsilon_m-\epsilon)+(2g_V^2+16g_T^2)\frac{\epsilon}{3}\right]. \quad (25)$$

Here $\epsilon_m=M/2$ is the maximum electron energy. Integrating over all energies the lifetime of the μ-meson becomes τ, where

$$\frac{1}{\tau}=4\pi\int_0^{\epsilon_m}\mathcal{P}_{(\mu)}\epsilon^2d\epsilon=\frac{\epsilon_m^5}{6(2\pi)^3}(g_S^2+4g_V^2+24g_T^2). \quad (26)$$

These are the results derived by Michel,[21] provided one sets

$$\rho=(3g_V^2+24g_T^2)/(g_S^2+4g_V^2+24g_T^2). \quad (27)$$

The two quantities τ and ρ are subject to experimental determination from the simple μ-decay. According to present experimental evidence,[22,23]

$$\tau=(2.22\pm0.02)\times10^{-6}\text{ sec}, \quad (28)$$

and

$$\rho=0.26\pm0.26. \quad (29)$$

The considerable uncertainty still attached to the value of ρ, as well as the fact that only two of the coupling constants can be determined so far, makes it worth while to explore further consequences of the theory. This leads us to the problem of inner bremsstrahlung which is the main subject of this paper.

IV. INNER BREMSSTRAHLUNG

The transition probability for the decay of a μ-meson into an electron plus a photon of energy-momentum $K=(W,\mathbf{K})$ is given by

$$\mathcal{P}_{(\gamma)}d^3pd^3K=(2\pi)^{-8}\int d^3k\int d^3k'\tfrac{1}{2}\sum_{\text{spins}}\sum_{\rho}|(f|R|i)|^2\times\delta(P-K-p-k-k')d^3pd^3K, \quad (30)$$

where now the matrix element $(f|R|i)$ must be obtained from second-order perturbation theory with a Hamiltonian that takes into account the electromagnetic interactions of the charged particles involved in the reaction. In (30), ρ denotes the state of linear polarization of the outcoming photon; we sum over this variable, since detectors are assumed to be insensitive to polarization.

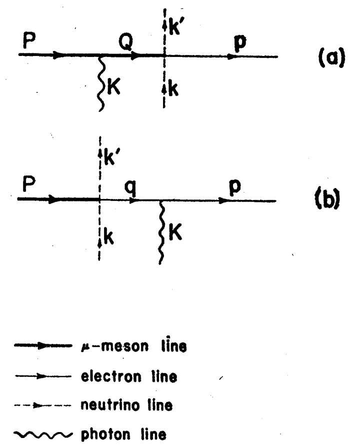

FIG. 1. Feynman graphs for inner bremsstrahlung in μ-decay. (a) Contribution due to meson acceleration, (b) Contribution due to electron acceleration.

The Hamiltonian is now written

$$H=H_{(\mu)}+H_{(\gamma)}, \quad (31)$$

the second term being the electromagnetic interaction. The latter consists of two terms,

$$H_{(\gamma)}=ie[(\bar{\Psi}\gamma_\mu\Psi)+(\bar{\psi}\gamma_\mu\psi)]A^\mu, \quad (32)$$

where A^μ is the transverse[24] electromagnetic potential field operator, and $e=(4\pi/137)^{\frac{1}{2}}$. The two terms involve the current vectors of the μ-meson field and the electron field. Correspondingly, the total amplitude for the transition desired will be a sum of the two amplitudes arising out of these two terms. The Feynman graphs corresponding to the two contributions are shown in Fig. 1. Evidently, in one case, the acceleration of the μ-meson into the intermediate state is responsible for the photon emission, while, in the other case, the emission is caused by the acceleration of the electron from the intermediate to the final state. One obtains the total amplitude by applying the Feynman rules[25] to the two graphs. The result is

$$(f|R|i)=e\sum_t g_t[\bar{\nu}(k)\Gamma_t\nu(k')]\times\left(\bar{u}(p)\left[\frac{(w\gamma)}{(2W)^{\frac{1}{2}}}\frac{(q\gamma)+im}{(qq)+m^2}\Gamma_t+\Gamma_t\frac{(Q\gamma)+iM}{(QQ)+M^2}\frac{(w\gamma)}{(2W)^{\frac{1}{2}}}\right]U(P)\right). \quad (33)$$

Here $\nu(k)$, $u(p)$, and $U(P)$ are properly normalized eigenspinors belonging to indicated momentum values, for instance, $(\bar{u}(p)u(p))=m/\epsilon$, etc.; $w=w(\rho)$ are two unit polarization vectors perpendicular to the photon

[21] L. Michel, Phys. Rev. **86**, 814 (1952), Eq. (1).
[22] W. E. Bell and E. P. Hincks, Phys. Rev. **84**, 1243 (1951).
[23] H. W. Hubbard, Thesis, University of California (1952), unpublished.
[24] For a justification of using the transverse field for computing transition amplitudes see, for instance, F. Coester and J. M. Jauch, Phys. Rev. **78**, 149 (1950), Sec. IV.
[25] See, for instance, F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

momentum and each other

$$w^0=0,\quad (ww)=1,\quad (wK)=0, \tag{34}$$

and q and Q are momenta in the intermediate states

$$q=p+K,\quad Q=P-K. \tag{35}$$

Squaring and summing over spins is carried out in the usual manner. As has already been observed, the integral over neutrino momenta has the same form as (15), the only difference being that now the photon momentum K also appears in the argument of the δ-function. The result (18) is still applicable, provided we set

$$G=P-p-K. \tag{36}$$

This will be understood in the following. The result of these manipulations can then be written as

$$\mathcal{P}_{(\gamma)}=\frac{1}{(2\pi)^8 16E\epsilon W}e^2\sum_t g_t^2\mathfrak{F}_t, \tag{37}$$

where

$$\mathfrak{F}_t=\sum_p J_{t,t'}\tfrac{1}{4}\,\mathrm{Tr}\left\{(p\gamma)\left[\frac{(w\gamma)}{(pK)}((p\gamma)+(K\gamma))\Gamma_t - \Gamma_t((P\gamma)-(K\gamma))\frac{(w\gamma)}{(PK)}\right](P\gamma)\left[\Gamma_{t'}((p\gamma)+(K\gamma))\frac{(w\gamma)}{(pK)} - \frac{(w\gamma)}{(PK)}((P\gamma)-(K\gamma))\Gamma_{t'}\right]\right\}. \tag{38}$$

Here we have written two different indices t and t' in order to indicate unambiguously which pairs of tensor indices are being contracted. There are, of course, no cross terms between different interaction types; and this is so for the same reason which has been pointed out in Sec. III. A property of the $\mathfrak{F}_t$, providing a valuable check at various stages of the calculation, is

$$\mathfrak{F}_t(p,P,K)=\mathfrak{F}_t(P,p,-K). \tag{39}$$

This symmetry has its origin in the relationship of the two graphs of Fig. 1 which contribute to the process.

Further steps in the calculation involve the application of the formulas (22)–(24) to the expressions (40) and finally the reduction of the resulting traces to a sum of products of invariants involving the vectors p, P, K, G, and w. By judicious application of (36), one obtains the following results:

$$\frac{1}{2\pi}\mathfrak{F}_S=-(GG)[(GG)+M^2]\Phi/W^2 -2(GG)(KG)^2/(pK)(PK), \tag{40}$$

$$\frac{3}{8\pi}\mathfrak{F}_V=\{-(GG)[(GG)+M^2/2]+M^4/2\}\Phi/W^2 -4(GG)+(KG)^2[-2(GG)+M^2]/(pK)(PK), \tag{41}$$

$$\frac{3}{32\pi}\mathfrak{F}_T=\{\tfrac{1}{2}(GG)[M^2-(GG)]+M^4-4(pK)(PK)\}\Phi/W^2 +4(GG)+(KG)^2[(GG)+4(pP)]/(pK)(PK) +4[(pK)^2-(PK)^2][(pG)+(PG)]/(pK)(PK). \tag{42}$$

Here we have used the abbreviation

$$\Phi\equiv W^2\sum_p\left[\frac{(wp)}{(pK)}-\frac{(wP)}{(PK)}\right]^2. \tag{43}$$

The function Φ does not depend on the photon energy, only on its direction of emission.

V. DISCUSSION OF RESULTS

In order to visualize the significance of these results, it is advantageous to rewrite them in the rest-system of the μ-meson as a function of ϵ, W, and the angle θ between the directions of propagation of the emitted electron and photon. The main angular dependence is contained in the factor Φ which can be expressed as

$$\Phi=\left(\frac{\beta\sin\theta}{1-\beta\cos\theta}\right)^2, \tag{44}$$

where β is the velocity of the emitted electron. This function has a sharp maximum in the "forward" direction $\theta=m/\epsilon$. It is convenient to express all the remaining angular dependence in terms of the function

$$W_0=\epsilon_m\frac{\epsilon_m-\epsilon}{\epsilon_m-\frac{1}{2}\epsilon(1-\beta\cos\theta)}. \tag{45}$$

Here, as before, $\epsilon_m=M/2$ is the maximum energy attainable by the electron. The significance of W_0 may be clarified by the remark that $W=W_0$ is the solution for W of the equation

$$(GG)=0, \tag{46}$$

which is an expression of the fact that the two neutrinos have been emitted in the same direction. This, in turn, means that the photon energy is a maximum, consistent with given ϵ and θ. In this notation one can write, for instance,

$$(GG)=-4\epsilon_m(\epsilon_m-\epsilon)(1-W/W_0) \tag{47}$$

and

$$(KG)=-2\epsilon_m(\epsilon_m-\epsilon)W/W_0, \tag{48}$$

etc. Making these substitutions, one then rewrites $\mathcal{P}_{(\gamma)}$ in terms of the desired variables. Since these substitutions are of a rather trivial nature, we shall not write down the results but merely call attention to some special cases.

Let us consider first the limiting case of small photon energies. This means that from (40)–(42) only the leading terms in $1/W$ are retained. The results are very simple, for comparison with (25) shows that

$$\mathcal{P}_{(\gamma)}d^3K=\frac{1}{2}\frac{e^2}{(2\pi)^3}\Phi\frac{dW}{W}d\Omega_\gamma\mathcal{P}_{(\mu)}, \tag{49}$$

where $d\Omega_\gamma$ is the solid angle element into which the photon is emitted. The significant feature of this relationship is that it is independent of the values of the coupling constants. That this is to be expected can be made plausible by means of an argument based on classical correspondence.[26] The relationship (49) can also be proved quite generally by using the quantum electrodynamics of spinor particles.[27] The function Φ may be said to represent the "classical" radiation pattern resulting from the sudden acceleration of a point charge.

Consider now the behavior of inner bremsstrahlung at the maximum photon energy $W=W_0$. It follows from (40) and (46) that at this energy the contribution of the scalar interaction vanishes. This situation is quite analogous to that of the ordinary μ-decay, as shown by (25). There the contribution of the scalar interaction vanishes at the upper end of the electron energy spectrum. Neither is the connection accidental: it has its root in the formula (22), for in both cases the emission of the two neutrinos in the same direction implies (46), so that the transition probability vanishes. Since this situation obtains at all angles θ, perhaps one has here, a method of obtaining further corroboration, as already shown by the existing experimental results[23] on μ-decay, of the preponderance of the scalar interaction.

Further information of a more quantitative nature may be obtained by looking at the actual form of the photon spectrum for given electron energies ϵ and angles θ. We shall be content here with giving the results for the simplest such case, *viz.*, $\theta=\pi$. Then $\Phi=0$ and $W_0=\epsilon_m$. One obtains, after some algebra,

$$\mathcal{P}_{(\gamma)}d^3pd^3K=\frac{e^2}{(2\pi)^7}d\epsilon dWd\Omega_e d\Omega_\gamma(\epsilon_m-\epsilon)\frac{W}{\epsilon_m}\left(A-B\frac{W}{\epsilon_m}\right), \tag{50}$$

where

$$A=\tfrac{1}{4}g_S^2(\epsilon_m-\epsilon)^2+\tfrac{1}{6}g_V^2(3\epsilon_m^2+2\epsilon^2-\epsilon_m\epsilon)+\tfrac{1}{3}g_T^2(6\epsilon_m^2+2\epsilon^2), \tag{51}$$

and

$$B=\tfrac{1}{4}g_S^2(\epsilon_m-\epsilon)^2+\tfrac{1}{3}g_V^2(\epsilon_m^2+\epsilon^2)+\tfrac{2}{3}g_T^2(\epsilon_m+\epsilon)^2. \tag{52}$$

$d\Omega_e$ is the solid angle element into which the electron is emitted. It is clear that a precise knowledge of the photon spectrum (50) at some or several electron energies would yield more than sufficient information to fix the values of all the coupling constants. Of particular interest in this connection may be the value of the spectrum at its high energy end as well as at its maximum.

The practical limitations imposed on such a measurement are, however, considerable, mainly on account of the smallness of the effect. The principal reason for this is that, in order to utilize most of the information supplied by the theory, one would have to do coincidence measurements between electrons and photons both of which occupy only a small part of the available momentum space. An estimate for the magnitude of the effect is easily obtained from the formula (50), for instance, where, for the sake of definiteness, one may assume that only the scalar interaction contributes to it. Putting $W=\epsilon_m/2$ at which energy the rate of transitions is a maximum, one gets

$$\mathcal{P}_{(\gamma)}d^3pd^3K=\frac{e^2}{4\pi}\frac{d\epsilon}{\epsilon_m}\frac{dW}{\epsilon_m}\frac{d\Omega_e}{4\pi}\frac{d\Omega_\gamma}{4\pi}\frac{3}{2\pi}\left(1-\frac{\epsilon}{\epsilon_m}\right)^3\frac{1}{\tau}. \tag{53}$$

It is clear that, even for the smallest energies ϵ and with an optimistic estimate for the energy intervals and solid angle factors, one needs a meson source of around 10^8 mesons produced per second in order to obtain a reasonable coincidence counting rate of the kind desired.

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[26] See reference 3, Sec. III.

[27] The proof is based on comparing the squared matrix element for an arbitrary graph with one obtained from it by inserting into an external spinor line an external photon line in the limit of small photon energy. See, for instance, R. Jost, Phys. Rev. **72**, 815 (1947), where this procedure is applied to estimating the cross section for the double Compton effect in the limit of small energy for one of the outcoming photons. For greater detail, the reader is referred to the author's thesis.