Stability of a Limiting Case of Plane Couette Flow*

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(Received February 5, 1953)

A fluid is supposed to be viscous and incompressible. The flow examined is $u=gy$; $v=0$ $(0\leqslant y<\infty;\ -\infty<x<\infty)$. Its stability is investigated by the method of small vibrations. Accepting the legitimacy of the usual expansions, the problem is reduced to the solution of a transcendental equation containing one parameter apart from the unknown. It is rigorously shown that the solutions of this equation are such as to make all modes of vibration of the flow damped.

I

IT is known experimentally that under seemingly similar conditions a fluid flow may, in general, be laminar or turbulent. The first instance of this phenomenon to be investigated experimentally in detail was the flow of a liquid through a pipe. It is known that the laminar flow through a pipe becomes increasingly "unstable" as the velocity of the fluid is increased. The resultant flow is turbulent. At sufficiently small velocities disturbances rapidly die out. O. Reynolds inferred that hydrodynamically similar laminar flows of incompressible fluids are stable or unstable according to as the "Reynolds' number" $R=ul/\nu$ is smaller or greater than a critical value.

The significance of the experimental results is a matter of some doubt. It is generally agreed with Reynolds that the breakdown of laminar flow and the onset of turbulence at high Reynolds numbers is initiated by a rapid increase of small disturbances.

The following hypotheses can be made: 1(a) The steady laminar flow is unstable as such, i.e., with respect to disturbances originating in the fluid. 1(b) The disturbances are carried into the fluid from outside. The experimentally observed phenomena may well be complicated by the coexistence of both these effects. 2(a) Instability (i.e., increase in time) is exhibited by certain disturbances of however small amplitude. 2(b) Instability is peculiar to certain small but finite disturbances.

It is natural first to attempt to confirm 1(a) and 2(a). This is the approach first taken by Rayleigh and Kelvin. 2(a) corresponds to neglecting certain inertia terms in the equations of motion of the perturbation to linearize them. Much work has been done in this direction.[1] There is some doubt in many results because of the approximations made. Recently[2] the case of the plane Poiseuille flow has been definitely settled through numerical methods in favor of instability of a type predicted by Lin.[1] Because of the amount of computing involved, this numerical approach seems impracticable to cover a wide range of the parameters.

The plane Couette flow is generally assumed to lead to no instability.[3] From Hopf's paper it seems not quite certain, however, that all roots of the transcendental equation involved were considered, each root corresponding to a mode of vibration. The analysis is complicated and it appears desirable to reinvestigate this case independently.

The limiting case of the plane Couette flow, where the moving boundary is at infinity, will be treated here. It will be shown that all modes of vibration are damped.

II. FORMULATION OF THE PROBLEM

The two-dimensional flow in the x–y plane of an incompressible viscous fluid is given by

$$\Delta\Psi_t+\Psi_y\Delta\Psi_x-\Psi_x\Delta\Psi_y=\nu\Delta\Delta\Psi,$$

$$u=\Psi_y,\quad v=-\Psi_x \text{ (condition of incompressibility).}$$

$\Psi(x, y, t)$ is the stream function, u and v are velocity components, and ν is the kinematic viscosity. We consider the main flow

$$u=gy;\quad v=0\ (0\leqslant y<\infty;\ -\infty<x<\infty;\ g\geqslant 0),$$

for which one may take $\Psi=\frac{1}{2}gy^2$. According to Squire,[4] two-dimensional disturbances are less stable than three-dimensional ones in the case of parallel flow. Hence, the perturbed stream function may be taken to be $\frac{1}{2}gy^2+\psi(xyt)$. Neglecting nonlinear terms in ψ, one obtains

$$\Delta\psi_t+gy^2\Delta\psi_x=\nu\Delta\Delta\psi.$$

We are concerned only with velocity perturbations originating inside the fluid. Therefore $\psi_x=\psi_y=0$ at $y=0$, and $\psi_x\rightarrow 0$, $\psi_y\rightarrow 0$ at infinity in the x–y plane. Hence at infinity ψ will tend to a constant, which one may take to be 0. It is therefore reasonable to assume[5] that

$$\psi(xyt)=\int_{-\infty}^{\infty}\omega(kyt)e^{ikx}dk,$$

$$\omega=\partial\omega/\partial y=0 \text{ at } y=0;$$

$$\omega\rightarrow 0,\quad \partial\omega/\partial y\rightarrow 0,\quad \text{when}\quad y\rightarrow\infty.$$

* Presented by B. Zondek in partial fulfilment of the requirements for the degree of Doctor of Philosophy in the Faculty of Pure Science, Columbia University, New York, New York.

[1] C. C. Lin, Quart. Appl. Math. **3**, Nos. 2, 3, 4 (1945). (An extensive list of references will be found here.)

[2] L. H. Thomas, Phys. Rev. **86**, 812 (1952).

[3] L. Hopf, Ann. Physik **44**, 1 (1914).

[4] H. B. Squire, Proc. Roy. Soc. (London) **A142**, 621 (1933).

[5] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Macmillan Company, New York, 1946), fourth edition, Sec. (9.7).

By separating variables (y and t) one obtains

$$\omega(kyt)=\sum_n c_n(k)\exp[-i\gamma_n(k)t]\varphi_n(y,k),$$

where $\gamma_n(k)$ and $\varphi_n(yk)$ are given by the following eigenvalue problem:

$$[\nu\{(d^2/dy^2)-k^2\}+i\gamma-ikgy][(d^2/dy^2)-k^2]\varphi(y)=0, \quad (1)$$

$$\varphi(0)=\varphi'(0)=0, \quad (2)$$

$$\varphi(y)\to 0;\quad \varphi'(y)\to 0,\text{ when } y\to\infty. \quad (3)$$

We shall not concern ourselves with the legitimacy of this eigenfunction expansion for ω. For the plane Couette flow with walls a finite distance apart, a proof has been given by Haupt.[6]

γ and φ are complex. k, ν, and g are real. It will be proved that for any k, ν, and g all eigenvalues γ satisfy $R(i\gamma)>0$, i.e., all perturbations are damped.

III

It is convenient here to make a list of some notations to be used henceforth.

Quantities: ν, g, k, γ, $\varphi(y)$ are defined in Sec. II.

$$z=ky,$$

$$b=g/\nu k^2\geqslant 0,$$

$$a=-1+i\gamma/\nu k^2,$$

$$\epsilon=b^{-\frac{1}{3}}=(\nu k^2/g)^{\frac{1}{3}}\geqslant 0,$$

$$\beta=-i\epsilon,$$

$$\alpha=-(a+1)\epsilon^2=-i\gamma/(\nu k^2g^2)^{\frac{1}{3}},$$

$$\sigma=\alpha+\epsilon^2,$$

$$c=(12^{1/6}/i\pi)\exp(-i\pi/6),$$

$$\alpha'=2^{\frac{2}{3}}3^{1/6}\pi^{-\frac{1}{2}},$$

$$C_m=(2^{4m}3^m m!)^{-1}\prod_{n=0}^{m}(9[2n-1]^2-4).$$

R and I mean real or imaginary parts, respectively. * means complex conjugate.

Notation defining contours of integration in the complex plane: Let z be a fixed point in the complex plane and t be the variable of integration.

$(z)_+$ or $(z)_-$: closed loop around z taken in the positive or negative sense, respectively.

$[z]_+$ or $[z]_-$: z by-passed so that $\arg(t-z)$ increases or decreases, respectively.

z; or; z: path starts or ends at z, respectively.

IV

The eigenvalue problem, Eqs. (1), (2), and (3), with the sign of k reversed has eigenvalues γ, for which sign $I(i\gamma)$ is reversed and $R(i\gamma)$ is unaffected. We shall therefore confine ourselves to $k\geqslant 0$.

Reduction to dimensionless form: Substituting $z=ky$ and $\rho(z)=\varphi(y)$ into Eqs. (1), (2), and (3) and bearing in mind the definitions of Sec. III, one obtains

$$[(d^2/dz^2)+a-ibz][(d^2/dz^2)-1]\rho(z)=0, \quad (4)$$

$$\rho(0)=\rho'(0)=0, \quad (5)$$

$$\rho(\infty)=\rho'(\infty)=0. \quad (6)$$

Reduction to a transcendental equation: Write

$$\vartheta(z)=[(d/dz)+1]\rho(z).$$

To satisfy Eqs. (5) and (6) one must have

$$\rho(z)=pe^{-z}+e^{-z}\int_\infty^z e^{\zeta}\vartheta(\zeta)d\zeta,$$

$$\vartheta(0)=0.$$

p is a constant of integration that is fixed by condition (5). We now have for $\vartheta(z)$ the following boundary value problem:

$$[(d^2/dz^2)+a-ibz][(d/dz)-1]\vartheta(z)=0, \quad (7)$$

$$\vartheta(0)=0, \quad (8)$$

$$\int_0^\infty e^{\zeta}\vartheta(\zeta)d\zeta \text{ converges.} \quad (9)$$

The solutions of Eq. (7) can be given as contour integrals (in close analogy to the treatment of Stokes' differential equation). This is conveniently done in two stages: Let

$$v(z)=[(d/dz)-1]\vartheta(z),$$

and put

$$v(z)=\int e^{ipz}f(p)dp.$$

The function $f(p)$ and the proper contours of integration in the complex p plane are determined by Eq. (7). We obtain

$$f(p)(-p^2+a)+bf'(p)=0.$$

Hence

$$f(p)=\exp[b^{-1}(\tfrac{1}{3}p^3-ap)],$$

apart from an irrelevant constant factor. Thus,

$$v(z)=\int\exp[b^{-1}(\tfrac{1}{3}p^3-ap)+ipz]dz,$$

and (apart from an arbitrary multiple of e^z)

$$\vartheta(z)=\int\exp[b^{-1}(\tfrac{1}{3}p^3-ap)+ipz](p+i)^{-1}dp.$$

Now b is positive (see Sec. III), and thus the integrand tends to zero strongly when $p\to\infty e^{i\varphi}$, $\infty e^{i\varphi'}$ or $\infty e^{i\varphi''}$,

[6] O. Haupt, Sitzber. math.-naturw. Kl. bayer. Akad. Wiss. München 2, 289 (1912).

where $5\pi/6<\varphi<7\pi/6$; $\pi/6<\varphi'<\pi/2$; $-\pi/2<\varphi''<-\pi/6$. To satisfy Eq. (9) the appropriate path of integration is $(\infty e^{i\varphi};\ [-i]_-;\ \infty e^{i\varphi'})$. Indeed, take $\varphi=11\pi/12$, $\varphi'=\pi/3$. Then along the contour one can make $17\pi/12\geqslant\arg(ip)\geqslant 5\pi/6$. Hence, on such a contour $\mathrm{R}(ip)\leqslant -m^2$ (a fixed negative upper bound). With $K(p)$ defined in an obvious way we have, when $z\to\infty$,

$$\vartheta(z)=\left|\int e^{ipz}K(p)dp\right|\leqslant \exp(-m^2z)\int\left|K(p)dp\right|$$

$$=\text{const.}\exp(-m^2z)\to 0.$$

Moreover m^2 can be chosen as large as we please, thus assuring condition (9). [The contour $(\infty e^{i\varphi};[-i]_+;\ \infty e^{i\varphi''})$ will lead to a divergent integral (9). This will be obvious from the relationship that these functions bear to the Hankel functions (see Sec. 5).]

The equation for the characteristic roots is then given by condition (8), namely,

$$0=\vartheta(0)=\int_{-\infty;\,[-i]_-;\ \infty e^{i\pi/3}}\exp[b^{-1}(\tfrac{1}{3}p^3-ap)](p+i)^{-1}dp.$$

Two convenient forms of this equation are obtained by the substitutions:

$$p=b^{\frac{1}{3}}s,\quad p=b^{\frac{1}{3}}t-i\quad (b^{\frac{1}{3}}>0).$$

They are

$$g_1(\sigma,\beta)\equiv\int_{-\infty;\,[\beta]_-;\ \infty e^{i\pi/3}}\exp(\tfrac{1}{3}s^3+\sigma s)(s-\beta)^{-1}ds=0. \tag{11}$$

$$f_1(\alpha,\beta)\equiv\int_{-\infty;\,[0]_-;\ \infty e^{i\pi/3}}\exp(\tfrac{1}{3}t^3+\beta t^2+\alpha t)t^{-1}dt=0. \tag{12}$$

For $\beta=0$ these are identical. We define for later use

$$g_1(\sigma,0)\equiv g_1(\sigma). \tag{13}$$

V

The functions f_1 and g_1 are related to the *Bessel functions of order one-third*. It is convenient to make use of the so-called "*modified Hankel functions of order one third*," $h_1(z)$ and $h_2(z)$.[7,8] They are connected with the Hankel functions by

$$h_1(z)=(\tfrac{2}{3}z^{\frac{3}{2}})^{\frac{1}{3}}H_{\frac{1}{3}}^{(1)}(\tfrac{2}{3}z^{\frac{3}{2}}), \tag{14}$$

$$h_2(z)=(\tfrac{2}{3}z^{\frac{3}{2}})^{\frac{1}{3}}H_{\frac{1}{3}}^{(2)}(\tfrac{2}{3}z^{\frac{3}{2}}). \tag{15}$$

They are univalued and are an independent set of solutions of "Stokes' equation"

$$u''(z)+zu(z)=0. \tag{16}$$

The well-known asymptotic series for the Hankel functions yield

$$h_1(z)\sim\alpha' z^{-\frac{1}{4}}\exp\left(\frac{2}{3}iz^{\frac{3}{2}}-\frac{5\pi i}{12}\right)[1+\sum_{m=1}(-i)^mC_mz^{-3m/2}], \tag{17}$$

valid for $-2\pi/3<\arg z<4\pi/3$.

$$h_2(z)\sim\alpha' z^{-\frac{1}{4}}\exp\left(-\frac{2}{3}iz^{\frac{3}{2}}+\frac{5\pi i}{12}\right)\times[1+\sum_{m=1}(i)^mC_mz^{-3m/2}], \tag{18}$$

valid for $-4\pi/3<\arg z<2\pi/3$. We also have the contour integrals

$$h_1(z)=c\int_{-\infty;\ \infty e^{i\pi/3}}\exp(\tfrac{1}{3}s^3+zs)ds, \tag{19}$$

$$h_2(z)=c^*\int_{-\infty;\ \infty e^{-i\pi/3}}\exp(\tfrac{1}{3}s^3+zs)ds. \tag{20}$$

Several symmetry relations are satisfied by these functions. We take note only of

$$[h_1(z^*)]^*=h_2(z), \tag{21}$$

$$[h_1'(z^*)]^*=h_2'(z). \tag{22}$$

VI

We now investigate the function $g_1(\sigma)$ [see Eqs. (13) and (11)]. From the contour integral (11) or otherwise it can be proved that

$$g_1(\sigma)=-e^{2\pi i/3}[g_1(\sigma^*e^{2\pi i/3})]^*. \tag{23}$$

We also need the function $g_2(\sigma)$ defined by

$$g_2(\sigma)=\int_{-\infty;\,[0]_+;\ \infty e^{-i\pi/3}}\exp(\tfrac{1}{3}s^3+\sigma s)s^{-1}ds. \tag{24}$$

It can be proved directly from the contour integrals [in close analogy to a similar formula involving h_1 and h_2 (see Sec. V)] that

$$g_1(\sigma e^{-2\pi i/3})=-g_1(\sigma)+g_2(\sigma)-2\pi i. \tag{25}$$

From Eqs. (11), (13), and (19) we see that

$$g_1(\sigma)=g_1(0)+c^{-1}\int_0^\sigma h_1(z)dz$$

and

$$g_1(0)=\int_{-\infty;\,[0]_-;\ \infty e^{i\pi/3}}\exp(\tfrac{1}{3}s^3)s^{-1}ds$$

$$=\tfrac{1}{3}\int_{(0)_-}e^tt^{-1}dt=-2\pi i/3.$$

[7] *Annals of the Computation Laboratory of Harvard University*, Vol. 2, "Tables of Modified Hankel Functions of Order One-Third and of their Derivatives."

[8] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944).

We have, therefore,

$$cg_1(\sigma)=-\tfrac{2}{3}\pi ic+\int_0^\sigma h_1(z)dz,$$
$$-\tfrac{2}{3}\pi ic=-\tfrac{2}{3}12^{1/6}e^{-\pi i/6}=-0.87358+0.50436i. \qquad (26)$$

This formula is useful to compute $g_1(\sigma)$ numerically from the table of h_1 and h_2.[7] A similar formula involving $h_2(z)$ gives $g_2(\sigma)$, but we do not need it.

Let us now investigate the function

$$c^{-1}\int_{\infty e^{i\pi/3}}^{\sigma} h_1(z)dz.$$

It follows from the asymptotic form of $h_1(z)$ [see Eq. (17)] that this integral converges. As a function of σ it tends to zero when $\sigma\to\infty e^{i\pi/3}$. Its derivative is $c^{-1}h_1(\sigma)$. The function $g_1(\sigma)$ has the derivative $c^{-1}h_1(\sigma)$ too. Moreover, $g_1(\sigma)\to 0$, when $\sigma\to\infty e^{i\pi/3}$. This follows from Eq. (11). (See similar arguments in Sec. IV.) For in Eq. (11) $\tfrac{1}{3}\pi\leqslant \arg s\leqslant\pi$, and when $\arg\sigma=\tfrac{1}{3}\pi$, we have $\tfrac{2}{3}\pi\leqslant\arg\sigma s\leqslant 4\pi/3$. Hence, $\mathrm{R}(\sigma s)<-m^2|\sigma|$ along the entire path of integration. Therefore,

$$cg_1(\sigma)=\int_{\infty e^{i\pi/3}}^{\sigma} h_1(z)dz. \qquad (27)$$

Similarly one obtains

$$c^*g_2(\sigma)=\int_{\infty e^{-i\pi/3}}^{\sigma} h_2(z)dz. \qquad (28)$$

Bearing in mind that $\arg c=-\tfrac{2}{3}\pi$ (see Sec. III) it is easily proved from Eqs. (25), (27), and (28) that

$$cg_1(\sigma)=e^{-2\pi i/3}\left[e^{\pi i/3}\int_{\infty e^{-\pi i/3}}^{\sigma} h_1(ze^{2\pi i/3})dz+\int_{-\infty}^{\sigma} h_2(ze^{2\pi i/3})dz\right]-2\pi ic. \qquad (29)$$

VII

The asymptotic series of $g_1(\sigma)$: From Eq. (27) one obtains, by integrating the leading term of Eq. (17)

$$\alpha'^{-1}cg_1(\sigma)\sim\sigma^{-\frac{3}{4}}\exp(\tfrac{2}{3}i\sigma^{\frac{3}{2}}-11\pi i/12) \qquad (30)$$

valid for: $-\tfrac{2}{3}\pi<\arg\sigma<4\pi/3$. We shall not need the higher terms. An asymptotic formula valid in a sector that includes the ray $\arg\sigma=-\tfrac{2}{3}\pi$ is obtained from Eq. (29). In the first place we have from Eqs. (17) and (18)

$$e^{-\pi i/3}h_1(ze^{2\pi i/3})\sim\alpha' z^{-\frac{1}{4}}\exp(-\tfrac{2}{3}iz^{\frac{3}{2}}-11\pi i/12)\times[1+\sum_{m=1} i^m C_m z^{-3m/2}] \qquad (31)$$

valid for $-4\pi/3<\arg z<\tfrac{2}{3}\pi$.

$$e^{-2\pi i/3}h_2(ze^{2\pi i/3})\sim\alpha' z^{-\frac{1}{4}}\exp(\tfrac{2}{3}iz^{\frac{3}{2}}-5\pi i/12)\times[1+\sum_{m=1}(-i)^m C_m z^{-3m/2}] \qquad (32)$$

valid for: $-2\pi<\arg z<0$.

Inserting Eqs. (31) and (32) into Eq. (29) and integrating term by term, we have

$$\begin{aligned}\alpha'^{-1}cg_1(\sigma)\sim&-i\sigma^{-\frac{3}{4}}\exp(\tfrac{2}{3}i\sigma^{\frac{3}{2}}-5\pi i/12)\\&\times[1-iA_1\sigma^{-\frac{3}{2}}-A_2\sigma^{-6/2}+iA_3\sigma^{-9/2}\cdots]\\&+i\sigma^{-\frac{3}{4}}\exp(-\tfrac{2}{3}i\sigma^{\frac{3}{2}}-11\pi i/12)\\&\times[1+iA_1\sigma^{-\frac{3}{2}}-A_2\sigma^{-6/2}-iA_3\sigma^{-9/2}\cdots]-2\pi ic\alpha'^{-1}\end{aligned} \qquad (33)$$

valid for $-4\pi/3<\arg\sigma<0$.

$$A_1=C_1+\tfrac{3}{4}=41/48,$$
$$A_2=C_2+(9/4)C_1+27/16=9241/4608,$$
$$A_3=C_3+(15/4)C_2+(135/16)C_1+405/64=5075225/663552.$$

VIII

The results of Secs. VI and VII are now used to obtain complete information about *the zeros of* $g_1(\sigma)$.

Proof that $g_1(\sigma)$ *has no zeros in the sector* $-\tfrac{1}{3}\pi\leqslant\arg\sigma\leqslant\pi$: To this end we determine the change in $\arg[g_1(\sigma)]$ going around a closed contour composed of the rays $\arg\sigma=\pi$ and $\arg\sigma=-\tfrac{1}{3}\pi$ and a large circular arc connecting them. Let

$$h_1(z)=\mathrm{R}h_1(x,y)+i\mathrm{I}h_2(x,y);\quad z=x+iy;$$
$$h_1'(z)=\mathrm{R}h_1'(x,y)+i\mathrm{I}h_1'(x,y).$$

Along the real axis ($y=0$), $\mathrm{I}h_1(x,0)$ satisfies Stokes' equation

$$(d^2/dx^2)\mathrm{I}h_1(x,0)+x\mathrm{I}h_1(x,0)=0.$$

Now from Eq. (19) or from the table[7] one finds that

$$\mathrm{I}h_1(0,0)<0,\quad (d/dx)\mathrm{I}h_1(x,0)|_{x=0}=\mathrm{I}h_1'(0,0)>0.$$

Hence, from Stokes' equation:

$$\mathrm{I}h_1(x,0)<0,\quad\text{when}\quad x\leqslant 0.$$

Thus from Eq. (26), when $\sigma=x\leqslant 0$,

$$\mathrm{I}[cg_1(x)]=0.504+\int_0^x \mathrm{I}h_1(x,0)dx>0.$$

We see from this that $\arg[g_1(\sigma)]$ changes by less than π when we go from $\sigma=0$ to $\sigma=-\infty$. Now, from Eq. (26),

$$\arg[cg_1(0)]=(5\pi/6)+2\pi n,$$

and from Eq. (30),

$$\arg[cg_1(-\infty)]=-(5\pi/3)+2\pi n',$$

so that

$$\arg[cg_1(\sigma)]\Big|_{\sigma=0}^{\sigma=-\infty}=-\tfrac{1}{2}\pi.$$

Going along a large circular arc in the negative sense, we have from Eq. (30)

$$\arg[cg_1(\sigma)]\Big|_{\sigma=-\infty}^{\sigma=\infty\exp(-\pi i/3)}=\pi.$$

Going back from $\infty e^{-\pi i/3}$ to the origin in the σ-plane, we have from the symmetry relation (23)

$$\arg[cg_1(\sigma)]\Big|_{\sigma=\infty \exp(-\pi i/3)}^{\sigma=0} = -\tfrac{1}{2}\pi.$$

Thus the total change of $\arg g_1(\sigma)$ vanishes, and the proof is established. All the zeros of $g_1(\sigma)$ are therefore in the sector $-\pi < \arg\sigma < -\frac{1}{3}\pi$.

IX

To obtain *an asymptotic expression suitable for numerical work*, substitute $t = -\frac{2}{3}\sigma^{\frac{3}{2}}$ in (33) ($-\pi < \arg t < \pi$, when $-4\pi/3 < \arg\sigma < 0$). This leads to

$$\begin{aligned} -i(3/4\pi)^{\frac{1}{2}} g_1(\sigma) t^{\frac{1}{2}} \sim & -(3\pi)^{\frac{1}{2}} t^{\frac{1}{2}} \\ & + \cos t[1 + A_1(3t/2)^{-1} - A_2(3t/2)^{-2} - A_3(3t/2)^{-3} \cdots] \\ & + \sin t[-1 + A_1(3t/2)^{-1} + A_2(3t/2)^{-2} \\ & - A_3(3t/2)^{-3} \cdots]. \end{aligned} \quad (34)$$

This is a real expression in t and thus has pairs of complex conjugate zeros in terms of t. In terms of σ there are pairs of zeros symmetrically placed with respect to the ray $\arg\sigma = -\frac{2}{3}\pi$. This also follows from Eq. (23).

Taking the leading terms of Eq. (34) we have the equation

$$(3\pi)^{\frac{1}{2}} t^{\frac{1}{2}} = \cos t - \sin t, \quad (35)$$

or

$$(\tfrac{3}{2}\pi)^{\frac{1}{2}} t^{\frac{1}{2}} = -\sin(t - \tfrac{1}{4}\pi); \; -\pi < \arg t < \pi. \quad (36)$$

Squaring Eq. (35) and setting $t = x + iy$, one obtains the pair of real equations

$$3\pi y + \cos(2x)\sinh(2y) = 0,$$
$$3\pi x - 1 + \sin(2x)\cosh(2y) = 0.$$

From this form of the equation it easily follows that for all large solutions, $y = O(\log x)$. Hence, for the large solutions $t^{\frac{1}{2}} \sim x^{\frac{1}{2}}$, and Eq. (36) becomes

$$(\tfrac{3}{2}\pi)^{\frac{1}{2}} x^{\frac{1}{2}} = -\sin(x - \tfrac{1}{4}\pi)\cosh y, \quad x^{\frac{1}{2}} > 0,$$
$$0 = \cos(x - \tfrac{1}{4}\pi)\sinh y.$$

The solutions are:

$$\begin{aligned} & x_k = (7\pi/4) + 2\pi k, \\ & y_k = \pm \operatorname{arc\,cosh}[(\tfrac{3}{2}\pi)^{\frac{1}{2}} x_k^{\frac{1}{2}}], \quad (k = 0, 1, \cdots), \\ & t_k \sim x_k + iy_k; \quad \sigma = e^{-2\pi i/3}(3t/2)^{\frac{2}{3}}; \quad -\pi < \arg t < \pi. \end{aligned} \quad (37)$$

By investigating $(d/dy)(y/x)$ it is easily seen that y/x is a monotonically decreasing function for $x \geqslant x_0 \approx 5$, and hence so is $|\arg t|$ and $|\arg\sigma + \frac{2}{3}\pi|$. So that if t_0 yields a damped perturbation ($R(\sigma) < 0$), so do all t_k.

X

We now want to gain an idea about the *accuracy of Eqs. (37)*. For $k = 0$ they yield $t_0 \approx 5.50 \pm 2.31i$. Calculating a first correction to this value from Eq. (34) using the terms up to t^{-3} gives $t_0 = 5.40 \pm 2.36i$. In terms of σ this gives the pair of zeros

$$\alpha_{0+} = \sigma_{0+} = -1.05 - 4.14i,$$
$$\alpha_{0-} = \sigma_{0-} = -3.06 - 2.98i,$$

with errors in the last figures retained.

This is the smallest pair of zeros obtainable from the asymptotic series (33). The larger ones should be given even more accurately by Eqs. (37). Whether these actually are *all* the zeros of $g_1(\sigma)$ (there might still be undetected small zeros) will be cleared up later. First we mention *an independent check on the preceding calculation*. This is furnished by the power series expansion of $cg_1(\sigma)$ around a point chosen suitably close to one of the zeros. The coefficients can be computed with the aid of the table in reference 7. If $\sigma = x + iy$, formulas (21) and (26) yield

$$\begin{aligned} cg_1(\sigma) = & -(\tfrac{2}{3})^{\frac{1}{3}} + \int_0^x Rh_2(\xi, -y)d\xi - \int_0^{-y} Ih_2(0, \eta)d\eta \\ & + i(\tfrac{2}{3})^{\frac{1}{3}} 3^{-\frac{1}{2}} - i\int_0^x Ih_2(\xi, -y)d\xi - i\int_0^{-y} Rh_2(0, \eta)d\eta, \end{aligned}$$

where $h_2(\sigma) = Rh_2(x, y) + iIh_2(x, y)$. The integrals can be evaluated for instance by the Euler-Maclaurin formula (reference 7 also tabulates the derivatives of h_1 and h_2). The derivatives

$$cg_1^{(n)}(\sigma) = h_1^{(n-1)}(\sigma) = [h_2^{(n-1)}(\sigma^*)]^*$$

are found in the table up to $n = 2$. The higher derivatives are given by recursion formulas that follow from Eq. (16).

In this way one obtains, for $\sigma = -1.0 - 4.1i$,

$$\begin{aligned} cg_1 &= 0.368\,015 + 0.225\,850i, \\ cg_1' &= 6.382\,182 + 1.164\,182i, \\ \tfrac{1}{2}cg_1'' &= 4.614\,268 + 4.854\,430i, \\ \tfrac{1}{6}cg_1''' &= 0.268\,173 + 4.555\,188i, \\ (1/24)cg_1^{iv} &= -1.539\,999 + 1.932\,570i. \end{aligned}$$

One obtains a corrected value for σ_{0+} which is

$$-1.0626 - 4.1288i.$$

XI

To complete the treatment of the limiting case $\epsilon = 0$, it remains to show that *formulas (37) actually furnish all the zeros of* $g_1(\sigma)$. We know that in terms of t, all zeros are in the sector $-\frac{1}{2}\pi < \arg t < \frac{1}{2}\pi$ (see Secs. VIII and IX). Now let $(3/2\pi)^{\frac{1}{2}} g_1(\sigma) = p(t)$. Then Eq. (34) gives

$$p(t) \sim i(6\pi)^{\frac{1}{2}} + t^{-\frac{1}{2}}[e^{-y}e^{i(x-\pi/4)} - e^{y}e^{-i(x-\pi/4)}],$$

where $t = x + iy$; and this is valid for: $-\pi < \arg t < \pi$.

Now let $a = x_k + \pi = 2\pi(k+1) + \frac{3}{4}\pi$ [see formulas (37)], k be a large positive integer, and b any large positive number. We now count the number of zeros of $p(t)$ inside a rectangle with vertices $a-ib$; $a+ib$; ib; $-ib$. We have to find the change in $\arg p(t)$ as we go around the perimeter. The symmetry of $g_1(\sigma)$ [see Eq. (23)] permits us to confine ourselves to that part of the perimeter that lies in the upper half t plane.

On the straight line going from a to $a+ib$,

$$p(t)\sim i(6\pi)^{\frac{1}{2}}+it^{-\frac{1}{2}}[e^{-y}+e^{y}],$$

and

$$\arg[p(t)]\Big|_a^{a+ib}=-\tfrac{1}{2}\arg(a+ib).$$

On the straight line joining $a+ib$ to ib:

$$p(t)\sim -t^{-\frac{1}{2}}e^{b}e^{-i(x-\pi/4)},$$

and

$$\arg[p(t)]\Big|_{a+ib}^{ib}=-\tfrac{1}{4}\pi+\tfrac{1}{2}\arg(a+ib)+2\pi(k+1)+\tfrac{3}{4}\pi$$
$$=\tfrac{1}{2}\arg(a+ib)+\tfrac{1}{2}\pi+2\pi(k+1).$$

On the straight line joining ib to 0,

$$\arg[p(t)]\Big|_{ib}^{0}=-\tfrac{1}{2}\pi.$$

This is so because $\arg t=\frac{1}{2}\pi$ corresponds to $\arg\sigma=-\frac{1}{3}\pi$ and this ray was investigated in Sec. VIII.

We thus find that the total increment in $\arg p(t)$ in going around the complete perimeter is $2\pi(2k+2)$. Hence there are $2k+2$ zeros of $p(t)$ inside it. This number of zeros is also given by Eq. (37), and thus none have been missed.

On the basis of this we can state: *In the limiting case* $\epsilon=0$, *all the characteristic roots are furnished by Eqs.* (*37*) *and lead to stable perturbations, i.e.,* $R(\sigma)=R(\alpha)<0$.

XII

The general case $\epsilon>0$: Equation (12) defines functions $\alpha(\beta)$. Denoting partial derivatives by subscripts, we have

$$d\alpha/d\beta=-f_\beta/f_\alpha.$$

Further we notice that $f_\beta=f_{\alpha\alpha}$ and

$$f_\alpha=\int_{-\infty;\ \infty e^{\pi i/3}}\exp(\tfrac{1}{3}t^3+\beta t^2+\alpha t)dt$$
$$=\int_{-\infty;\ \infty e^{\pi i/3}}\exp(\tfrac{1}{3}r^3+\sigma r)dr\,\exp(-\sigma\beta-\tfrac{1}{3}\beta^3)$$
$$=c^{-1}h_1(\sigma)\exp(-\sigma\beta-\tfrac{1}{3}\beta^3),$$

where $t=r-\beta$. Moreover,

$$f_\beta=f_{\alpha\alpha}=c^{-1}[h_1'(\sigma)-\beta h_1(\sigma)]\exp(-\sigma\beta-\tfrac{1}{3}\beta^3).$$

Hence,

$$d\alpha/d\beta=-[h_1'(\sigma)/h_1(\sigma)]+\beta,$$

or

$$d\alpha/d\epsilon=i[h_1'(\alpha+\epsilon^2)/h_1(\alpha+\epsilon^2)]-\epsilon. \qquad (38)$$

TABLE I. Comparison of the asymptotic formula (40) for $w(\sigma)$ with the direct calculation of $w(\sigma)$ from the table of reference 7.

σ	w (computed from tables in reference 7)	$-i\sigma^{\frac{1}{2}}$
$-2i$	$1.064+0.688i$	$1+i$
$-4i$	$1.417+1.348i$	$1.414+1.414i$
$4i$	$-1.417+1.473i$	$-1.414+1.414i$
$-6i$	$1.733+1.689i$	$1.732+1.732i$
$3-3i$	$0.742+1.860i$	$0.768+1.903i$

(Similar considerations lead to expressions for the rates of change of the characteristic roots in the case of plane Couette flow with walls a finite distance apart.)

It is perhaps of interest to note that one can obtain a second-order algebraic differential equation for α or more conveniently for σ. From Eq. (38) we have

$$(d\sigma/d\epsilon-\epsilon)h_1(\sigma)=ih_1'(\sigma).$$

Differentiating with respect to ϵ and using $h_1''(\sigma)=-\sigma h_1(\sigma)$ [see Eq. (16)], one obtains

$$\frac{d^2\sigma}{d\epsilon^2}+i\frac{d\sigma}{d\epsilon}\left[\sigma-\left(\frac{d\sigma}{d\epsilon}-\epsilon\right)^2\right]-1=0. \qquad (39)$$

As ϵ is varied, each of the characteristic roots α will trace out a curve in the complex α-plane according to Eq. (38). The starting values of α at $\epsilon=0$ lie in the left half α-plane ($R(\alpha)<0$). Moreover, the possible singularities of Eq. (38) are given by the roots of $h_1(\alpha+\epsilon^2)=0$. The zeros of $h_1(\sigma)$ are however known[7,8] to lie all on the ray $\arg\sigma=-\frac{2}{3}\pi$ for which $R(\alpha)=R(\sigma-\epsilon^2)<0$. To establish now that $R(\alpha)<0$ for all $\epsilon\geqslant 0$, it is sufficient to show that

$$R\left(\frac{d\alpha}{d\epsilon}\right)=R[i\{h_1'(\sigma)/h_1(\sigma)\}-\epsilon]<0,$$

when $R(\sigma)\geqslant 0$. For in that case $\alpha(\epsilon)$ cannot cross the imaginary axis in the α-plane to get into the region $R(\alpha)\geqslant 0$. ϵ being positive, it is sufficient to verify that $I[h_1'(\sigma)/h_1(\sigma)]>0$, when $R(\sigma)\geqslant 0$.

Asymptotically we have from Eq. (17)

$$w(\sigma)\equiv h_1'(\sigma)/h_1(\sigma)\sim -i\sigma^{\frac{1}{2}}, \qquad (40)$$

valid for $-\frac{2}{3}\pi<\arg\sigma<4\pi/3$, so that for large σ: $I(w)>0$, when $R(\sigma)\geqslant 0$. For small σ, one can verify from the tables[7] that

$$Rh_1\cdot Ih_1'-Rh_1'\cdot Ih_1>0\ [\text{for } I(\sigma)\geqslant 0],$$

and

$$Rh_2'\cdot Ih_2-Rh_2\cdot Ih_2'>0\ [\text{for } I(\sigma)\leqslant 0].$$

Actually the asymptotic formula (40) is sufficiently accurate throughout most of the relevant portion of the tables. This can be seen from the values of Table I.

Thus we have established that for all $\epsilon\geqslant 0$ *one has* $R(\alpha)<0$, i.e., *the fluid flow examined is stable towards infinitesimal perturbations originating inside the fluid.*