Space-Charge Limited Emission in Semiconductors

W. SHOCKLEY AND R. C. PRIM
Bell Telephone Laboratories, Murray Hill, New Jersey

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A situation analogous to thermionic emission into vacuum can occur in semiconductors. A semiconductor analog for a plane parallel vacuum diode may consist of two layers of n type semiconductor bounding a plane parallel slab of pure semiconductor. The current density analogous to Child's law is $J=9\kappa\epsilon_0\mu V^2/8W^3$, where κ=dielectric constant, ϵ_0=mks permittivity, μ=mobility, V=applied voltage, and W=thickness of pure region. The condition prevailing at the space-charge maximum is analyzed taking into account diffusion due to random thermal motion. Brief discussions are given of the effect of fixed space charge, the dependence of mobility upon electric field strength and the role of space-charge limited emission in a new class of unipolar transistors.

I. CHILD'S LAW ANALOG

ONE of the basic equations of vacuum electronics is Child's law,[1] which gives the current density for space-charge limited emission. Child's law is an approximate expression, valid for conditions in which the voltage is large compared to kT/e. In order to deal with lower voltages or with the situation prevailing at the potential energy maximum, it is necessary to deal with the problem by the methods of statistical mechanics.[2]

In this article we shall show that there is a situation in transistor physics analogous to space-charge limited emission in vacuum and that for this situation there is an approximate equation for current density analogous to Child's law. There is also an exact solution that gives the situation near the potential maximum.

The situation of interest may be described in terms of an n-i-n structure which consists of a sandwich with n type material bounding a plane parallel layer of pure or intrinsic material. For concreteness we shall think in terms of a single crystal of germanium containing donor rich ends and a pure center. Under conditions of thermal equilibrium the densities of electrons n, holes p, donors N_d, the net charge density ρ and the potential energy for electrons will be qualitatively as represented in Fig. 1.

The potential energy rise from each n type region is seen to be due to a dipole layer: the positive part is due to unbalanced donors within the n type region and the negative part is due to electrons in the i region. Deep within the i region, holes are present in equal numbers with electrons. Where holes and electron densities are nearly equal, the deviations from equality vary as $\exp(\pm x/L_D)$ where L_D is the Debye length[3] and in mks units is

$$L_D=(\kappa\epsilon_0 kT/2en_i)^{1/2}, \tag{1.1}$$

where

$$\kappa=\text{the dielectric constant of germanium,} \tag{1.2a}$$
$$\epsilon_0=\text{mks permittivity,} \tag{1.2b}$$
$$kT=\text{thermal energy,} \tag{1.2c}$$
$$e=|e|=\text{the electronic charge,} \tag{1.2d}$$
$$n_i=\text{electron density in intrinsic material.} \tag{1.2e}$$

For germanium

$$\kappa=16, \tag{1.3}$$

and at room temperature $n_i\doteq 3\times 10^{13}$ cm^{-3} so that $L_D\doteq 6\times 10^{-5}$ cm. The situation of Fig. 1 corresponds to an i region many Debye lengths thick.

When a voltage is applied to the structure, the situation shown in Fig. 2 results. The hole density has been neglected in this case, as it may be if it is small compared to the electron density. The hole current flowing from the positively biased n region is similar to the reverse saturation current in a $p-n$ junction and will be small if the n region is heavily doped and has long lifetime.[4]

The conditions prevailing in the i region may be derived from Poisson's equation

$$\kappa\epsilon_0 dE/dx=\rho, \tag{1.4}$$

(where E is the electric field and ρ is the charge density and is negative), and the equation for current density

$$J=Dd\rho/dx-\mu\rho E, \tag{1.5}$$

where J is current density in the minus x direction and D and μ (both positive numbers) are diffusion constant and mobility. D and μ are related by the Einstein relationship

$$eD=kT\mu. \tag{1.6}$$

The ratio of diffusion current to drift current is

$$(Dd\rho/dx)/\mu\rho E=(d\ln\rho/dx)/(eE/kT). \tag{1.7}$$

Throughout most of the i region $\ln\rho$ varies gradually so that

$$d\ln\rho/dx\approx 1/W. \tag{1.8}$$

[1] C. D. Child, Phys. Rev. **32**, 492 (1911).
[2] I. Langmuir and K. T. Compton, Revs. Modern Phys. **3**, 237 (1931).
[3] W. Shockley, Bell System Tech. J. **28**, 435 (1949), for a derivation see p. 441.
[4] See reference 3, p. 461.

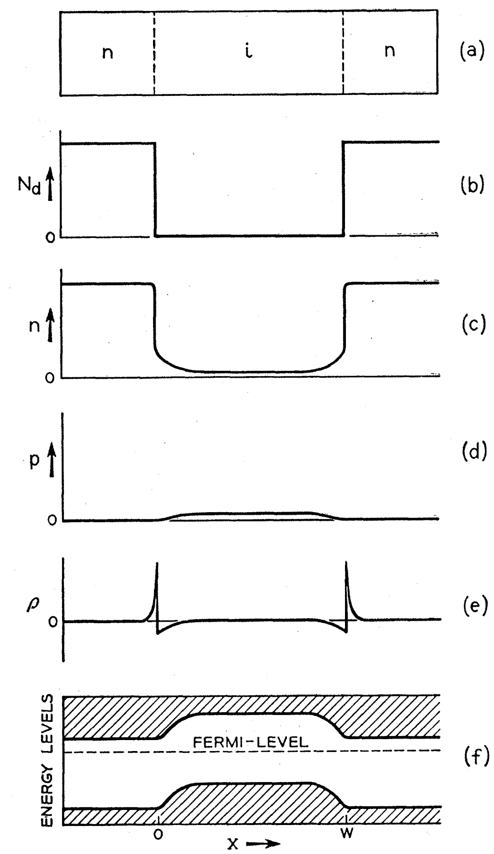

FIG. 1. The thermal equilibrium state of a *n-i-n* structure.

On the other hand,

$$eE/kT \approx eV/kTW, \quad (1.9)$$

where V is the applied voltage, so that if V is large compared to kT/e ($\doteq$25 millivolts at room temperature) the current is carried chiefly by drift.

If the current is carried by drift, then ρ may be obtained in terms of J from (1.5), this value of ρ inserted in (1.4) and the resulting equation integrated to give

$$\tfrac{1}{2}E^2 = Jx/\kappa\epsilon_0\mu + \text{const.} \quad (1.10)$$

Space-charge limited emission requires that $E=0$ at $x=0$, the left edge of the i region. Hence the "const" in (1.10) is zero. We shall also set the electrostatic potential $\psi(x)=0$ at $x=0$. The applied voltage is then $V=\Psi(W)$. Integration of (1.10) to obtain Ψ gives

$$\Psi = (8J/9\kappa\epsilon_0\mu)^{1/2}x^{3/2}, \quad (1.11)$$

$$V = (8J/9\kappa\epsilon_0\mu)^{1/2}W^{3/2}. \quad (1.12)$$

The current density and transit time across the region are

$$J = 9\kappa\epsilon_0\mu V^2/8W^3, \quad (1.13)$$

$$\text{transit time} = 4W^3/3\mu V. \quad (1.14)$$

The above equations constitute the analog to Child's law.

These equations are in error because diffusion and the dependence of mobility upon electric field have been neglected. For large applied voltages, the latter effect is more important. We shall formulate a method of solution for the latter case under somewhat more general conditions.

II. CASE OF A *p-n-p* STRUCTURE

In Fig. 3 we illustrate a case in which the mobile charges are holes and the fixed charges are donors. The change from electrons to holes emphasizes the added freedom of transistor electronics compared to conventional electronics and simplifies the subsequent equations in regard to signs. The p regions are regarded as heavily doped so that the voltage drop through them is negligible. If the charge density due to donors is ρ_f and that due to holes is ρ, then the equation for current

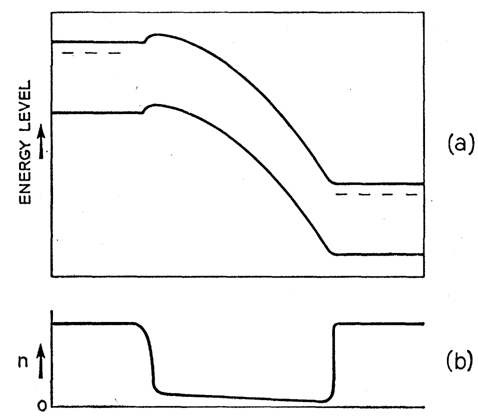

FIG. 2. Space-charge limited emission in a *n-i-n* structure.

in the plus x direction is

$$J = \rho(x)v[E(x)], \quad (2.1)$$

where $v(E) = \mu_p E$ for low fields and μ_p is the low field mobility of holes. For larger values of E, empirical or theoretical expressions must be used for $v(E)$.[5] Poisson's equation then becomes

$$\kappa\epsilon_0 dE/dx = \rho_f + J/v(E). \quad (2.2)$$

If ρ_f is independent of x, this equation may be reduced to quadratures by writing it as

$$\kappa\epsilon_0 dE/[\rho_f + J/v(E)] = dx, \quad (2.3)$$

so that the coefficient of dE on the left-hand side is a function of E alone.

Before integrating this equation we may note that for small applied potentials, the structure will behave similarly to a $p-n-p$ transistor with a floating base. Conditions of space-charge limited emission will occur

[5] E. J. Ryder and W. Shockley, Phys. Rev. **81**, 139 (1951); W. Shockley, Bell System Tech. J. **30**, 990 (1951).

first when the space-charge layer at the reversely biased junction penetrates the n region. This phenomenon is referred to as *punch-through* since the space charge punches through the middle layer. The voltage at which it occurs is called the *punch-through voltage*. For voltages greater than the punch-through voltage, the electrons may be neglected and any additional potential will produce a space-charge limited flow of holes. The critical voltage is obtained by integrating (2.2) with $J=0$ and is

$$V_f=\rho_f W^2/2\kappa\epsilon_0, \tag{2.4}$$

corresponding to a potential distribution

$$\Psi=-\rho_f x^2/2\kappa\epsilon_0, \tag{2.5}$$

which gives $\Psi=0$ and $E=0$ at $x=0$.

For the case in which constant mobility μ_p prevails, the equation for E may be conveniently integrated in terms of the transit time s from $x=0$, to x, where

$$s=\int_0^x dx/v=\int_0^x dx/\mu_p E. \tag{2.6}$$

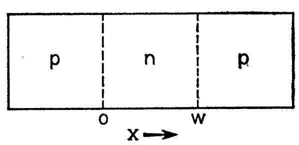

FIG. 3. A p-n-p structure.

This procedure leads to

$$E(s)=(J/\mu_p\rho_f)(\xi-1), \tag{2.7}$$

$$\Psi(s)=-(J^2\kappa\epsilon_0/\mu_p^2\rho_f^3)(\tfrac{1}{2}\xi^2-2\xi+\ln\xi+3/2), \tag{2.8}$$

$$x(s)=(J\kappa\epsilon_0/\mu_p\rho_f^2)(\xi-\ln\xi-1), \tag{2.9}$$

$$\xi\equiv\exp(s\mu_p\rho_f/\kappa\epsilon_0), \tag{2.10}$$

$$V(t)=(J^2\kappa\epsilon_0/\mu_p^2\rho_f^3)(\tfrac{1}{2}\eta^2-2\eta+\ln\eta+3/2), \tag{2.11}$$

$$W(t)=(J\kappa\epsilon_0/\mu_p\rho_f^2)(\eta-\ln\eta-1), \tag{2.12}$$

$$\eta\equiv\exp\beta\equiv\exp(t\mu_p\rho_f/\kappa\epsilon_0), \tag{2.13}$$

where t is the transit time through the n layer of thickness W and V is the applied voltage.

Equations (2.11) and (2.12) relate J to W and V with β, or its equivalent η, as a parameter. From them J can be plotted as a function of V and W.

The relationship between J and V can be conveniently expressed in terms of V_f and a current unit J_f. The current unit,

$$J_f=9\kappa\epsilon_0\mu_p V_f^2/8W^3, \tag{2.14}$$

is the current that would flow if V_f were applied to a structure of the same spacing but with the n region replaced by an i region. In terms of J_f the current density is

$$J=(32J_f/9)[e^\beta-\beta-1]^{-1}. \tag{2.15}$$

Figure 4 shows the relationship between J and V. On this plot, the Child's law analog would pass through the point (1, 1) with a slope of 1/2. This is the asymptote of the solution for large values of J and V.

Space-charge limited currents in germanium p-n-p structures have been investigated experimentally by Dacey.[6] He finds the behavior in qualitative agreement with that described here. In order to get quantitative agreement, however, he finds it necessary to refine the treatment taking into account the correct dependence of v upon E.

III. SOLUTION INCLUDING DIFFUSION

For the p-i-p or n-i-n structures it is possible to obtain an exact solution in the region in which v is proportional to E. For this case Eqs. (1.4) and (1.5), modified for p-i-p structure with μ and D applying to holes, may be integrated once to obtain

$$Jx=-\kappa\epsilon_0 D dE/dx+\tfrac{1}{2}\kappa\epsilon_0\mu E^2. \tag{3.1}$$

If distance, potential, electric field, and charge density are measured in the following units:

$$L_r=(\eta D/J)^{1/3}, \tag{3.2}$$

$$V_r=kT/e, \tag{3.3}$$

$$E_r=kT/eL_r, \tag{3.4}$$

$$\rho_r=\kappa\epsilon_0 E_r/L_r, \tag{3.5}$$

where

$$\eta=2\kappa\epsilon_0 kT/e \tag{3.6}$$

then (3.1) becomes

$$dy/dz-(1/2)y^2+z=0, \tag{3.7}$$

where

$$y=E/E_r, \tag{3.8}$$

$$z=x/L_r. \tag{3.9}$$

This equation has a solution for y, denoted by E_s, which has the correct approach to the Child's law analog for large positive values of z:

$$E_s(z)\rightarrow(2z)^{1/2}. \tag{3.10}$$

The corresponding electrostatic potential is taken as

$$V_s=\Psi/V_r=-\int_0 E_s(z)dz. \tag{3.12}$$

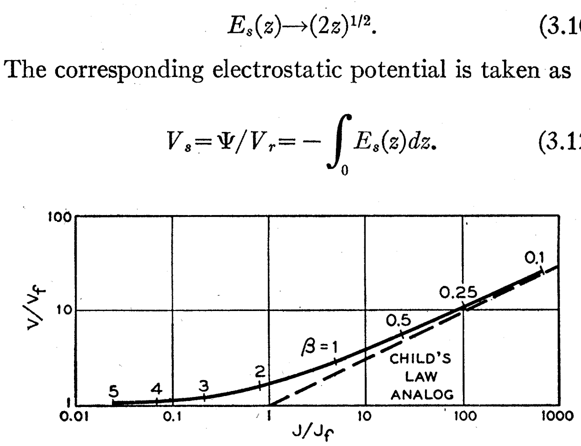

FIG. 4. Current-voltage relationship for p-n-p structure neglecting diffusion and effect of electrons.

[6] G. C. Dacey, Phys. Rev. 90, 759 (1953).

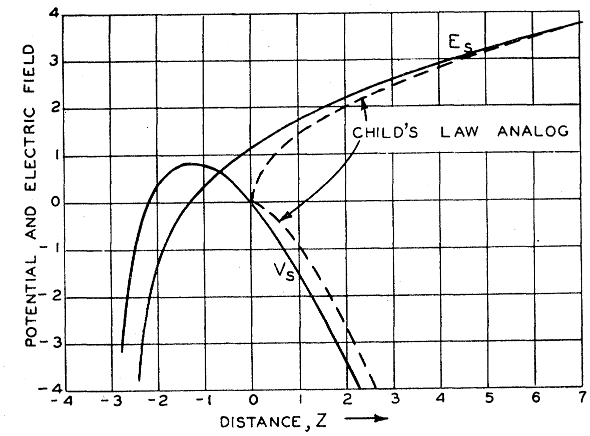

FIG. 5. Field and potential near the space-charge maximum as functions of distance in reduced units for exact solution and Child's law analog.

The procedures employed in integrating (3.7) to obtain the solution and the equations for the solution are given in Appendix II.

The dependence of E_s and V_s upon z is shown in Fig. 5 for the exact solution. Also shown is the approximate solution given by the Child's law analog. For most purposes the error is seen to be quite small. At large values of z, the approximate and exact E curves are asymptotically equal. If Child's law is extrapolated back from this point to $E=0$, an error of about $1.3L_r$ in estimating the location of the maximum will be made. The error in estimating the potential will be less than kT/e over most of the range. The reason for the good agreement between exact and approximate solutions is that except near the maximum the current is carried preponderately by drift.

To the left of the maximum, the exact solution closely approximates a Boltzmann distribution. If the hole distribution were accurately a Boltzmann distribution, then the height of the potential maximum above the Fermi level in the P region would be obtainable from[7]

$$\rho=eN_vf_p=\rho_r dE_s/dz, \qquad (3.13)$$

where N_v is the effective density of states in the valence band, f_p is the Fermi factor for holes and $dE_s/dz \doteq 1.3$ is evaluated at the maximum. Approximating f_p by the Boltzmann factor for the energy difference, denoted by $e\Delta V$, between valence-band edge and the Fermi level gives

$$e\Delta V=kT\ln(eN_v/1.3\rho_r). \qquad (3.14)$$

The fact that the hole density falls off slightly faster with increasing x than would a Boltzmann distribution causes this expression to overestimate the height of the maximum by something less than $0.7kT$, as is discussed in Appendix I.

[7] W. Shockley, *Elections and Holes in Semiconductors* (D. Van Nostrand Company, Inc., New York, 1950), for a discussion in this notation see p. 241.

IV. SOME IMPLICATIONS OF THE VIEWPOINT

The analogy between thermionic space-charge limited emission and space-charge limited emission in semiconductors suggests that amplifying structures should be possible that bear a closer resemblance to vacuum tubes than do point-contact transistors, filamentary transistors, or *p-n* junction transistors. In these types of transistors both hole currents and electron currents flow through the same regions of space and interact with each other in a significant way. In this sense these transistors are *bipolar* devices, whereas a vacuum tube is *unipolar* since the working current flows through a space containing no other charged particles. Although the mode of operation and the basic structure of transistors are fundamentally different from those of vacuum tubes, it is possible to design *unipolar transistors.* There are, of course, two classes of unipolar transistors, corresponding to positive and negative carriers, compared to one class of vacuum tubes. One type of unipolar transistor, referred to as an *analog transistor*, may be described as a literal translation of a vacuum tube triode into semiconductor language, and there are other types as well.[8] These possibilities give further evidence that the physics of semiconductors can furnish a sound base for an extensive engineering science of transistor electronics.

APPENDIX I

Before presenting an exact solution to Eq. (3.7) in Appendix II, certain physical features of the solution for diffusion over a potential maximum will be presented. For this purpose we shall consider the flow of holes and shall assume that the electrostatic potential $\psi(x)$ is known and has its maximum at $x=0$. The equation for current then becomes

$$J=-Dd\rho/dx-\mu\rho d\psi/dx. \qquad (A1.1)$$

The appropriate solution is

$$\rho(x)=(J/D)[\exp(-e\psi/kT)]\int_x^C \exp(e\psi/kT)dx, \qquad (A1.2)$$

where C corresponds approximately to the right-hand edge of middle layer. This solution correctly represents current drifting to the right beyond the maximum as may be seen by writing ψ in the form

$$\psi(x_1+\delta x)\doteq\psi(x_1)-E(x_1)\delta x, \qquad (A1.3)$$

and inserting this in the integral and letting $C\rightarrow\infty$; this leads to

$$\rho(x_1)=(J/D)\exp[-q\psi(x_1)/kT](kT/eE_1) \times\exp[q\psi(x_1)/kT]=J/\mu E_1. \qquad (A1.4)$$

To the left of the potential maximum, the integral is practically independent of x so that $\rho(x)$ is propor-

[8] W. Shockley, Proc. Inst. Radio Engrs. **40**, 1289 and 1365 (1952).

tional to the first exponential and thus is given by a Boltzmann distribution. If the potential $\psi(x)$ is symmetrical about its maximum, then at the maximum itself the value of $\rho(x)$ is half of what it would be on the basis of a Boltzmann distribution. Put in another way, the potential is less by $(kT/q)\ln 2$ than would be deduced by estimating it by assuming that $\rho(x)$ is given by a Boltzmann distribution. If the potential is unsymmetrical, however, as it is for Fig. 5, then the integral has more than half its maximum value at the potential maximum; this is the justification for the statement at the end of Sec. III.

APPENDIX II

This Appendix presents the integration of the differential Eq. (3.7) leading to the solutions plotted in Fig. 5.

The mathematical problem is uniquely set (as is readily apparent from a rough direction field sketch) by asking for that solution of

$$dy/dz-\tfrac{1}{2}y^2+z=0 \tag{A2.1}$$

for which $y\sim(2z)^{1/2}$ as $z\rightarrow+\infty$.

First, we introduce the new dependent variable ζ through the substitution

$$y\equiv-2(d/dz)\ln|\zeta|. \tag{A2.2}$$

This transforms (A2.1) into a linear equation of higher order:

$$d^2\zeta/dz^2-\tfrac{1}{2}z\zeta=0. \tag{A2.3}$$

The simultaneous change of dependent and independent variables defined by

$$g\equiv\tfrac{2}{3}\{z^3/2\}^{1/2} \tag{A2.4}$$

and

$$\zeta\equiv g^{1/3}\zeta_1 \tag{A2.5}$$

reduces (A2.3) to the modified Bessel's equation of order 1/3,

$$\frac{d^2\zeta_1}{dg^2}+\frac{1}{g}\frac{d\zeta_1}{dg}-\left\{1+\frac{1}{9g^2}\right\}\zeta_1=0,$$

for which a general solution is

$$\zeta_1=AI_{1/3}(g)+BK_{1/3}(g), \tag{A2.6}$$

where A and B are arbitrary constants. Substituting (A2.6) into (A2.5) we have

$$\zeta=Ag^{1/3}I_{1/3}(g)+Bg^{1/3}K_{1/3}(g). \tag{A2.7}$$

By use of (A2.4), (A2.2) can be written as

$$y=-2\frac{(d/dg)\ln|\zeta|}{(d/dg)(z)}=-\frac{(6g)^{1/3}}{\zeta}\frac{d\zeta}{dg}. \tag{A2.8}$$

Making use of

$$(d/dg)[g^nI_n(g)]=g^nI_{n+1}(g)+(2n/g)g^nI_n(g)$$

and

$$(d/dg)[g^nK_n(g)]=-g^nK_{n+1}(g)+(2n/g)g^nK_n(g),$$

one obtains by substituting (A2.7) into (A2.8),

$$y=-(6g)^{1/3}\left[\frac{AI_{4/3}(g)-BK_{4/3}(q)}{AI_{1/3}(g)+BK_{1/3}(g)}+\frac{2}{3g}\right]. \tag{A2.9}$$

Now the asymptotic behavior of the modified Bessel functions as $g\rightarrow+\infty$ (and hence $z\rightarrow+\infty$) is

$$\begin{aligned}I_n(g)&\sim(1/2\pi g)^{1/2}e^g,\\ K_n(g)&\sim(\pi/2g)^{1/2}e^{-g}.\end{aligned} \tag{A2.10}$$

Therefore, for $A\neq0$

$$y\sim-(6g)^{1/3}=-(2z)^{1/2},$$

so that the requirement that

$$y\sim(2z)^{1/2}=(6g)^{1/3}$$

necessitates the choice $A=0$. For $B\neq0$, the solution (A2.9) then reduces to

$$y=(6g)^{1/3}\left[\frac{K_{4/3}(g)}{K_{1/3}(g)}-\frac{2}{3g}\right], \tag{A2.11}$$

where $g\equiv\frac{2}{3}(z^3/2)^{1/2}$. By use of (A2.10) it is readily ascertained that this solution is asymptotic to $(6g)^{1/3}$ [which is equal to $(2z)^{1/2}$] as $g\rightarrow+\infty$. It is therefore the desired field function $E_s(z)$.

The potential function $V_s(z)$ will now be computed. By definition,

$$V_s(z)\equiv-\int_0^z E_s(z_1)dz_1. \tag{A2.12}$$

Making use of (A2.8) we replace this by

$$\begin{aligned}V_s&=-\int_0^z-2\frac{(d/dg_1)\ln|\zeta(g_1)|}{(d/dg_1)[z_1]}dz_1\\ &=2\int_0^g\frac{d}{dg_1}\ln|\zeta(g_1)|\cdot dg_1\\ &=2\ln|\zeta(g)/\zeta(o)|.\end{aligned} \tag{A2.12a}$$

Substitution of (A2.17) (with $A=0$) gives

$$V_s=2\ln\left|\frac{g^{1/3}K_{1/3}(g)}{\lim_{g_1\rightarrow0}g_1^{1/3}K_{1/3}(g_1)}\right|. \tag{A2.13}$$

Now, for small g_1,

$$K_{1/3}(g_1)=\frac{\pi}{3^{1/2}}\left[\frac{(g_1/2)^{-1/3}}{\Gamma(2/3)}-\frac{(g_1/2)^{1/3}}{\Gamma(4/3)}+O(g_1^{5/3})\right], \tag{A2.14}$$

so that

$$\lim_{g_1\rightarrow0}g_1^{1/3}K_{1/3}(g_1)=\frac{\pi2^{1/3}}{3^{1/2}\Gamma(2/3)}$$

and (A2.13) leads to

$$V_s=2\ln\left|\frac{3^{1/2}\Gamma(2/3)}{\pi 2^{1/3}}g^{1/3}K_{1/3}(g)\right|. \qquad \text{(A2.15)}$$

Now for any $z>0$, g is a positive real number and (A2.11) and (A2.15) permit straightforward plotting of E_s and V_s as functions of z. However, for $z<0$ g is imaginary and (A2.11) and (A2.15) cannot easily be used for computation. Because of the rather messy branch point at $g=0$, the simplest and most straightforward procedure is to obtain a solution of (A2.1) useful for $z<0$ and continuous with (A2.15) at $z=0$. To this end we change the independent variable in (A2.1) to

$$u\equiv -z,$$

obtaining

$$dy/du+\tfrac{1}{2}y^2+u=0. \qquad \text{(A2.16)}$$

The variable change (A2.2) now becomes

$$y\equiv 2(d/du)\ln|\zeta|, \qquad \text{(A2.17)}$$

and in conjunction with the double transformation,

$$w\equiv\tfrac{2}{3}(u^3/2)^{1/2}, \qquad \text{(A2.18)}$$

$$\zeta\equiv w^{1/3}\bar{\zeta}, \qquad \text{(A2.19)}$$

leads to the ordinary Bessel's equation of order 1/3 (for $\bar{\zeta}$ as a function of w) and hence to the solution

$$\zeta=A_1w^{1/3}J_{1/3}(w)+B_1w^{1/3}Y_{1/3}(w), \qquad \text{(A2.20)}$$

where A_1 and B_1 are arbitrary constants. Substituting (A2.20) into (A2.17) and using

$$\frac{d}{dw}[w^nJ_n(w)]=-w^nJ_{n+1}(w)+\frac{2n}{w}w^nJ_n(w)$$

and

$$\frac{d}{dw}[w^nY_n(w)]=-w^nY_{n+1}(w)+\frac{2n}{w}w^nY_n(w),$$

we obtain the solutions

$$y=-(6w)^{1/3}\left[\frac{A_1J_{4/3}(w)+B_1Y_{4/3}(w)}{A_1J_{1/3}(w)+B_1Y_{1/3}(w)}-\frac{2}{3w}\right]. \qquad \text{(A2.21)}$$

The constants A_1 and B_1 must now be so chosen that

$$\lim_{w\to 0}(\text{A2.21})=\lim_{g\to 0}(\text{A2.11}).$$

For small g,

$$K_{1/3}(g)=\frac{\pi}{3^{1/2}}\left[\frac{(g/2)^{-1/3}}{\Gamma(2/3)}-\frac{(g/2)^{1/3}}{\Gamma(4/3)}+O(g^{5/3})\right]$$

and

$$K_{4/3}(g)=\frac{\pi}{3^{1/2}}\left[-\frac{(g/2)^{-4/3}}{\Gamma(-1/3)}-\frac{(g/2)^{2/3}}{\Gamma(2/3)}+O(g^{4/3})\right].$$

From these it follows that

$$\lim_{g\to 0}(\text{A2.11})=\frac{(2/3)^{2/3}\Gamma(2/3)}{\Gamma(4/3)}=E_s(o). \qquad \text{(A2.22)}$$

Also, for small w

$$J_{4/3}(w)=\frac{(w/2)^{4/3}}{\Gamma(7/3)}-\frac{(w/2)^{10/3}}{\Gamma(10/3)}+O(w^{16/3}), \qquad \text{(A2.23)}$$

$$J_{1/3}(w)=\frac{(w/2)^{1/3}}{\Gamma(4/3)}-\frac{(w/2)^{7/3}}{\Gamma(7/3)}+O(w^{13/3}), \qquad \text{(A2.24)}$$

$$Y_{4/3}(w)=\frac{1}{3^{1/2}}\left[\frac{2(w/2)^{-4/3}}{\Gamma(-1/3)}-\frac{2(w/2)^{2/3}}{\Gamma(2/3)}+O(w^{4/3})\right], \qquad \text{(A2.25)}$$

and

$$Y_{1/3}(w)=\frac{1}{3^{1/2}}\left[-\frac{2(w/2)^{-1/3}}{\Gamma(2/3)}+\frac{(w/2)^{1/3}}{\Gamma(4/3)}+O(w^{5/3})\right]. \qquad \text{(A2.26)}$$

When these are used in (A2.21), it can be shown that

$$\lim_{w\to 0}(\text{A2.21})=-\frac{3^{1/2}A_1+B_1}{2B_1}\frac{(2/3)^{2/3}\Gamma(2/3)}{\Gamma(4/3)}.$$

To make this limit equal to $E_s(o)$ as given by (A2.22), it is necessary that

$$-(3^{1/2}A_1+B_1)/2B_1=1,$$

or

$$B_1=-3^{-1/2}A_1.$$

From (A2.21) we then obtain

$$E_s=-(6w)^{1/3}\left[\frac{Y_{4/3}(w)-3^{1/2}J_{4/3}(w)}{Y_{1/3}(w)-3^{1/2}J_{1/3}(w)}-\frac{2}{3w}\right], \qquad \text{(A2.27)}$$

where $w\equiv\frac{2}{3}[(-z)^3/2]^{1/2}$. The associated potential function is obtained by substituting (A2.20) (with $B_1=-3^{-1/2}A_1$) into (A2.12a) and making use of (A2.24) and (A2.26):

$$V_s=2\ln|\zeta(w)/\zeta(o)|$$

$$=2\ln\left|\frac{3^{1/2}\Gamma(2/3)}{2^{4/3}}w^{1/3}[3^{1/2}J_{1/3}(w)-Y_{1/3}(w)]\right|. \qquad \text{(A2.28)}$$

For any $z<0$, w is a positive real number so that (A2.27) and (A2.28) are suitable for computing E_s and V_s as functions of z. They thus complement (A2.11) and (A2.15) which are suitable for use when $z>0$. Finally, of course, for the joining point $z=0$, (A2.12) and (A2.22) provide

$$E_s(o)=(2/3)^{2/3}\Gamma(2/3)/\Gamma(4/3),$$
$$V_s(o)=0.$$