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The Construction of Potentials in Quantum Field Theory*

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An inductive method, based on the formal solution of the Schrödinger equation in the form given by Lippmann and Schwinger, is used to construct scattering potentials in quantum field theories of various types. The method is applied to a linear theory without nucleon pair production, to a general nonlinear theory without pair production, and finally to a general nonlinear theory with pair production. The nth order potential is obtained by the solution of a Schrödinger type equation involving the potential of lower order; the series of potentials so obtained is, however, not a power series in the coupling constant. Rough estimates indicate that some of the problems of convergence associated with the usual perturbation expansion are obviated by this method. Simple illustrative examples are given for the linear and nonlinear theories.

I. INTRODUCTION

THE most straightforward attempts to calculate scattering cross sections in field theory have involved power-series expansions of the scattering matrix in the coupling constant. In the case of meson theories, at least, there seems to be little hope that the power series method will lead to useful results. Other methods such as the radiation damping theory of Heitler[1] also do not appear promising.

On the other hand, the usefulness and validity of the approaches of Bethe and Salpeter[2] and of Tamm[3] and Dancoff[4] remain to be determined. These methods have been applied primarily to problems of the scattering of two particles without the creation of new particles. The interaction between the two particles is obtained as a power series in the coupling constant which is used to solve a Schrödinger equation. This in turn gives the scattering matrix.

In the present work, a new approach is used to obtain the interaction potential between two particles. The method makes use of the integral-equation expression for the Schrödinger equation given by Lippmann and Schwinger[5] and of the algebraic techniques for handling this developed by Chew and Goldberger[6] and by Watson.[7] The method is related to that of Tamm and Dancoff in that it leads to an interaction potential which is diagonal in particle occupation numbers but nondiagonal in the momentum states of the interacting particles. Just as in the Tamm-Dancoff and Bethe-Salpeter methods, the potential is obtained as a sum of an infinite number of terms. However, in contrast to these theories, the series is not a power-series in the coupling constant.[8] In particular, the nth term in the series for the potential is obtained in terms of the solution of the "Schrödinger equation" which results from using for the "interaction potential" the first $(n-1)$ terms in the series. Because the successive potentials are diagonal in particle occupation numbers, these successive "Schrödinger type" equations do not involve the complications of field theory and can in principle be considered soluble. (We note that there are several powerful variational methods available for the handling of such problems.)

The present study was initiated as a result of the work of Chew[9] on the scattering of mesons by nucleons. Chew has applied the Tamm-Dancoff method to obtain the g^2 potential. His calculation of the scattering with pseudovector coupling in the pseudoscalar meson theory was in not unsatisfactory agreement with present experimental results.[10]

To simplify the presentation of the material we shall first derive the potential for a coupling linear in the meson field variables and without nucleon pair production. The results will then be extended to general nonlinear interactions (still without nucleon-pair production). Finally the production of nucleon pairs will be taken into account. We further remark that although we have meson theory most specifically in mind, the results are generally applicable to other field theories such as electrodynamics.

We shall not in this paper consider the renormalization problem; preliminary considerations indicate, however, that the usual renormalization procedures may be applicable. The detailed investigation of this problem will be the content of a later paper.

II. PRELIMINARY CONSIDERATIONS

We begin the discussion by presenting some formal relations which will be repeatedly used. We suppose the Schrödinger equation to have the form

$$(H_0+H')\psi=E\psi, \tag{1}$$

where E is the energy of the system, H_0 is the energy of two uncoupled fields under consideration, and H' is

* Supported in part by a grant from the National Science Foundation.
[1] W. Heitler, Proc. Cambridge Phil. Soc. **37**, 291 (1941).
[2] E. Salpeter and H. Bethe, Phys. Rev. **84**, 1232 (1951).
[3] I. Tamm, J. Phys. (U.S.S.R.) **9**, 449 (1945).
[4] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
[5] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[6] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).
[7] K. M. Watson, Phys. Rev. **89**, 575 (1953).
[8] M. Lévy [Phys. Rev. **88**, 72 (1952)] has also devised a (different) method of constructing successive interactions by means of integral equations.
[9] G. F. Chew, Phys. Rev. **89**, 591 (1953).
[10] Anderson, Fermi, Nagle, and Yodh, Phys. Rev. **86**, 793 (1952).

the coupling between the fields. For instance H_0 operating on a state containing mesons in momentum states $k_1, k_2, \cdots k_n$ with the wave function $\chi(k_1, \cdots, k_n)$, gives

$$H_0\chi=[\omega_1+\omega_2+\cdots+\omega_n]\chi, \tag{2}$$

where $\omega_1=(k_1^2+\mu^2)^{\frac{1}{2}}$, etc. ($\mu$ is the meson rest mass and we use units in which $\hbar=c=1$).

We suppose ψ in Eq. (1) to describe a scattering event which originates in an eigenstate of H_0, say χ_a. In terms of the Møller[11] wave matrix Ω, we write

$$\psi=\Omega\chi_a, \tag{3}$$

where Ω satisfies the Lippmann-Schwinger[5] equation,

$$\Omega=1+\frac{1}{E+i\eta-H_0}H'\Omega, \tag{4}$$

which is supposed to operate on the state χ_a. (In this expression η is a small positive parameter used to specify the contour of integration which is set equal to zero after the integration is done.) Chew and Goldberger[6] have introduced the solution

$$\Omega=1+\frac{1}{E+i\eta-H_0-H'}H' \tag{5}$$

to Eq. (4), which may be readily verified by using the operator relation

$$A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}=B^{-1}(B-A)A^{-1}. \tag{6}$$

We shall need a straightforward generalization of Eqs. (4) and (5). Let a and q be matrix operators and suppose that a has an inverse. Then the unique solution to

$$\omega=1+(1/a)q\omega \tag{7}$$

is

$$\omega=1+\frac{1}{a-q}q, \tag{A}$$

where "1" is the identity operator and it is supposed that relation (A) has a well-defined meaning. We shall frequently use the converse to relation (A); namely, that ω as defined in relation (A) satisfies the integral Eq. (7).

We shall encounter and require a prescription for evaluating quantities such as

$$P\equiv Q_1\frac{1}{a-q}Q_2, \tag{8}$$

where Q_1 and Q_2 are two other matrix operators. Using Eq. (6), P can be written as

$$P=Q_1\left[\frac{1}{a}+\frac{1}{a-q}q\frac{1}{a}\right]Q_2$$
$$=Q_1\omega(1/a)Q_2, \tag{B}$$

where ω is the solution to Eq. (7). In general q and a will be functions of field variables and consequently non-diagonal in particle occupation numbers. Our primary task will be to express all such quantities P in terms of an operator q which is *diagonal* in occupation numbers and an operator a which has the form

$$a=E+i\eta-H_0.$$

Equation (7) is then reduced to a scattering-type problem and for present purposes will be considered soluble. If the matrix elements of Q_1 and Q_2 are known, then P will be considered as evaluated.

Our next general relation is

$$\frac{1}{a-q}q=\frac{1}{a-L}q+\frac{1}{a-L}L, \tag{C}$$

where

$$L=q(1/a)q. \tag{9}$$

Relation (C) can be proved by expanding $(a-L)^{-1}$ in a power series in L and $(a-q)^{-1}$ in a series in q, then comparing both sides of Eq. (C) term by term with L expressed in terms of q by Eq. (9). The proof can also be carried out more elegantly by adding the identity operator to both sides of Eq. (C) and showing that both sides satisfy the same integral Eq. (7).[12]

Our final formal relation is obtained by setting $q=q_1+q_2$ (where q_1 and q_2 are both matrix operators) in the left side of Eq. (C). Then

$$\frac{1}{a-q_1-q_2}(q_1+q_2)=\frac{1}{a-q_2-M}(q_2+M)$$
$$+\frac{1}{a-q_2-M}q_1\left[1+\frac{1}{a-q_2}q_2\right], \tag{D}$$

where

$$M=q_1\frac{1}{a-q_2}q_1. \tag{10}$$

Relation (D) is obtained from relation (C) as follows:

$$\frac{1}{a-q_1-q_2}(q_1+q_2)=\frac{1}{(a-q_2)-q_1}q_1+\frac{1}{(a-q_2)-q_1}q_2$$
$$=\frac{1}{a-q_2}q_2+\frac{1}{(a-q_2)-q_1}q_1\left[1+\frac{1}{a-q_2}q_2\right], \tag{11}$$

using Eq. (6). By relation (C), we have

$$\frac{1}{(a-q_2)-q_1}q_1=\frac{1}{a-q_2-M}(q_1+M), \tag{12}$$

on identifying $(a-q_2)$ and (q_1) with a and q of relation

[11] C. Møller, Kgl. Danske. Videnskab. Selskab, Mat.-fys. Medd. **23**, No. 1 (1945).

[12] Relations (C) and (D) are special forms of a solution to the Schrödinger equation given in Appendix (B) of reference (7), where methods of deriving such equations are considered in more detail.

(C). Simplifying with the use of Eq. (6), relation (D) follows.

In what follows, we shall show how to construct a potential in quantum field theories by sequential application of relation (D).

III. LINEAR COUPLING WITHOUT NUCLEON PAIR PRODUCTION

The interaction H' in Eq. (1) is assumed to be a homogeneous linear polynomial in the meson field variables ϕ_σ ($\sigma=1, 2, \cdots N$) and it is supposed that H' is diagonal in nucleon occupation numbers so no nucleon pairs can be created. The index "σ" on ϕ_σ may refer to isotopic spin components, vector components, field variables evaluated at the position of different nucleons, etc.

Defining

$$a\equiv E+i\eta-H_0,$$

Eq. (4) has the solution (5):

$$\Omega=1+\frac{1}{a-H'}H' =1+\frac{1}{a-\Delta_0}H'+\frac{1}{a-\Delta_0}\Delta_0. \qquad (13)$$

The last step follows from relation (C) and the definition

$$\Delta_0=H'(1/a)H'. \qquad (14)$$

We shall now develop some general theorems relating to expressions of the type of Eq. (13) which will allow us to express the part of Ω which refers to scattering without particle creation or annihilation in terms of operators which themselves are diagonal in occupation numbers.

We note the following self-evident lemma.

Lemma I

A homogeneous polynomial $G(\phi)$ of degree r in the field variable ϕ (by ϕ we mean the set "ϕ_σ") has only matrix elements for the production or absorption of a net number of mesons which is even or odd depending upon whether r is even or odd.

It follows from Lemma I that the term

$$\frac{1}{a-\Delta_0}H'$$

in Eq. (13) does not contribute to a scattering event in which no particles are produced or absorbed, since this quantity is odd in H' and thus in ϕ. Thus for calculating such a scattering, we need only (the diagonal part of)

$$\Omega_S(0)=1+\frac{1}{a-\Delta_0}\Delta_0. \qquad (15)$$

Comparing with relation (A) and Eq. (7), we see that $\Omega_S(0)$ satisfies a Schrödinger equation with Δ_0 as a potential.

We now split Δ_0 into two parts:

$$\Delta_0=V_0+U_0, \quad V_0=\mathrm{DP}\Delta_0, \quad U_0=\mathrm{NDP}\Delta_0, \qquad (16)$$

where the symbol $\mathrm{DP}\Delta_0$ means the diagonal part of Δ_0 with respect to meson occupation numbers and "$\mathrm{NDP}\Delta_0$" means the nondiagonal part of Δ_0 with respect to meson occupation numbers. (This notation will be used frequently in what follows.)

If we were to approximate Eq. (15) by setting $U_0=0$, then V_0 would be the potential in the Schrödinger equation which then has the form used by Chew[9] to discuss meson-nucleon scattering.[13]

To proceed, we define by induction from Eq. (16):

$$\begin{aligned} \Delta_n&=U_{n-1}\frac{1}{a-\mathcal{V}_{n-1}}U_{n-1},\\ U_n&=\mathrm{NDP}\Delta_n,\\ V_n&=\mathrm{DP}\Delta_n,\\ \mathcal{V}_n&=\mathcal{V}_{n-1}+V_n=\sum_{e=0}^{n}V_e. \end{aligned} \qquad (17)$$

We wish to show that in the limit as n approaches infinity that $\mathcal{V}_n$ is the desired interaction potential to be used in the Schrödinger equation which describes the scattering event (if the series defining $\mathcal{V}_n$ converges).

Since the $\mathcal{V}_n$'s are diagonal in meson occupation numbers, we may suppose the matrix elements of $\mathcal{V}_n$ to have been evaluated as numbers and to be *no longer functions* of the field variables. Then U_n is a homogeneous polynominal of degree 2^{n+1} in the field variables. This follows by induction since U_n contains twice as many ϕ's as does U_{n-1} and since U_0 is of second degree in ϕ.

Lemma II

For some value of n let us suppose that U_{n-1} can only produce (or absorb) a net number of meson pairs equal to 2^{n-1}; i.e., we suppose the difference between the number n_p of creation operators and the number n_a of annihilation operators is

$$\tfrac{1}{2}|n_p-n_a|=2^{n-1}.$$

Then U_n can only produce or absorb a net number of 2^n meson pairs (that is, it has no other matrix elements).

The proof follows immediately from the form of Δ_n in Eq. (17). Since by definition U_n cannot be diagonal in occupation numbers, it follows that each of its factors of U_{n-1} must either produce or absorb 2^{n-1} meson pairs. Then U_n produces or absorbs twice this number of pairs (or 2^n pairs).

[13] The present method has been employed in reference 7 to derive the potential in the "optical model" for meson-nucleus scattering when meson absorption can occur.

Theorem I: U_n (for all n) can actually only produce or absorb a net number of meson pairs equal to 2^n.

The proof goes by induction from Lemma II since U_0 can obviously only produce or absorb one meson pair (i.e., $2^0=1$).

A useful theorem for the evaluation of the U_n's is the following:

Theorem II: In U_n all the field variables must either produce or absorb mesons. There are no virtual emissions and reabsorptions in U_n. (Remembering that we have considered the $\mathcal{V}_n$'s to be evaluated, so that they are no longer functions of the field variables.)

The proof follows from counting the number of ϕ's occurring in U_n. As remarked above, U_n is a degree 2^{n+1} in ϕ. Since U_n must create or absorb 2^n *pairs* of mesons (i.e., 2^{n+1} mesons), there are no ϕ's left over to perform virtual creations and reabsorptions.

The importance of this theorem is that for the evaluation of V_n all the 2^n emission operators must stand on the right of the 2^n absorption operators (except for the smallest value of n, where the order can be inverted).

Lemma III

All functions of a given U_n which do not otherwise contain the field variables and which are odd in U_n contain no matrix elements which are diagonal in the meson occupation numbers.

The proof is trivial: Each U_n absorbs or emits only 2^n pairs of mesons and can be considered as a single emission or absorption operator for this number of meson pairs. By the argument of Lemma I an odd number of such emission or absorption operators must lead to a net emission or absorption of mesons.

We are finally led to the following theorem.

Theorem III: The potential to be used in the Schrödinger equation which describes the original scattering event is

$$\mathcal{V}=\lim_{n\to\infty}\mathcal{V}_n=\sum_{e=0}^{n}V_e, \tag{18}$$

assuming proper convergence.

The proof proceeds by induction. Assume that the scattering is described by

$$\Omega_S=\mathrm{DP}[\Omega_S(n)],$$

where

$$\Omega_S(n)=1+\frac{1}{a-\mathcal{V}_n-U_n}[\mathcal{V}_n+U_n]. \tag{19}$$

We use relation (D), identifying q_2 with $\mathcal{V}_n$ and q_1 with U_n. Then

$$\Omega_S(n)=1+\frac{1}{a-\mathcal{V}_n-\Delta_{n+1}}[\mathcal{V}_n+\Delta_{n+1}]+\frac{1}{a-\mathcal{V}_n-\Delta_{n+1}}U_n\left[1+\frac{1}{a-\mathcal{V}_n}\mathcal{V}_n\right]. \tag{20}$$

The last term may be discarded by Lemma III, since it is of odd order in U_n and can give no contribution to the scattering cross section (we are assuming scattering without creation or absorption of mesons). Writing [by Eq. (17)] $\Delta_{n+1}=V_{n+1}+U_{n+1}$, the scattering is also given by

$$\mathrm{DP}[\Omega_S(n+1)]=\Omega_S,$$

where

$$\Omega_S(n+1)=1+\frac{1}{a-\mathcal{V}_{n+1}-U_{n+1}}[\mathcal{V}_{n+1}+U_{n+1}], \tag{21}$$

which has the form of Eq. (19) with n increased by unity. Since $\Delta_0=V_0+U_0$, Eq. (15) has the form of Eq. (19) with $n=0$. Proceeding step by step to sufficiently large values of n, if the matrix elements of U_n decrease to zero with sufficient rapidity that the series defining $\mathcal{V}_n$ converges, we obtain in the limit

$$\mathcal{V}=\sum_{e=0}^{n\to\infty}V_e \tag{22}$$

and

$$\Omega_S=1+\frac{1}{a-\mathcal{V}}\mathcal{V}. \tag{23}$$

Now Ω_S contains all matrix elements of Ω which are diagonal in occupation numbers and so contains all the information required to completely describe the desired scattering cross section.

By relation (A) Ω_S satisfies the Lippmann-Schwinger equation,

$$\Omega_S=1+\frac{1}{E+i\eta-H_0}\mathcal{V}\Omega_S. \tag{24}$$

For the evaluation of the V_n's as well as the U_n's we use relation (B),

$$\Delta_{n+1}=U_n\frac{1}{a-\mathcal{V}_n}U_n$$
$$=U_n\omega_n(1/a)U_n, \tag{25}$$

where ω_n satisfies

$$\omega_n=1+(1/a)\mathcal{V}_n\omega_n. \tag{26}$$

This describes the scattering of the virtual and real mesons in an "intermediate state." Because in the definition of V_n Eq. (17) the first U_n contains only creation operators (except for the smallest n value, as remarked previously, where the order can be inverted) and the second U_n only absorption operators, the structure of the V_n's (and U_n's) is relatively simple compared (for instance) to the complexity of the Feynman diagrams for a high order process. Thus the matrix elements of V_n can be written directly in terms of the ω_n's and integrals over virtual meson states by only enumerating the order of reabsorption for a given order of emission. Also, because of the large number of meson states upon which $(a-\mathcal{V}_n)$ operates $(\simeq 2^{n+1})$, except for small n values, a will be quite large in magnitude and a reasonable approximation might be to neglect $\mathcal{V}_n$ entirely for all but the lowest n-values in Eq. (25). This would make the evaluation of the V_n's relatively simple. Also, in this case $\mathcal{V}$ becomes a power series in g^2, but of the form

$$\mathcal{V}=O(g^2)+O(g^4)+O(g^8)+O(g^{16})+O(g^{32})+\cdots. \quad (26')$$

In this approximation, the terms contributing to $\mathcal{V}$ are particularly simple. Making use of the defining equation for V_n, we find

$$V_n=a[(1/a)H']^{2^n}[(1/a)H']^{2^n},$$

where by theorem II, the first 2^n operators H' must create mesons, the last 2^n operators annihilate mesons. It is interesting to note that the operators $1/a$ which appear in this expression will contribute factors roughly of the form

$$1/(2^n!)^2$$

multiplied by dimensionless factors depending on the convergence of the integrals over the momenta of the virtual mesons. In evaluating the contribution to V_n, we will in general be able to create the 2^n particles in $2^n!$ ways corresponding to the ways of permuting the creation operators. A corresponding factor will come from the annihilation operators, neglecting cancellations due to reordering of noncommuting spin and isotopic spin matrices. Accordingly, aside from other factors, the factorial dependence of the V_n arising from the multiplicity of contributions in high order is at least cancelled by the behavior of the energy denominators.

Aside from this approximation the form (25) is surprisingly simple. This is to be compared, for instance, with the greater complexity of the perturbation or Tamm-Dancoff methods, which do not lead to a decomposition into the U_n's and the V_n's. The simplification results from considering explicitly the "scattering" of virtual mesons before reabsorption. It is also of interest to note from Eq. (17), that the difference between the power of g occurring in the numerator and the combined powers of g in the denominators is just g^2 in every Δ_n (and thus $\mathcal{V}_n$). This suggests that, as the coupling becomes larger, the successive terms may not increase with g as rapidly as might otherwise be indicated (if the quantities $(a-\mathcal{V}_n)^{-1}$ had no singularities, then every term in $\mathcal{V}$ would be of order g^2 in the strong coupling limit).

The effect just mentioned is apparent in the calculation of the first two terms of Eq. (22) for the nuclear forces, as is done in the Appendix. Here, the multiple scattering of the virtual exchanged meson is shown to arise from V_1. The second term is of order g^4 for weak coupling but becomes of order g^2 for large values of g. The nuclear force problem discussed in the Appendix is given to illustrate techniques of handling the formal algebraic expressions which have been used above.

Problems of meson production and absorption may also be easily handled by the present techniques. In this case, it is of course necessary to keep some of the off-diagonal terms in Ω which were discarded in analyzing processes which are diagonal in occupation numbers (actually, only a small finite number of these terms need be kept in any case of practical importance). By repeating inductive arguments such as those given previously, one may easily show that [Eq. (13)]

$$\Omega=\Omega_S[1+(1/a)F],$$

where Ω_S is given by Eq. (24). F is a non-diagonal operator, which may be broken into parts for producing (or absorbing) one meson, two mesons, etc. For instance, for the production of a single meson when no meson is initially present (i.e., by nucleon-nucleon collisions) we have directly from Eq. (13)

$$\frac{1}{a-\Delta_0}H'=\Omega_S(0)\frac{1}{a}H'. \quad (27)$$

It therefore follows that

$$F^{(1)}=H', \quad (28)$$

where $F^{(1)}$ is that part of F which has matrix elements for the production of just one meson. Equation (28) results since H' must obviously produce a meson. Since we wish just one meson produced, the remaining factor, $\Omega_S(0)$, must be diagonal. But $\mathrm{DP}[\Omega_S(0)]=\Omega_S$ so Eq. (28) follows. For single meson production by the collision of a meson with a nucleon, we must generalize Eq. (28):

$$F^{(1)}=\left[1+U_0\frac{1}{a-V_0}\right]H'. \quad (28')$$

It is of course implied that we keep only those matrix elements on the right-hand side of this equation which refer to the production of a single meson. Equation (28′) obviously reduces to Eq. (28) for those initial states which do not contain a meson. The general expression for $F^{(1)}$ may easily be worked out. The absorption of mesons may be obtained from the above using the detailed reversibility theorem,[14] or directly by re-expressing the above quantities with the nondiagonal operators to the left rather than to the right of the scattering operators. The generalizations of the above arguments to the nonlinear and nucleon pair theories are straightforward and will not be given in detail.

IV. NONLINEAR INTERACTIONS

Nonlinear interactions such as those obtained by Berger *et al.*[15] or by Drell and Henley[16] by transforming

[14] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, New York, 1952), p. 528.
[15] Berger, Foldy, and Osborn, Phys. Rev. **87**, 1061 (1952).
[16] S. Drell and E. Henley, Phys. Rev. **88**, 1053 (1952).

the linear pseudoscalar interaction may also be handled by the methods here employed. If the matrix elements of the interaction term can be evaluated by some such method as Glauber's,[17] then it is not necessary to expand the interaction in powers of g. We again assume that the interaction is diagonal in nucleon occupation numbers, so no nucleon pairs are formed.

In the Schrödinger Eq. (1) H' will be assumed to be nonlinear in ϕ. We first break H' into two parts.

$$H'=H_1+H_2, \tag{29}$$

where H_1 is odd in ϕ and H_2 is even in ϕ. Thus the wave matrix Ω satisfies [see Eq. (4)]

$$\begin{aligned}\Omega&=1+\frac{1}{E+i\eta-H_0}(H_1+H_2)\Omega\\&=1+\frac{1}{a-H_1-H_2}(H_1+H_2)\end{aligned} \tag{30}$$

(where again $a=E+i\eta-H_0$). We use relation (D), identifying H_1 with q_1 and H_2 with q_2 to express Ω as

$$\Omega=1+\frac{1}{a-H_2-D_0}(H_2+D_0)+\frac{1}{a-H_2-D_2}H_1\left[1+\frac{1}{a-H_2}H_2\right], \tag{31}$$

where

$$D_0=H_1\frac{1}{a-H_2}H_1. \tag{32}$$

Now, by Lemma I, since H_1 is odd in ϕ, H_1 produces (or absorbs) a net odd number of mesons. Furthermore H_2, being even in ϕ, produces (absorbs) an even number of mesons only. Since the last term of Eq. (31) is odd in H_1, it contains an odd number of production events, each producing an odd number of mesons, plus any number of events producing an even number of mesons (because of H_2). The net number of produced mesons arising from this term is therefore odd and it consequently cannot be diagonal in meson occupation numbers. It will not contribute to a scattering event in which no particles are produced and may be dropped. We may thus calculate the scattering from

$$\Omega_S(0)=1+\frac{1}{a-H_2-D_0}(H_2+D_0). \tag{33}$$

The "potential," H_2+D_0, in Eq. (33) has only diagonal matrix elements or matrix elements for the production of mesons in multiples of two. Since the interaction, H_1+H_2, with which we started in Eq. (30), had diagonal matrix elements or matrix elements for the production of one meson, two mesons, three mesons, etc., we have increased the minimum multiplicity of virtual mesons present. It is clear that this process may be iterated, each step increasing the multiplicity of mesons in the *off diagonal* matrix elements of the interaction.

To proceed more generally, we first define

$$\begin{aligned}v_0&=\mathrm{DP}(H_2+D_0),\\U_0(e)&=\mathrm{NDP}(e2)(H_2+D_0),\\U_0(o)&=\mathrm{NDP}(o2)(H_2+D_0).\end{aligned} \tag{34}$$

The notation DP and NDP is as used before. The symbol "(em)" is interpreted to mean the matrix elements which produce (or absorb) *even multiples* of m-mesons. The symbol "(om)" likewise is to be interpreted as those matrix elements which produce (or absorb) *odd multiples* of m mesons. Thus in Eq. (34), where $m=2$, $U_0(e)$ produces (or absorbs) only even multiples of two mesons (except zero) and $U_0(o)$ produces (or absorbs) only odd multiples of two mesons.

As in the last section, we define by induction an infinite set of such quantities as those in Eq. (34). Let us suppose that we have already defined three quantities (for some value of n) $\mathcal{V}_n$, $U_n(o)$ and $U_n(e)$. $\mathcal{V}_n$ is assumed to be a scattering interaction (that is, diagonal in meson occupation numbers). The U's are supposed by hypothesis to be only nondiagonal in meson occupation numbers and $U_n(o)$ is a quantity of the type $(o2^{n+1})$ and $U_n(e)$ is of the type $(e2^{n+1})$. This means that $U_n(o)$ can produce (or absorb) mesons only in odd multiples of the number 2^{n+1} whereas $U_n(e)$ can produce (or absorb) mesons only in even multiples of 2^{n+1}. Now define

$$\begin{aligned}D_{n+1}&=U_n(o)\frac{1}{a-\mathcal{V}_n-U_n(e)}U_n(o),\\K_{n+1}&=U_n(e)+D_{n+1}.\end{aligned} \tag{35}$$

Lemma IV

The quantity K_{n+1} can produce (or absorb) mesons only in multiples of 2^{n+2}.

It follows from its definition that $U_n(e)$ can produce only in multiples of 2^{n+2} mesons. From the structure of D_{n+1} in the definition (35) it also follows that D_{n+1} can also produce (or absorb) mesons only in multiples of 2^{n+2}. Thus the Lemma IV is proven. We can therefore define

$$\begin{aligned}v_{n+1}&=\mathrm{DP}K_{n+1},\\U_{n+1}(e)&=\mathrm{NDP}(e2^{n+2})K_{n+1},\\U_{n+1}(o)&=\mathrm{NDP}(o2^{n+2})K_{n+1},\\\mathcal{V}_{n+1}&=\mathcal{V}_n+v_{n+1}.\end{aligned} \tag{36}$$

If we define

$$\mathcal{V}_0=v_0,$$

and start with Eq. (34), we can now construct by induction all the quantities in Eq. (36) to any order in

[17] R. J. Glauber, Phys. Rev. **84**, 395 (1951).

n. In particular,

$$\mathcal{V}_n=\sum_{e=0}^{n} v_e. \quad (37)$$

The following theorem finally gives the potential $\mathcal{V}$.

Theorem IV: Assuming suitable convergence, the interaction potential is

$$\mathcal{V}=\lim_{n\to\infty}\mathcal{V}_n=\sum_{e=0}^{\infty} v_e. \quad (38)$$

We proceed by induction. Let us assume for some n that $\Omega_S(n)$ has the form

$$\Omega_S(n)=1+\frac{1}{a-\mathcal{V}_n-U_n(e)-U_n(o)}\times[\mathcal{V}_n+U_n(e)+U_n(o)]. \quad (39)$$

Using relation (D), in which q_2 is interpreted as $\mathcal{V}_n+U_n(e)$ and q_1 as $U_n(o)$, we have

$$\Omega_S(n)=1+\frac{1}{a-\mathcal{V}_n-U_n(e)-D_{n+1}}[\mathcal{V}_n+U_n(e)+D_{n+1}]$$
$$+\frac{1}{a-\mathcal{V}_n-U_n(e)-D_{n+1}}\times U_n(o)\left[1+\frac{1}{a-\mathcal{V}_n-U_n(e)}(\mathcal{V}_n+U_n(e))\right]. \quad (40)$$

The last term may be dropped, since it corresponds to an odd number of productions in odd multiplicities of 2^{n+1} mesons and therefore has no matrix elements diagonal in meson occupation numbers. From the definitions (35) and (36) we write

$$U_n(e)+D_{n+1}=v_{n+1}+U_{n+1}(e)+U_{n+1}(o),$$

so the first two terms in Eq. (40) have the form

$$\Omega_S(n+1)=1+\frac{1}{a-\mathcal{V}_{n+1}-U_{n+1}(e)-U_{n+1}(o)}\times[\mathcal{V}_{n+1}+U_{n+1}(e)+U_{n+1}(o)]. \quad (41)$$

Since the first Eq. (33) has the form of Eq. (39) with $n=0$ [by the definitions (34)], we obtain Eq. (41) after n iterations of the process just described. If for sufficiently large n, the matrix elements of the U's become negligible and if the series for $\mathcal{V}$ converges, we obtain finally

$$\Omega_S=1+\frac{1}{a-\mathcal{V}}\mathcal{V}, \quad (42)$$

which satisfies the Schrödinger equation,

$$\Omega_S=1+(1/a)\mathcal{V}\Omega_S. \quad (43)$$

Since $\mathcal{V}$ is diagonal in meson occupation numbers, it is a potential in the ordinary sense and Eq. (43) can be considered soluble.

Our problem is still solved only in the formal sense, however, since the definitions of the D_{n+1}'s as given in Eq. (35) contain the nondiagonal operators $U_n(e)$ in the energy denominator. To complete the solution we note that by relation (B) D_{n+1} can be written as

$$D_{n+1}=U_n(o)\omega_n(1/a)U_n(o), \quad (44)$$

where ω_n satisfies

$$\omega_n=1+(1/a)[\mathcal{V}_n+U_n(e)]\omega_n. \quad (45)$$

To solve this equation we must proceed just as we did in the solution to Eq. (30), etc. The process here is clearly much more complicated than was that of Sec. III in which the linear theory was studied, but still leads to a means of constructing the potentials in terms of integral equations which do not involve emissions or absorptions. In any actual problem, one will cut off the series defining $\mathcal{V}$ at a certain multiplicity of virtual mesons. The V_n's occurring in $\mathcal{V}$ can be defined to the same order in virtual mesons. Since the multiplicities increase with n as 2^n, the calculation is not as complicated as might appear, if only low multiplicities are kept. We illustrate this by developing the method of evaluating

$$v_0=\mathrm{DP}\left[H_2+H_1\frac{1}{a-H_2}H_1\right]. \quad (46)$$

Defining

$$H_d=\mathrm{DP}H_2$$

and

$$H_2=H_d+H_2', \quad (47)$$

it is apparent that H_2' must create at least two mesons. Using relation (D) we obtain

$$H_1\frac{1}{a-H_2}H_1=H_1\frac{1}{a-H_d-R}H_1+H_1\frac{1}{a-H_d-R}H_2'\frac{1}{a-H_d}H_1, \quad (48)$$

where

$$R=H_2'\frac{1}{a-H_d}H_2'. \quad (49)$$

The quantity R is clearly calculable in terms of the solution to an integral equation with H_d as a potential by means of relation (B). The NDP(R) creates at least four mesons. If, for instance, we do not permit such high multiplicities this can be neglected and we need keep only

$$\mathrm{DP}[R]\equiv R_d,$$

and the expression (48) can be evaluated using relation (B). The calculation can obviously be extended to higher multiplicities.

To give a specific illustration, we calculate the potential which results from applying the Dyson transformation[18] to the symmetric pseudoscalar coupling term in pseudoscalar meson theory. To order g^2 the interaction H' is

$$H' = \frac{g}{2M}\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi} + \frac{g^2}{2M}\boldsymbol{\phi}\cdot\boldsymbol{\phi}. \tag{50}$$

If we compare this result with the defining equation (29) for H_1 and H_2 and make use of Eq. (46), we find

$$v_0 = \mathrm{DP}\left[\frac{g^2}{2M}\boldsymbol{\phi}\cdot\boldsymbol{\phi} + \frac{g^2}{4M^2}\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi} \times \frac{1}{a-(g^2/2M)\boldsymbol{\phi}\cdot\boldsymbol{\phi}}\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right]. \tag{51}$$

Consistent with the approximation just discussed, we shall restrict ourselves to diagonal elements of $\boldsymbol{\phi}\cdot\boldsymbol{\phi}$ in evaluating the integral operator $[a-(g^2/2M)\boldsymbol{\phi}\cdot\boldsymbol{\phi}]^{-1}$. If we evaluate the second term of v_0 for p-state mesons, then the s-state potential $(g^2/2M)\mathrm{DP}[\boldsymbol{\phi}\cdot\boldsymbol{\phi}]$ cannot scatter the incoming meson or the virtual mesons created in p-states by the first operator $\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi}$ if nucleon recoil is neglected. For s-state mesons, the two operators $\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi}$ can create and absorb a virtual p-state meson, the s-state meson being scattered by the potential $(g^2/2M)\mathrm{DP}[\boldsymbol{\phi}\cdot\boldsymbol{\phi}]$.

This process gives a contribution to the s-state interaction of the same form and sign as the first term of v_0. Accordingly we have approximately

$$v_0 = \mathrm{DP}\left[\lambda\frac{g^2}{2M}\boldsymbol{\phi}\cdot\boldsymbol{\phi} + \frac{g^2}{4M^2}\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi}\frac{1}{a}\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right], \tag{52}$$

where λ is of the order of 2. The second term of this is the p-state potential considered by Chew[9] which gives strong scattering in the isotopic spin $\frac{3}{2}$ and spin $\frac{3}{2}$ state; the first term is a strongly repulsive potential which gives a rather smaller s-wave scattering (roughly 50 millibarns at 100 Mev).[19]

V. NUCLEON PAIR PRODUCTION

We suppose the interaction H' to be in general nonlinear in both the meson field variables and in the pairs of nucleon field variables. We may formally obtain the "interaction potential" from the results of the previous section by either of two methods: First, we may ignore the nucleon field variables and diagonalize the interaction in meson occupation numbers only, just as was done in Sec. IV. The operator $\mathcal{V}$ obtained in this manner is now a nonlinear operator in the nucleon field variables and is the type discussed in Sec. IV. This operator is a gain broken into parts even and odd in the production (or absorption) of *nucleon pairs*. We can proceed to diagonalize it in nucleon states just as was done for the meson states.

A more straightforward method for calculation is to formally include in out set of field variables ϕ_σ ($\sigma=1, 2, \cdots N$) each pair of nucleon field variables $\bar{\psi}O\psi$ (where O is a matrix operating on the components of ψ). This can be done since we have not made use of the commutation rules for the ϕ's. For purposes of argument we can say that a "particle" is emitted when either a nucleon pair or a meson is created. Splitting H' into parts even and odd in "particle" creation we can develop the potential as in Sec. IV. There is no point in a detailed development, since it follows the methods used in Sec. IV.

We shall illustrate the method by showing how to proceed for an H' which is linear in ϕ and linear in $\bar{\psi}O\psi$ (i.e., of the usual sort).

We set

$$H' = H_e' + H_o' \tag{53}$$

where H_e' produces (or absorbs) one meson and one nucleon pair (even in "particle" creation) and H_o' produces (or absorbs) one meson and no nucleon pairs. Using relation (D)

$$\begin{aligned}\Omega &= 1 + \frac{1}{a-H'}H' \\ &= 1 + \frac{1}{a-H_e'-D_o}[H_e'+D_o] \\ &\quad + \frac{1}{a-H_e'-D_o}H_o'\left[1+\frac{1}{a-H_e'}H_e'\right],\end{aligned} \tag{54}$$

where

$$D_o = H_o'\frac{1}{a-H_e'}H_o'. \tag{55}$$

The second term in Eq. (2) can be discarded and $H_e'+D_o$ can be broken into a "diagonal part," a part producing an even number of pairs of "particles" and a part producing an odd number of pairs of "particles." The process can now be repeated, etc. The terms in the "potential" can be defined by means of integral equations as in the previous sections.

VI. DISCUSSION

We have seen that potentials can be obtained in a straightforward manner in quantum field theories. Since the potential is given as an infinite series of terms the question of convergence remains to be studied. It is evident that the potentials of the type discussed here will always converge if a strict power series expansion (in the coupling constant) converges, since each term in the potential series can be expanded into a power series in g^2. It is also quite possible that the series given here may converge even when the power series expansion is not convergent.

[18] F. J. Dyson, Phys. Rev. **73**, 929 (1948).

[19] G. F. Chew (private communication) has called our attention to this small value of the s-wave scattering.

The results of our studies strongly suggest that a great deal of the apparent complexity of the S matrix in field theory is associated with the scattering of real and virtual particles in intermediate states. When these effects are identified and isolated, the complexity of the analysis is strikingly reduced.

It is to be noted that the potential $\mathcal{V}$ will be hermitean when and only when true absorption and emission of particles is not energetically possible. When creation or absorption is possible some of the energy denominators will be singular and the $i\eta$ term will contribute an anti-hermitean part to $\mathcal{V}$. Otherwise $i\eta$ is redundant for the construction of the potentials and may be neglected. In this case bound state problems may be handled by dropping the additive identity operator on the right-hand side of Eq. (26) and (43) and solving the resulting eigenvalue problem.

Finally, although meson theory has been most specifically considered, the results are of course applicable to other types of field theories.

We are indebted to Professor G. F. Chew and Professor F. Low for several interesting discussions concerning problems related to the present one.

APPENDIX

Nuclear Forces in the Isotopic Spin Zero State for the Symmetric Scalar Theory

We consider this case explicitly to illustrate the development of the method for a simple problem. For the two nucleon system, the coupling term of Eq. (1) is

$$H'=H'(1)+H'(2), \tag{A1}$$

where $H'(i)$ is the coupling term for the ith nucleon. For simplicity we shall in this problem treat the nucleons as infinitely heavy so that their energy is unchanged by the emission of mesons. The operator

$$-a=-E+H_0 \tag{A2}$$

will then give just the sum of the energies of the virtual mesons present.

The first two terms in the potential series are [Eq. (17)]

$$\begin{aligned} V_0&=H'\frac{1}{a}H'^*+H'^*\frac{1}{a}H', \\ V_1&=H'\frac{1}{a}H'\frac{1}{a-V_0}H'^*\frac{1}{a}H'^*, \end{aligned} \tag{A3}$$

where the starred operators create mesons, the unstarred annihilate mesons. The sum of these is particularly simple if in the term

$$H'\frac{1}{a-V_0}H'^* \tag{A4}$$

appearing in V_1 we retain in V_0 only that part which first annihilates a meson; this is equivalent to restricting the number of mesons present to two or less. If this approximation is made, then it is easily shown that

$$\begin{aligned} H'\frac{1}{a-V_0}H'^*&=V_0'+V_0'\frac{1}{a-V_0'}V_0' \\ &\equiv T, \end{aligned} \tag{A5}$$

where

$$V_0'=H'(1/a)H'^* \tag{A6}$$

and T is the scattering matrix associated with V_0'. With this result we obtain for the first two terms in the nuclear potential,

$$V^{(1)}=V_0+V_1=H'\left(1+\frac{1}{a}T\right)\frac{1}{a}H'^*. \tag{A7}$$

This result differs from the lowest order potential V_0' in that the meson created by H'^* goes into the scattering state

$$\omega_S=1+(1/a)T, \tag{A8}$$

which describes the scattering of the meson by the two nucleons Only if the scattering is small and we can set $\omega_S=1$ is the lowest order potential an adequate approximation. The principal difficulties of evaluating $V^{(1)}$ are now those of solving the multiple scattering equation for ω_S.

In terms of the potentials v_1 and v_2 which scatter the meson at nucleons (1) and (2), the equation for ω_S is

$$\omega_S=1+(1/a)(v_1+v_2)\omega_S. \tag{A9}$$

The solution to this equation can be given in terms of the set of equations discussed by Watson[7] and Brueckner:[20]

$$\omega_S=1+(1/a)(T_1\omega_1+T_2\omega_2), \tag{A10}$$

where

$$\omega_1=1+(1/a)T_2\omega_2,\quad \omega_2=1+(1/a)T_1\omega_1, \tag{A11}$$

and T_i is the scattering matrix for the potential v_i. These coupled integral equations are difficult in general to solve; under a reasonable qualitative approximation we can, however, obtain exact solutions. For the symmetric scalar theory, the potential for the scattering of a meson is

$$(\mathbf{k}|v|\mathbf{k}_0)=-\tfrac{1}{2}g^2(1+\boldsymbol{\tau}_1\cdot\mathbf{l})e^{i(\mathbf{k}_0-\mathbf{k})\cdot r_1}/[(\omega\omega_0)^{\frac{1}{2}}(\omega+\omega_0)], \tag{A12}$$

where $\mathbf{l}$ is the isotopic angular momentum operator for the meson and $\boldsymbol{\tau}$ is the nucleon isotopic spin operator. In this form the integral equation relating v_1 to T_1 is not easily solved; if, however, we make the replacement

$$\omega+\omega_0\rightarrow(\omega\omega_0)^{\frac{1}{2}}, \tag{A13}$$

then we obtain directly

$$(\mathbf{k}|T_1|\mathbf{k}_0)=-\sigma_1 e^{i(\mathbf{k}_0-\mathbf{k})\cdot r_1}/(\omega\omega_0), \tag{A14}$$

[20] K. A. Brueckner, Phys. Rev. 89, 834 (1953).

where

$$\sigma_1 = \tfrac{1}{2}g^2(1+2\Delta+\boldsymbol{\tau}_1\cdot\mathbf{l})/[(1-2\Delta)(1+\Delta)] \quad \text{(A15)}$$

and

$$\Delta = \tfrac{1}{2}g^2(2\pi)^{-3}\int d\mathbf{k}/\omega_k^3. \quad \text{(A16)}$$

Substituting this result into the coupled equations of Eq. (A11) we find

$$G_1 = e^{i\mathbf{k}_0\cdot\mathbf{r}_1}/\omega_0 + \sigma_2 I(r) G_2,$$
$$G_2 = e^{i\mathbf{k}_0\cdot\mathbf{r}_2}/\omega_0 + \sigma_1 I(r) G_1, \quad \text{(A17)}$$

and

$$(\mathbf{k}|\omega_S|\mathbf{k}_0) = (2\pi)^3\delta(\mathbf{k}_0-\mathbf{k}) + (1/\omega)(\sigma_1 G_1 e^{-i\mathbf{k}\cdot\mathbf{r}_1} + \sigma_2 G_2 e^{-i\mathbf{k}\cdot\mathbf{r}_2}) \quad \text{(A18)}$$

where

$$G_i = (2\pi)^{-3}\int \frac{(\mathbf{k}|\Omega_i|\mathbf{k}_0)}{\omega} e^{i\mathbf{k}\cdot\mathbf{r}_i} d\mathbf{k} \quad \text{(A19)}$$

and

$$I(r) = (2\pi)^{-3}\int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\omega^3} d\mathbf{k}. \quad \text{(A20)}$$

Eliminating G_2 in Eq. (A17) we obtain for G_1

$$(1-I^2\sigma_2\sigma_1)G_1 = (1/\omega_0)e^{i\mathbf{k}_0\cdot\mathbf{r}_1} + (\sigma_2 I/\omega_0)e^{i\mathbf{k}_0\cdot\mathbf{r}_2}, \quad \text{(A21)}$$

with a similar result for G_2. The construction of the inverse of the operator $(1-I^2\sigma_2\sigma_1)$ is in general rather complicated; for the case of total isotopic spin zero, however, the operators σ_2 and σ_1 are diagonal with the matrix elements

$$\sigma_1 = \sigma_2 = -\tfrac{1}{2}g^2/(1+\Delta) \equiv \rho. \quad \text{(A22)}$$

With this result the wave matrix ω_S is

$$(\mathbf{k}|\omega_S|\mathbf{k}_0) = (2\pi)^3\delta(\mathbf{k}_0-\mathbf{k}) - \frac{1}{\omega^2\omega_0{}^2}\frac{\rho}{1-I^2\rho^2} \times\{e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}_1} + e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}_2} - \rho I[e^{i(\mathbf{k}_0\cdot\mathbf{r}_1-\mathbf{k}\cdot\mathbf{r}_2)} + e^{i(\mathbf{k}_0\cdot\mathbf{r}_2-\mathbf{k}\cdot\mathbf{r}_1)}]\}. \quad \text{(A23)}$$

Finally the potential is

$$V^{(1)} = \frac{-3g^2}{(2\pi)^6}\int \frac{d\mathbf{k}_0 d\mathbf{k}}{2\omega_0(\omega_0\omega)^{\frac{1}{2}}}(e^{i\mathbf{k}\cdot\mathbf{r}_1} - e^{i\mathbf{k}\cdot\mathbf{r}_2}) \times(\mathbf{k}|\omega_S|\mathbf{k}_0)(e^{-i\mathbf{k}_0\cdot\mathbf{r}_1} - e^{-i\mathbf{k}_0\cdot\mathbf{r}_2}), \quad \text{(A24)}$$

where we have set $\boldsymbol{\tau}_1 = -\boldsymbol{\tau}_2$ (for the singlet state) and $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_1 = 3$. In evaluating the integrals over $\mathbf{k}_0$ and $\mathbf{k}_1$ we encounter expressions of the form of

$$\frac{1}{(2\pi)^6}\int \frac{d\mathbf{k}d\mathbf{k}_0}{(\omega\omega_0)^{5/2}} e^{i\mathbf{k}\cdot(\mathbf{r}_a-\mathbf{r}_b)} e^{i\mathbf{k}_0\cdot(\mathbf{r}_c-\mathbf{r}_d)}. \quad \text{(A25)}$$

Consistent with the approximation [Eq. (A13)] we have already made in obtaining the scattering matrices, we shall approximate this integral very roughly by

$$\frac{1}{(2\pi)^6}\int \frac{d\mathbf{k}d\mathbf{k}_0}{\omega^2\omega_0{}^2} e^{i\mathbf{k}\cdot(\mathbf{r}_a-\mathbf{r}_b)} e^{i\mathbf{k}\cdot(\mathbf{r}_c-\mathbf{r}_d)} = Y(\mathbf{r}_a-\mathbf{r}_b) I(\mathbf{r}_c-\mathbf{r}_d), \quad \text{(A26)}$$

where

$$Y(r) = (1/4\pi)e^{-\mu r}/r \quad \text{(A27)}$$

is the Yukawa potential and $I(r)$ is defined by Eq. (A20).

With this approximation we obtain for the potential:

$$V^{(1)} = 3g^2 Y(r)\left[\frac{1-2\rho I(r)}{1-\rho I(r)}\right] - 3g^2 Y(0)\left[\frac{1-\rho I(r)-\rho I(0)}{1-\rho I(r)}\right]. \quad \text{(A28)}$$

The second term contains mass renormalization effects since it remains finite as $r\rightarrow\infty$; if we subtract these as self energies and replace $I(r)Y(0)$ by $I(0)Y(r)$ as was done in Eq. (A26), we obtain, using Eq. (A22) for ρ,

$$V^{(1)} = 3g^2 Y(r)[\{1+\tfrac{1}{2}g^2 I(0)\} \times\{1+\tfrac{1}{2}g^2 I(0) - \tfrac{1}{2}g^2 I(r)\}]^{-1}. \quad \text{(A29)}$$

The dependence on $g/[1+\frac{1}{2}g^2 I(0)]$ which appears in this result is probably a charge renormalization effect although the identification as such is not unambiguous; if we replace

$$g/(1+\tfrac{1}{2}g^2 I(0)) \rightarrow g', \quad \text{(A30)}$$

then

$$V^{(1)} = 3g'^2 Y(r)[1-gg'I(r)]^{-1}. \quad \text{(A31)}$$

This potential increases more rapidly with decreasing r than does the Yukawa potential; the denominator is always positive, however, so that it does not introduce new singularities.