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Specific Primary Ionization of H_2, He, Ne, and A by High Energy Electrons*

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Monoenergetic electrons, magnetically separated from the continuous spectrum of a radioactive β-ray source, are directed through a series of 3 thin-window G-M counters. A measurement of the efficiency of the first counter by the coincidence method yields information from which the specific primary ionization of its contained gas is calculated.

Measurements of H_2, He, Ne, and A over the range of incident electron energies 0.2 to 1.6 Mev have been made and the data compared with the Bethe theory of primary ionization. By adjustment of two constants contained in the theoretical formula, it is possible to fit the data for each of the four gases within the experimental uncertainties. The adjusted formulas yield extrapolated values of the specific primary ionization which are fairly consistent with the work of others at lower and higher energies.

INTRODUCTION

MEASUREMENTS of the specific primary ionization of gases by charged particles have been conducted by a number of investigators[1–7] over a considerable range of incident-particle energies. There have been large discrepancies between the results obtained with different methods of measurement, particularly for electrons in the neighborhood of minimum ionization, and the experimental errors have thus far precluded a critical quantitative comparison between the measurements and the theory.

In view of the many applications of knowledge concerning the ionization of gases in various fields of experimental physics, it was considered worth while to conduct a new set of precise measurements on several gases in the neighborhood of minimum ionization using β-particles from a radioactive source as the primary ionizing radiation. The measurements reported here extend over a sufficient range of energies to make possible a determination of the parameters which enter into the Bethe theory of primary ionization and indicate the degree to which the theory may be relied upon as a tool for extrapolating experimental data.

EXPERIMENTAL PROCEDURE

The experimental procedure consists of measuring the probability that a G-M counter filled with the gas under investigation will be discharged by a β-ray having a known energy and a well-defined path length in the counter. Given the discharge probability (or efficiency) of the counter together with the path length and the gas pressure, one may calculate the specific primary ionization of the contained gas. A discussion of the required calculations will be found near the end of this section of the paper.

The experimental arrangement (as shown in Fig. 1) consists essentially of a 180° β-ray spectrograph (evacuated to 10^{-2} mm Hg) and a set of G-M counters C_1, C_2, and C_3. A radioactive source at S (Pr^{144} in equilibrium with Ce^{144}) gives rise to a continuous spectrum of β-rays ranging in energy from zero to 3 Mev. The series of baffles, B, selects from the continuous spectrum a beam of β-rays with a ±5 percent energy spread which pass through the counter train via a series of 1-mil aluminum windows W_1, W_2, and W_3. The magnetic field, which determines the momentum of the β-rays entering W_1, is measured by means of a ballistic galvanometer connected to a flip coil F.

The counter C_1 was filled with a gas or mixture of gases whose primary ionization was to be measured. The gas pressure was chosen so that the β-rays in the spectrograph energy range were counted with an efficiency of about 0.5, a condition which results in the least required operating time for determining the primary ionization to a given statistical accuracy. Counters C_2 and C_3 filled with a self-quenching argon-butane mixture served to count the number of β-ray transversals of counter C_1.

The counters were connected to an external circuit which simultaneously recorded the threefold coincidences ($C_1C_2C_3$) and twofold coincidences (C_2C_3). As the walls of C_1 permit the penetration of β-rays only through windows W_1 and W_2, any ray which is to give rise to either type of coincidence must traverse C_1 within the solid angle subtended by the two windows. The efficiency of C_1 for counting rays of this selected group is given by

$$\epsilon=\frac{(C_1C_2C_3)-(C_1C_2C_3)_0}{(C_2C_3)-(C_2C_3)_0}, \tag{1}$$

where the quantities with "0" subscripts represent the cosmic-ray background coincidence rates recorded with the magnetic field of the spectrograph reduced to zero. The background rates were less than 10 percent of the

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* Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.

[1] E. J. Williams and F. R. Terroux, Proc. Roy. Soc. (London) **126**, 289 (1929).

[2] P. T. Smith, Phys. Rev. **36**, 1293 (1930).

[3] J. T. Tate and P. T. Smith, Phys. Rev. **39**, 272 (1932).

[4] W. E. Danforth and W. E. Ramsey, Phys. Rev. **49**, 854 (1936).

[5] M. G. E. Cosyns, Nature **138**, 284 (1936).

[6] M. G. E. Cosyns, Nature **139**, 802 (1937).

[7] F. L. Hereford, Phys. Rev. **74**, 574 (1948).

β-ray counting rates during all of the measurements reported here.

It was necessary to operate the counter C_1 as a resistance-quenched counter with all of the gases used in the investigation. As a series resistance of the order of 10^9 ohms was required for proper quenching, the dead time of C_1 was of the order of 10^{-3} sec. In order to obviate the necessity of correcting the measured efficiencies for the dead-time effect, a dc coupled anticoincidence circuit was introduced between the wire of C_1 and the coincidence selection circuits so as to prevent the recording of any coincidence (C_2C_3) or $(C_1C_2C_3)$ which occurred at a time when the wire potential of C_1 was more than 5 volts below its normal (quiescent) potential. This imposed the condition that counter C_1 be completely recovered from any preceding discharge before a coincidence could be recorded and thus rendered the experimentally determined efficiencies independent of the dead time of C_1.

The background rates used in Eq. (1) to evaluate the efficiencies were corrected to compensate for the fractional inactive time of the recording circuits imposed by the anticoincidence circuit during the β-ray data runs. This correction was accomplished by multiplying the measured background rates by the ratio between the coincidence rates (C_1C_2) measured at each field setting with and without the anticoincidence circuit in operation.

As mentioned above, only those β-rays which traverse C_1 within the solid angle of windows W_1 and W_2 contribute to the measured counting rates. These windows are 0.250 in. in diameter and the plane surfaces on which they are mounted are separated by a distance, $L_0 = 0.688''$. For β-rays of the energy range considered here, the mean scattering angle in W_1 is sufficiently large that one may consider each point of the window as an isotropic source of rays emanating into the counter. Calculated on this basis, the mean path length L, of rays traveling between windows W_1 and W_2 turns out to be only 1.3 percent greater than the direct pathlength L_0.

A second path-length perturbation arises from a slight, approximately spherical distortion of window W_1, which resulted from subjecting the window to a one atmosphere pressure differential before the counter assembly was attached to the spectrograph. This distortion has the effect of reducing the average path length to a value approximately 0.5 percent less than that calculated on the basis of scattering alone. The combined effects of scattering and window curvature yield a mean path length differing from the value L_0 by less than 0.8 percent. Since the error introduced into the final result by setting the effective mean path equal to L_0 is of the same order of magnitude as the statistical errors, the path-length correction has been neglected in all of the calculation.

Corrections for the energy loss suffered by the β-rays in penetrating the first window were made on the basis of the range *vs* energy curves for electrons in Al. For the lowest energy β-rays considered in the present investigation (0.2 Mev), this correction amounts to approximately 9 percent, while for the highest energy rays (1.6 Mev), the correction is approximately one percent.

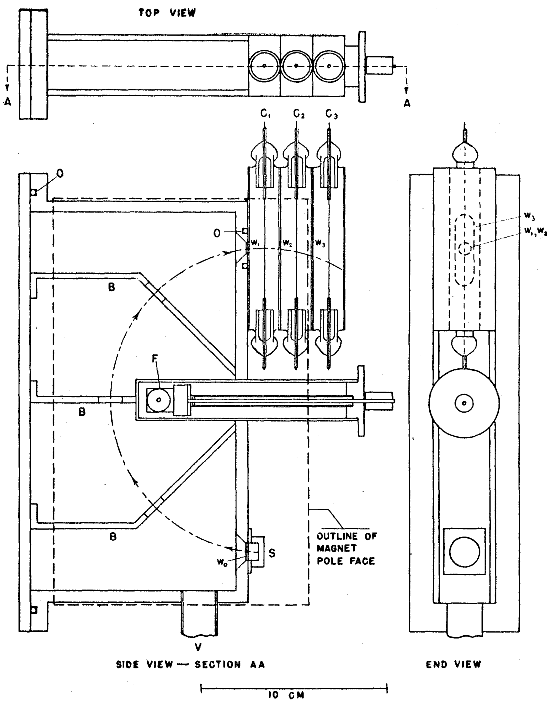

FIG. 1. Apparatus for determining the specific primary ionization of gases. S—beta-ray source; B—spectrograph baffles; F—flip coil for measuring magnetic field; C_1, C_2, C_3—GM counters, W_1, W_2, W_3—0.001 in. Al windows separating counters; V—forepump connection for evacuating spectrograph; O—"O" ring seals.

The statistical distribution of the number of ions produced by monoenergetic rays traversing the well-defined path length between the collimating windows W_1 and W_2 is given by the Poisson law. Accordingly, if x denotes the average number of ionizing collisions per traversal, the probability that no ion pair will be produced in a traversal is equal to e^{-x}. Provided that (1) the counter undergoes a complete discharge whenever at least one free electron is present within its volume, and (2) no secondaries are released from the wall material by the primary rays, the counting efficiency ϵ is

$$\epsilon = 1 - e^{-x}. \tag{2}$$

The specific primary ionization S (at N.T.P.) is given in terms of x by the formula

$$S = xT/273LP. \tag{3}$$

where L is the path length in cm, P the gas pressure in atmospheres, and T the temperature of the gas in degrees K.

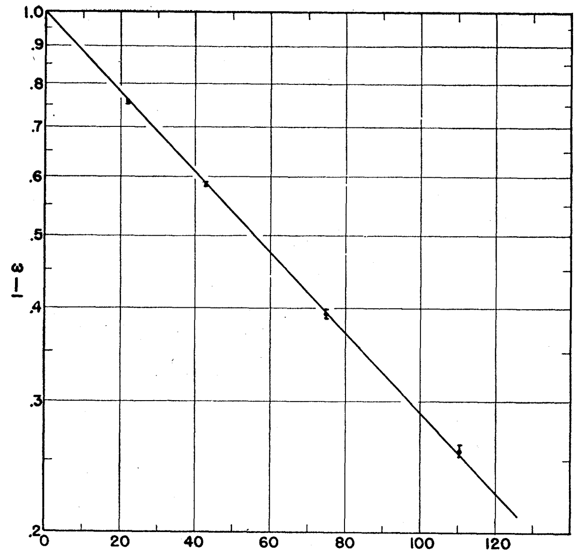

FIG. 2. Variation of the quantity $(1-\epsilon)$ as a function of the pressure of H_2 contained in counter C_1 (0.855 Mev β-rays).

In order to insure the fulfillment of provision (1), the investigation was restricted to gases known to have electron attachment coefficient sufficiently small to be neglected. In addition, care was taken to ascertain that the counter C_1 was operated in a region of its characteristics where increases of from 50 to 100 volts in operating potential yielded no significant change in the measured efficiency.

Spectroscopically pure gases supplied by the Linde Air Products Company were used throughout the investigation. The pressures of gases introduced into the counter C_1 were measured with a mercury manometer constructed of glass tubing of sufficiently large diameter (0.5 in.) to eliminate errors arising from unequal meniscus shapes in the two columns of the manometer. An antiparallax method was used in comparing the heights of the mercury columns with a fixed scale, graduated in mm. Errors in the measured specific primary ionization values, caused by inaccuracies in pressure readings, were less than or equal to the statistical errors.

Counter C_1 exhibited very flat efficiency *vs* voltage curves (slopes less than 1 percent per 100 volts) with pure H_2 fillings but yielded unsatisfactory performance when operated with the rare gases unmixed with a secondary gas component. (With pure rare-gas fillings, the counter broke down into continuous discharge under applied potentials only slightly above the starting threshold.) It was found, however, that the addition of a small percentage of H_2 to any of the rare-gas fillings permitted the counter to function as well as when filled with pure H_2. Therefore, to avoid the difficulties encountered with the pure rare gases, the specific primary ionization measurements on the latter were performed with small partial pressures of H_2 added to C_1. With the mixtures employed, the corrections required to compensate the measurements for the added quantities of H_2 amounted to approximately 10 percent, 16 percent, and 5 percent for He, Ne, and A respectively. The precise percentages, of course, vary slightly with the β-ray energy.

As a test of the over-all reliability of the experimental procedure, a curve of efficiency *vs* pressure was taken on the counter C_1 with H_2 as the filling gas. Figure 2 is a semilogarithmic plot of the measured values of $(1-\epsilon)$ as a function of the hydrogen pressure. It is seen that within the statistical errors the experimental points lie along a straight line which extrapolates to $(1-\epsilon)=0$ at $P=0$ as expected from Eqs. (2) and (3). If secondary electrons from the walls of C_1 contributed any significant amount to the counting rate of C_1, the efficiency *vs* pressure data would be expected to extrapolate to a value $(1-\epsilon)$ less than unity at $P=0$. On the basis of these results, the experimental errors due to wall-effect are considered to be negligible.

EXPERIMENTAL RESULTS

The results of the specific primary ionization measurements on H_2, He, Ne, and A are presented in Table I. The experimental errors indicated in the table are the statistical standard deviations which in all cases are closely comparable with the independent uncertainties associated with the pressure and path-length determinations. The relative magnitude of the specific primary ionization values obtained for any individual gas are considered to be accurate within the statistical errors. Uncertainties in the absolute values, which depend upon the precision of the pressure and path-length determinations, are conservatively estimated at ±3 percent. Repeat runs on H_2, He, and A, after removing the original gas filling and introducing new fillings of the same pressures, indicated that the measurements were reproducible within the statistical uncertainty.

COMPARISON OF THE EXPERIMENTAL RESULTS WITH THE THEORY

The theory of primary ionization has been treated both classically and quantum mechanically in various degrees of approximation by a number of authors. The most recent and comprehensive treatment of the subject —given by Bethe[8]—yields the following formula for

TABLE I. Measured values of the specific primary ionization of H_2, He, Ne, and A for various β-ray energies.

Energy (Mev)	p/Mc	Specific primary ionization-ions/cm at N.T.P.			
		H_2	He	Ne	A
0.205	0.98	8.45±0.095	7.56±0.16	18.45±0.36	41.7±0.77
0.500	1.71	6.04±0.065	5.58±0.066	13.4 ±0.17	30.5±0.34
0.855	2.49	5.44±0.054	5.08±0.043	12.5 ±0.14	27.7±0.31
1.17	3.15	5.30±0.053	...	...	...
1.55	3.90	5.32±0.059	5.02±0.060	12.4 ±0.13	27.8±0.31

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[8] H. A. Bethe, *Handbuch der Physik* (Julius Springer Verlag, Berlin, 1933), Vol. **24**, No. 1, p. 515.

the variation of specific primary ionization with the velocity of the incident particle:

$$S=2\pi r_0^2mc^2Nz^2\frac{Za}{I_0}\frac{1}{\beta^2}\left[\ln\frac{2mc^2\beta^2}{I_0(1-\beta^2)}+b-\beta^2\right], \tag{4}$$

where S = number of primary ions per cm of path, $r_0=e^2/mc^2$ (classical electron radius), m = electron rest mass, c = velocity of light, N = number of atoms per cc at N.T.P., z = incident particle charge, Z = atomic number of gas, β = ratio of incident particle velocity to the velocity of light, I_0 = ionization potential of the gas, and a, b = dimensionless constants dependent upon the electronic structure of the gas.

For the particular case of atomic hydrogen ($I_0=13.5$ ev, $Z=1$), Bethe has calculated the values of the constants in this equation to be: $a=0.285$, $b=3.04$. For gases other than atomic hydrogen, the calculation of the constants cannot be readily accomplished without introducing rather rough approximations regarding the form of the wave functions of the bound electrons.

In the derivation of Eq. (4) the assumption is made that the incident particle is undeviated by individual collision with the atoms of the gas. This approximation leads to rather large errors at low energies but should not seriously affect the validity of the formula at electron energies greater than a few kv.

TABLE II.

	H_2	He	Ne	A
A	0.356±0.03	0.422±0.038	1.13±0.09	2.37±0.27
C	4.35 ∓0.02	3.98 ∓0.02	9.64∓0.11	21.8 ∓0.20

The form of Eq. (4) indicates that for a given gas, S depends only upon the velocity v and the charge z of the incident ionizing particle. By making the substitution,

$$\beta=(p/Mc)[(p/Mc)^2+1]^{-\frac{1}{2}}, \tag{5}$$

where p is the momentum and M the rest mass of the incident particle, and lumping the constant coefficients in Eq. (4), one obtains the expression:

$$S=A\left\{\left(\frac{p}{Mc}\right)^{-2}\left[\left(\frac{p}{Mc}\right)^2+1\right]\ln\left(\frac{p}{Mc}\right)^2-1\right\}+C\left(\frac{p}{Mc}\right)^{-2}\left[\left(\frac{p}{Mc}\right)^2+1\right]. \tag{6}$$

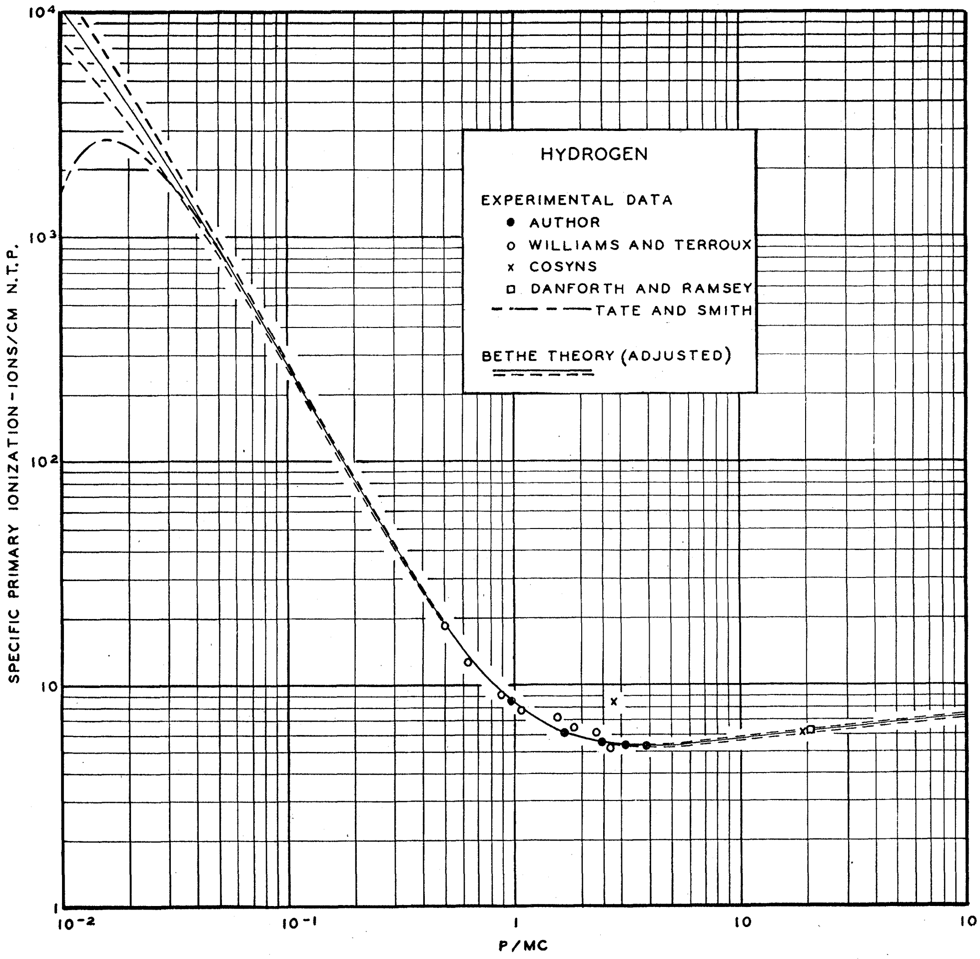

FIG. 3. Experimental results on hydrogen plotted *vs* p/Mc. The data of Williams and Terroux,[1] Tate and Smith,[3] Danforth and Ramsey[4] and Cosyns[5,6] are shown for comparison. The solid curve represents the least-squares fit of the Bethe formula [Eq. (6)] to the present results. The dashed curves neighboring the solid curve are the extremes of the family of theoretical curves which fit all points of the present data within one standard deviation.

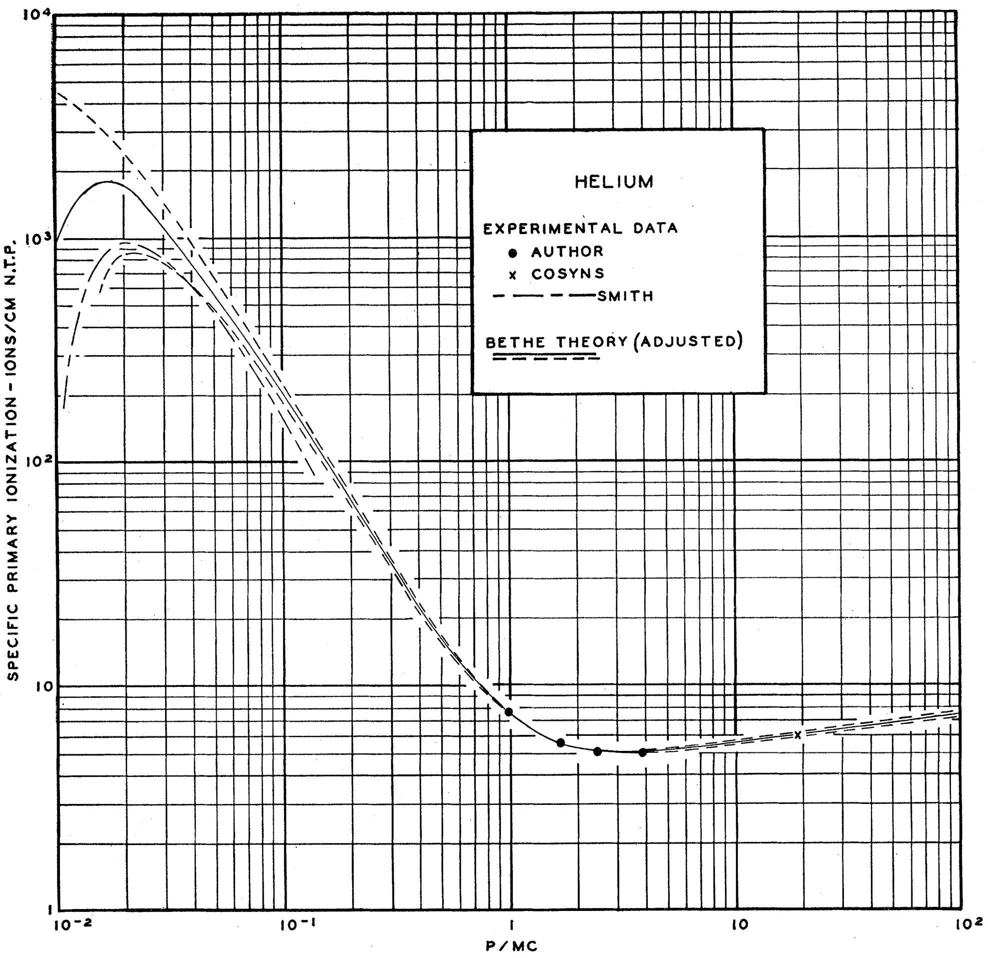

Fig. 4. Experimental results on helium plotted *vs* p/Mc. The data of Smith[2] and Cosyns[5] are also shown. The same remarks as in the caption of Fig. 3 apply to the solid and dashed curves.

The above form of the Bethe formula is preferred for ease in comparison with the experimental data. In terms of the constants of Eq. (4) the lumped constants A and C in Eq. (6) have the form:

$$A=2\pi r_0^2mc^2Nz^2Za/I_0, \qquad (7)$$

$$C=A[b+\ln(2mc^2/I_0)]. \qquad (8)$$

In order to compare the theory with the experimental results the constants A and C have been empirically determined for each of the gases investigated so that Eq. (6) yields the best agreement with the experimental results. The values of A and C determined by least squares fits are given in Table II, and plots of Eq. (6) evaluated with the tabulated constants are shown in Figs. 3–6 (solid curves). The dashed curves also shown in the figures indicate the extremes of a family of theoretical curves which fit all of the experimental values determined in the present investigation within one standard deviation. The amounts by which the constants corresponding to the two extreme curves deviate from the least squares values are indicated in Table II by the increments to the right of the $\pm$ and $\mp$ signs. The parameters for one of the extreme curves are obtained by taking the upper signs and those for the other extreme curve by taking the lower signs. It should be noted that the two extreme curves in each of Figs. 3–6 cross over one another in the neighborhood of $p/Mc=2$.

The parameters A and C calculated for atomic hydrogen from Eqs. (7) and (8) using the theoretical values of a and b are $A=0.145$, $C=2.08$. If the above values are multiplied by the factor 2 (to give the effect of doubling the number of atoms/cm^3) the resulting constants $A=0.290$ and $C=4.15$ should represent a rather gross approximation to the constants for molecular hydrogen. Comparison with the empirical constants for H_2 given in Table II indicates that the A's agree within 25 percent and the C's within 5 percent. The discrepancy between the A's is not at all surprising in view of the difference in the electronic structures of atomic and molecular hydrogen.

DISCUSSION

(a) Hydrogen

The least squares fit of the Bethe formula (solid curve Fig. 3) to the H_2 data obtained in the present investigation merges smoothly with the experimental data of Tate and Smith[3] at $p/Mc=0.054$ (electron energy of 750 v) but diverges gradually at lower energies toward primary ionization values in excess of the Tate and Smith values. The experimental points of Cosyns[5] and of Danforth and Ramsey[4] for cosmic-ray mesons (average value of $p/Mc \simeq 19$) agree with the solid curve within the experimental errors. The cloud-chamber data of Williams and Terroux[1] scatter somewhat broadly above and below the curve indicating the presence of relatively large experimental uncertainties, however, the over-all consistency with the present results is quite satisfactory.

Hereford[7] has conducted a low pressure counter study of the primary ionization of H_2 by β-rays and cosmic-ray mesons. These results (not plotted in Fig. 3) are consistent with the other work as regards the ionization of cosmic rays but show considerable discrepancies with the present work in the β-ray energy range. A value obtained by Hereford for a continuous spectrum of β-rays of near-minimum ionization ($2.6 < p/Mc < 5.8$) is about 15 percent lower than the present data in the same range. On the other hand, a point obtained by Cosyns[6] for 1-Mev electrons ($p/Mc = 2.8$) is about 50 percent higher than the present value. The most likely explanation for these discrepancies seems to be that in the earlier low pressure counter work dealing with β-rays the path length was not well defined and that scattering was not adequately treated in calculations of the effective pathlength.

(b) Helium

In the case of helium, there is a rather large discrepancy between the solid curve and the low energy data of Smith.[2] This seems rather surprising in view of the relatively high accuracy with which the formula joins the high energy data with the low energy data for the other gases. The cosmic-ray point obtained by Cosyns[5] for helium agrees with the solid curve within the experimental uncertainties.

The gradual relativistic increase in primary ionization (at high energies) predicted by the Bethe formula is confirmed in both H_2 and He by the consistency of the cosmic-ray data of Cosyns[5] and of Danforth and Ramsey,[4] with the theoretical extrapolation of the β-ray data.

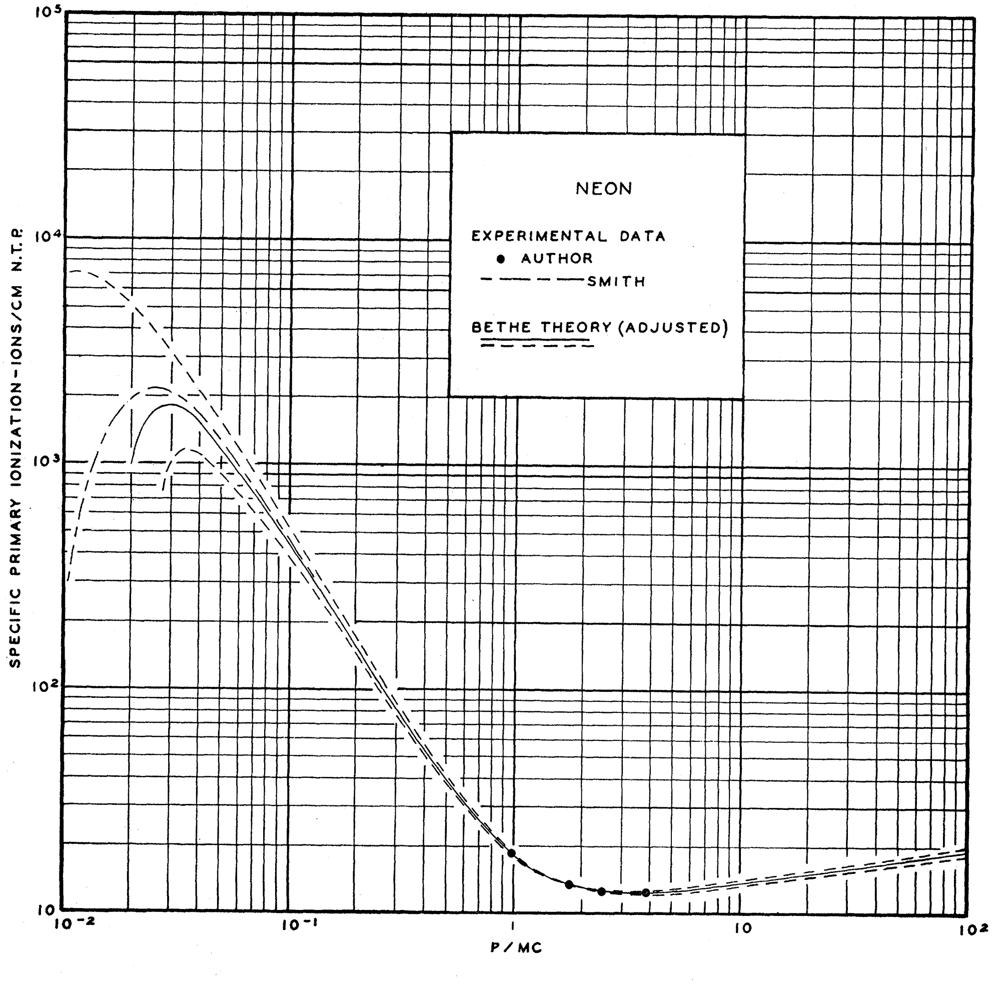

FIG. 5. Experimental results on neon plotted *vs* p/Mc. A plot of Smith's data[2] adjusted as indicated in the text is also shown. The same remarks as in the caption of Fig. 3 apply to the solid and dashed curves.

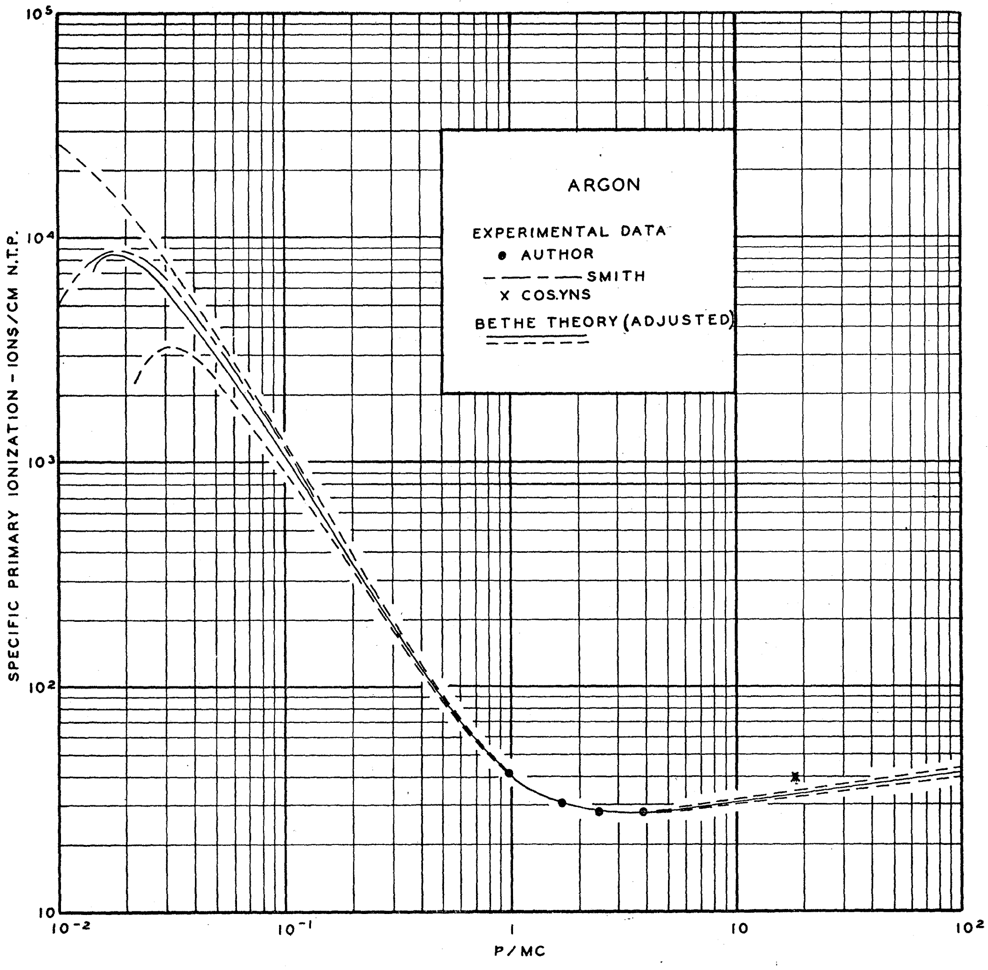

Fig. 6. Experimental results for argon plotted *vs* p/Mc. The data of Smith[2] (adjusted as indicated in the text) and Cosyns[5] are also shown. The same remarks as in the caption of Fig. 3 apply to the solid and dashed curves.

(c) Argon and Neon

In order to compare Smith's results[2] on argon and neon with the present measurements, it is necessary to observe that the quantity measured by Smith is not in general exactly equivalent to specific primary ionization. While the specific primary ionization, S, measured in the present experiment is defined as the number of ionizing collisions per unit path length, the "ionization probability," P, measured by Smith may be defined as the number of electrons released in ionizing collisions per unit of path length. If R denotes the average number of electrons removed from an atom (or molecule) per ionizing collision, we have the relationship

$$S = P/R. \tag{9}$$

The mass-spectrographic measurements of Bleakney[9] show that the R values for H_2 and He are equal to unity within approximately one percent for incident electrons in the energy range zero to 500 ev. Consequently, the Smith data for H_2 and He have been plotted directly in Figs. 3 and 4 without compensation for multiple ionization. An analysis of Bleakney's Ne and A data indicate that $R = 1.06$ for 500-ev electrons in Ne and $R = 1.10$ for 500-ev electrons in A. As the R values for both of the heavy gases vary quite slowly with increasing energy in the range 300 to 500 ev, we have adopted the procedure of extrapolating R as a constant above 500 volts ($p/Mc = 0.048$) in transforming the Smith A and Ne data[2] into the curves attributed to Smith in Figs. 5 and 6. The uncertainty as to the accuracy of the transformed curves in the energy range 500 to 4500 ev ($0.048 < p/Mc < 0.133$) is of little importance in view of a comparable uncertainty in the position of the extrapolated theoretical curve in the low energy region. The fact that the Bethe formula provides the indicated degree of fit over the energy range from approximately 1 kev to 1.6 Mev may be regarded as a substantial verification of the correctness of the form of the theory at energies exceeding 1 kev.

CONCLUSION

The comparison made between the existing experimental results and the adjusted Bethe formula indicate a rather comprehensive agreement over an extremely

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[9] W. Bleakney, Phys. Rev. **36**, 1303 (1930); **35**, 1180 (1930).

large energy range. Although it is unfortunate that the parameters in the Bethe formula have not been calculated from theory for any of the gases investigated here, the formula appears to be a reliable tool for interpolating between and extrapolating beyond measured values of specific primary ionization at energies in excess of a few kev.

At very low energies, there is a systematic departure of the theory from the experimental results, which must be attributed to the weakness of the present theory. The only cases where the Bethe formula extrapolations of the present data are inconsistent are the experimental results of others in the neighborhoods of $p/Mc=0.05$ and $p/Mc=19$ occur in the case of He, where the low energy data of Smith[2] fall below the theoretical curve, and in the case of A where the cosmic-ray point of Cosyns[5] lies above the curve. The decision as to whether these discrepancies are attributable to the theory or to the experimental technique is difficult to make at the present time.

The data obtained in the present investigation, in addition to serving as a useful guide toward further development of the theory of ionization, should be of value both in the design of low efficiency G-M counters and in the analysis of cloud-chamber photographs containing the tracks of fast particles.

Measurements of the specific primary ionization of several other simple gases and of some of the complex organic vapors utilized in G-M counters will be carried out in the near future.

The writer wishes to express his appreciation to Dr. W. F. G. Swann, Director of the Bartol Research Foundation, for several helpful discussions during the course of this work.

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Two-Electron Self-Consistent Field

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Stevenson's extension of the Hartree self-consistent field (s.c.f.) method, to take some account of polarization interaction between two outer electrons, is worked out numerically, up to a point, for H$^-$. The two-electron wave function is a product of two one-electron radial functions and a function X of the angle between the two radii. Several integrals must be calculated and a boundary value problem for X must be solved prior to the stage of ordinary Hartree calculations. These fundamental computations are made here, but the routine solution of the radial Hartree equations has been omitted; instead the Stevenson corrections to the total energy are estimated by perturbation methods. These corrections seem just about large enough to yield an electron affinity for H$^-$ (something not achieved by s.c.f. or variation methods without polarization). However, a good deal of labor is involved beyond that required for s.c.f. solutions, and it is not certain that this particular method of including polarization effects is to be preferred to a combination of variation and s.c.f. calculations.

INTRODUCTION

ABOUT 15 years ago Stevenson published an extension of the Hartree self-consistent field (s.c.f.) equations, designed to handle cases of two electrons outside closed shells, including allowance for spatial symmetry properties and mutual polarization energy.[1] So far as the present author knows, no calculations have been carried out by this method, save for some approximate ones reported by Stevenson himself in a later paper.[2]

In view of continuing interest in such atomic structures (mainly in the form of loosely bound negative ions—specific references appear below), it seemed worth while to work out one trial example. The aim is to assess the difficulties and the gains of the Stevenson method, in comparison with both conventional s.c.f. and variation procedures. For such a trial the H$^-$ ion seems ideally suited. Its properties have importance in solar spectroscopy, its electron affinity and wave functions have been found with considerable precision by a lengthy variational method, and yet neither the s.c.f. nor the simplest variational calculation[3] yield a positive electron affinity.

Stevenson's paper gives equations applicable to any configuration of the two electrons, but for H$^-$ we shall consider only the simple case of two $1s$ orbits, without exchange. The pertinent equations are (pp. 596–598 of reference 1)

$$\chi_i''(r_i)+[\epsilon_i+2V(r_i)+C/r_i^2-2U_i(r_i)]\chi_i(r_i)=0, \quad (1)$$

$$(i=1,\,2),$$

$$\frac{d}{dx}[(1-x^2)X'(x)]+[\epsilon_3-U_3(x)]X(x)=0, \quad (2)$$

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[1] A. F. Stevenson, Proc. Roy. Soc. (London) **A160**, 588 (1937).

[2] A. F. Stevenson, Phys. Rev. **56**, 586 (1939).

[3] That is, the one using spherically symmetrical hydrogen-like wave functions.