Note that owing to the extraction of a δ-function factor in the derivation of (3.7) from (2.10), it is *not* true that the Fredholm determinant $d(\lambda)$ of (3.7) is equal to $(0|S|0)$. Consequently, for a static field, the iterated solution, $(\mathbf{p}|\mathbf{R}_I|\mathbf{q})$, (Born approximation) is *not* the same as the Fredholm solution. This is in agreement with the conclusion of Jost and Pais.[1]

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Spherically Symmetric Solutions in Nonsymmetrical Field Theories. I. The Skew Symmetric Tensor*

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The spherically symmetric form of the skew tensor $g_{\check{ik}}$, given by Papapetrou, is not sufficiently general. A more general form is found. This necessitates a reconsideration of the program of spherically symmetric solutions in nonsymmetrical field theories initiated by Papapetrou. The present paper makes a beginning in this direction.

The new form of the spherically symmetric tensor $g_{\check{ik}}$ is derived from a consideration of the infinitesimal rotation of a sphere about a diameter. It is hoped to use this form to obtain nonstatic solutions in nonsymmetrical field theories which will correspond to solutions of a radiating star in general relativity.

I. INTRODUCTION

Papapetrou[1] initiated the study of rigorous (nonapproximate) solutions in the various unified field theories by showing that the skew symmetric tensor $g_{\check{ik}}$ with only $g_{\check{14}}=w(r, t)$ and $g_{\check{23}}=v(r, t)\sin\theta$ as the surviving components, is spherically symmetric. This form of the tensor was the starting point for a number of investigations in this direction. Papapetrou himself worked out the solutions of the field equations of Schrödinger.[2] Rigorous solutions of the field equations of Einstein and Strauss[3] were given by Wyman.[4] Bandyopadhyaya[5] gave a simple solution of the latest unified field theory of Einstein.[6] Recently Bonner[7] has satisfactorily solved the problem of finding static spherically symmetric solutions in Einstein's unified field theory. All these investigations began with the form of $g_{\check{ik}}$ found by Papapetrou.

A nonsymmetrical tensor field can be split up into its symmetrical and skew symmetrical parts. We write

$$g_{ik}=g_{\underline{ik}}+g_{\check{ik}},$$

the bar or the hook below the suffixes distinguishing the two parts, respectively.

For spherically symmetric solution the form of $g_{\underline{ik}}$ is well known from general relativity. In order to find the spherically symmetric form of $g_{\check{ik}}$, Papapetrou considered the rotation of a sphere about a diameter POP' and compared the values of the various components of $g_{\check{ik}}$ before and after the rotation at the point P on the sphere and the axis of rotation. Since a rotation through a right angle will interchange the components perpendicular to OP, these would vanish at a point on the axis of rotation. Hence Papapetrou's method will naturally give the components of $g_{\check{ik}}$ along the radial direction only. That is why only $g_{\check{14}}$ and $g_{\check{23}}$ (which correspond to the radial components of magnetic and electric field, respectively) survived in his tensor. In what follows we consider an infinitesimal rotation of a sphere and compare the values of $g_{\check{ik}}$ at a point not on the axis of rotation. We shall, of course, recover Papapetrou's components; but we shall also find that there are some other components of $g_{\check{ik}}$ which are nonzero.

II. INFINITESIMAL ROTATIONS OF A SPHERE

In this section we shall be interested in the two-dimensional geometry of the surface of a sphere of radius a. The fundamental quadratic form ψ on this surface is given by

$$\psi=a^2d\theta^2+a^2\sin^2\theta d\varphi^2=g_{22}(dx^2)^2+g_{33}(dx^3)^2. \quad (2.1)$$

The contravariant components ξ^μ, $(\mu=2, 3)$ of an infinitesimal transformation which would represent a motion of the sphere into itself, satisfy the following

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[1] A. Papapetrou, Proc. Roy. Irish Acad. **A52**, 69 (1948).

[2] For field theory of Schrödinger, see E. Schrödinger, Proc. Roy. Irish Acad. **A51**, 163 (1947).

[3] A. Einstein and E. G. Strauss, Ann. Math. **47**, 731 (1946).

[4] M. Wyman, Can. J. Math. **2**, 427 (1950).

[5] G. Bandyopadhyaya, Nature **167**, 648 (1951).

[6] A. Einstein, *Meaning of the Relativity* (Methuen, London, 1950), Appendix II.

[7] W. B. Bonner, Proc. Roy. Soc. (London) **A209**, 353 (1951); **A210**, 427 (1952).

equations of Killing:[8]

$$\cos\theta\xi^2+\sin\theta(\partial\xi^3/\partial\varphi)=0,$$
$$\sin^2\theta(\partial\xi^3/\partial\theta)+(\partial\xi^2/\partial\varphi)=0, \quad (2.2)$$
$$\partial\xi^2/\partial\theta=0.$$

From the third of these equations, we have $\xi^2=X_3$, where X_3 is a function of φ alone. Indicating by primes derivatives with respect to the arguments, from the first two we have

$$\partial\xi^3/\partial\varphi=-X_3\cot\theta, \quad \partial\xi^3/\partial\theta=-X_3'\operatorname{cosec}^2\theta, \quad (2.3)$$

for which the condition of consistency is

$$X_3''+X_3=0, \quad \text{or} \quad X_3=A\cos(\varphi+B). \quad (2.4)$$

We thus arrive at the following final form of the contravariant components ξ^μ of an infinitesimal rotation of a sphere about a diameter:

$$\xi^2=A\cos(\varphi+B), \quad (2.5)$$

$$\xi^3=-A\sin(\varphi+B)\cot\theta+C, \quad (2.6)$$

A, B, and C being arbitrary constants.

III. SPHERICALLY SYMMETRIC TENSOR FIELDS

The condition for a spherically symmetric tensor field is the following: there is a frame of reference such that, after an arbitrary rotation around the center of symmetry, the new components g'_{ik} are the same functions of the new coordinates x'^μ as the g_{ik} are of x^μ. Taking the center of symmetry as the origin, let x^μ be a set of polar coordinates r, θ, φ and the time coordinate t. Consider an infinitesimal rotation given by

$$x'^\mu=x^\mu+\xi^\mu\delta\epsilon, \quad (3.1)$$

where

$$\xi^1=0, \quad \xi^2=A\cos(\varphi+B),$$
$$\xi^3=-A\sin(\varphi+B)\cot\theta+C, \quad \xi^4=0, \quad (3.2)$$

and $\delta\epsilon$ is an infinitesimal. It is clear from general considerations that arbitrary transformations of the coordinates r and t will leave the spherical symmetry of a tensor expression undisturbed, and so in (3.2) transformations of the coordinates θ and φ alone are considered by stipulating $\xi^1=\xi^4=0$. The criterion of spherical symmetry of a tensor field g_{ik} is

$$g'_{ik}(x'^\mu)=g_{ik}(x'^\mu). \quad (3.3)$$

Substituting from (3.1) the values of x'^μ in the right-hand member of (3.3), applying Taylor's theorem, and retaining terms only up to the first power of $\delta\epsilon$, we can write this criterion in the form

$$g'_{ik}(x'^\mu)=g_{ik}(x^\mu)+g_{ik,\mu}\xi^\mu\delta\epsilon. \quad (3.4)$$

Applying the tensor law of transformation to $g_{ik}(x^\mu)$, we find, again correct to the first power in $\delta\epsilon$, that

$$g'_{ik}(x'^\mu)=g_{ik}(x^\mu)-[\xi^\alpha{}_{,i}g_{\alpha k}+\xi^\alpha{}_{,k}g_{i\alpha}]\delta\epsilon. \quad (3.5)$$

With the help of Eq. (3.5), the criterion (3.4) for the spherical symmetry of the field can be written in the form,

$$\xi^\alpha{}_{,i}g_{\alpha k}+\xi^\alpha{}_{,k}g_{i\alpha}+g_{ik,\alpha}\xi^\alpha=0. \quad (3.6)$$

The first two terms in the left-hand member of this equation arise out of the tensor law of transformation of g_{ik}, while the third term may be called the "transport term." It arises because the value of g_{ik} at a point x^μ is compared with the transformed g_{ik} at the point which after the transformation possesses the coordinates x^μ (and therefore, originally had the coordinates $x^\mu-\xi^\mu\delta\epsilon$).

We now split up the g_{ik} field into its symmetrical and skew parts,

$$g_{ik}=g_{\underline{ik}}+g_{\check{ik}}.$$

The general form of the spherically symmetric $g_{\underline{ik}}$, known from general relativity, is

$$g_{\underline{ik}}=\begin{bmatrix}-\alpha & 0 & 0 & a\\ 0 & -\beta & 0 & 0\\ 0 & 0 & -\beta\sin^2\theta & 0\\ a & 0 & 0 & \gamma\end{bmatrix}, \quad (3.7)$$

where α, β, γ, and a are functions of r and t. We now obtain the form of $g_{\check{ik}}$ which will satisfy the criterion (3.6).

It is easy to verify that, as a result of the Eq. (3.6) both $g_{\check{14}}$ and $g_{\check{23}}\operatorname{cosec}\theta$ have to satisfy only one equation of the form

$$z_{,\alpha}\xi^\alpha=0, \quad \text{or} \quad \xi^2(\partial z/\partial\theta)+\xi^3(\partial z/\partial\varphi)=0. \quad (3.8)$$

This is a differential equation for z in terms of the variables θ and φ, and with ξ^2 and ξ^3 given by (3.2), its general solution would be

$$z=f(w,r,t), \quad (3.9)$$

$$w=A\sin(\varphi+B)\sin\theta+C\cos\theta, \quad (3.10)$$

f being an arbitrary function of its arguments. We assume this arbitrary function to be expressed as a product of a function of (r, t) and a function of (θ, φ). Since in spherically symmetric fields the transformation of the variables r and t can be studied independently of the transformations of the variables θ, φ, this assumption will not contradict the requirements of spherical symmetry. Further, we shall find that under this assumption we shall be able to obtain an expression for the spherically symmetric g_{ik} which will be more general than that found by Papapetrou. This assumption introduces a good deal of simplification in the procedure of solving the field equations. The form of the functions of (θ, φ) will now be determined by the criterion (3.6) of spherical symmetry, while the form of the function of (r, t) will be decided by the field equations which

[8] See L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1949), pp. 241, 242.

govern the $\check{g}_{ik}$ field. Therefore, from (3.8) we can write

$$\check{g}_{14}=H(r,t)h(w);\quad \check{g}_{23}=E(r,t)\sin\theta k(w).$$

H, E, h, and k are arbitrary functions of their arguments.

Writing the remaining components of $\check{g}_{ik}$ as products of a function of (r, t) and a function of (θ, φ), it will be found that criterion (3.6) gives the following forms for these components:

$$\check{g}_{12}=p(r,t)v(\theta,\varphi),\quad \check{g}_{13}=-p(r,t)u(\theta,\varphi)\sin\theta;$$
$$\check{g}_{24}=q(r,t)v(\theta,\varphi),\quad \check{g}_{34}=-q(r,t)u(\theta,\varphi)\sin\theta.$$

Of course, all the functions of (r, t) will be arbitrary; but $u(\theta, \varphi)$ and $v(\theta, \varphi)$ have to satisfy the equations

$$-A\sin(\varphi+B)\operatorname{cosec}\theta u+\xi^2(\partial v/\partial\theta)+\xi^3(\partial v/\partial\varphi)=0,\quad (3.11)$$

$$A\sin(\varphi+B)\operatorname{cosec}\theta v+\xi^2(\partial u/\partial\theta)+\xi^3(\partial u/\partial\varphi)=0.\quad (3.12)$$

We can combine these two equations to obtain

$$\xi^\alpha[\partial(u^2+v^2)/\partial x^\alpha]=0,\quad (3.13)$$

and

$$\xi^\alpha[\partial(\tan^{-1}(u/v))/\partial x^\alpha]+A\sin(\varphi+B)\operatorname{cosec}\theta=0.\quad (3.14)$$

From Eq. (3.13) we immediately find that

$$u^2+v^2=f^2,\quad (3.15)$$

where $f=f(w)$ is an arbitrary function of w. Equation (3.14) can be integrated to yield

$$v/u=A\cos(\varphi+B)\{\partial w/\partial\theta\}^{-1},$$

so that we ultimately find

$$v=A\cos(\varphi+B)(A^2+C^2-w^2)^{-\frac{1}{2}}\cdot f(w),\quad (3.16)$$

$$u=(A\sin(\varphi+B)\cos\theta-C\sin\theta)\times(A^2+C^2-w^2)^{-\frac{1}{2}}\cdot f(w),\quad (3.17)$$

where w is given by the Eq. (3.10). Thus the final expression for the tensor field $\check{g}_{ik}$ which satisfies the criterion (3.6) is

$$\check{g}_{ik}=\begin{bmatrix} 0 & pv & -pu\sin\theta & Hh \\ -pv & 0 & Ek\sin\theta & qv \\ pu\sin\theta & -Ek\sin\theta & 0 & -qu\sin\theta \\ -Hh & -qv & qu\sin\theta & 0 \end{bmatrix},\quad (3.18)$$

p, q, E, H are arbitrary functions of r and t, h and k are arbitrary functions of w, and u, v, w are given by (3.17), (3.16), and (3.10), respectively. It should be noted that (3.18) is not the most general solution of (3.6), because of the assumption which separated the variables (r, t) and (θ, φ).

IV. THE POLARIZATION OF THE $\check{g}_{ik}$ FIELD

We shall now consider in detail the dependence of these components on the variables θ and φ. This dependence is mainly given by the functions u, v, and w. Consider first the function

$$w=A\sin(\varphi+B)\sin\theta+C\cos\theta.\quad (4.1)$$

If the point P, (r, θ, φ), of the sphere of radius r, at which these components of $\check{g}_{ik}$ are evaluated, has the usual Cartesian coordinates (x, y, z), then (4.1) can be written in the form

$$w=(ax+by+cz)/r,\quad (4.2)$$

where the constants a, b, c can be easily expressed in terms of the earlier constants A, B, C. If OQ is the diameter of the sphere (center O) with direction-cosines proportional to a, b, c, from (4.2) w is seen to be proportional to the cosine of the angle POQ. The occurrence of the arbitrary functions of this variable w in spherically symmetric tensors can now be easily understood, if we realize that OQ was the axis about which the infinitesimal rotation was given to the sphere to derive the criterion (3.6). This interpretation of w also points out a marked difference between the two factors $E(r, t)\sin\theta$ and $k(w)$ of $\check{g}_{23}$, or between the factors $H(r, t)$ and $h(w)$ of $\check{g}_{14}$. Let us choose our Cartesian axes of coordinates in such a way that OZ is along the axis of rotation OQ. Then we have $A=0$, $B=0$, and

$$w=C\cos\theta.\quad (4.3)$$

If, however, the coordinate axes are oriented in such a way that OY is along the axis of rotation, we would get $B=0$, $C=0$, and

$$w=A\sin\theta\sin\varphi.\quad (4.4)$$

Thus the functional form of w (as a function of θ and φ) depends on the orientation of the axis of rotation. Further, the *form* (4.3) of w will remain the same for rotations round OZ only, while the form (4.4) of w will remain invariant for all rotations round OY only. We say that w is an invariant spherically symmetric expression with a polarization. For w given by (4.3) we may say that "the axis of polarization" is the z axis, while in (4.4) it is the y axis. It is now easy to see that

$$\check{g}_{14}=H(r,t)\quad\text{and}\quad \check{g}_{23}=E(r,t)\sin\theta\quad (4.5)$$

are spherically symmetric unpolarized components of $\check{g}_{ik}$, while

$$\check{g}_{14}=H(r,t)h(w),\quad \check{g}_{23}=E(r,t)\sin\theta k(w),$$

are spherically symmetric with a polarization. The $\check{g}_{ik}$, with only nonvanishing components given in (4.5), was first obtained by Papapetrou.

Now consider the functions u and v. They also show a polarization. If $A=0$, $B=0$ we find

$$u=f(\cos\theta),\quad v=0.\quad (4.6)$$

If, however, $B=0$, $C=0$, we find

$$u=\sin\varphi\cos\theta(1-\sin^2\theta\sin^2\varphi)^{-\frac{1}{2}}\cdot f,$$
$$v=\cos\varphi(1-\sin^2\theta\sin^2\varphi)^{-\frac{1}{2}}\cdot f. \qquad (4.7)$$

For u and v given by (4.6), we find that

$$g_{\underset{\vee}{12}}=0,\quad g_{\underset{\vee}{13}}=-pf\sin\theta,$$
$$g_{\underset{\vee}{24}}=0,\quad g_{\underset{\vee}{34}}=-qf\sin\theta. \qquad (4.8)$$

If we accept the interpretation of $g_{\underset{\vee}{ik}}$ as given by the first ordered solution,[7] we see that (4.8) will represent a polarized spherical wave with the electric vector in the plane of the point P and the z axis. In our notations (4.8) gives a spherically symmetric tensor field which is polarized, the axis of polarization being the z axis.

We would like to put down together the conclusions that we have drawn from the discussions of this section. Tensor fields may satisfy our criterion of spherical symmetry, (3.6), in two different ways. Some of them satisfy (3.6) for all possible values of the parameters A, B, C of the infinitesimal transformations. An illustration of such a field is Papapetrou's tensor field with two nonvanishing components, $g_{\underset{\vee}{14}}=H(r,t)$, $g_{\underset{\vee}{23}}=E(r,t)\sin\theta$. Such tensor fields are spherically symmetric without any polarization. On the other hand, there are some other tensor-fields which satisfy the criterion (3.6) through the appearance of the parameters A, B, C in the very expressions of the components of the fields. Thus the functional forms of the various components carry, as it were, an impression of the axis about which the infinitesimal rotation was given to derive the criterion (3.6). Such fields are spherically symmetric and polarized. We have called the particular axis of rotation the "axis of polarization" of the field. An illustration of such a field is the electromagnetic tensor $F_{\alpha\beta}$ used to describe the spherically symmetric field of a radiating star in general relativity.[9] We have found above a general spherically symmetric skew tensor field with any arbitrary axis of polarization. Both the above illustrations are particular cases of the field found here.

Since we are at liberty to choose the orientation of our coordinate axes, we shall not be specializing our $g_{\underset{\vee}{ik}}$ in any way if we take the axis of polarization of $g_{\underset{\vee}{ik}}$ as the z axis. We put $A=0$, $B=0$, $C=1$ and find our $g_{\underset{\vee}{ik}}$ to be of the form,

$$g_{\underset{\vee}{14}}=H(r,t)h(\theta),\quad g_{\underset{\vee}{23}}=E(r,t)\sin\theta k(\theta),\quad g_{\underset{\vee}{12}}=0,\quad g_{\underset{\vee}{24}}=0,$$
$$g_{\underset{\vee}{13}}=-p(r,t)\sin\theta f(\theta),\quad g_{\underset{\vee}{34}}=-q(r,t)\sin\theta f(\theta).$$

This is the simplest and, at the same time, the most general form of $g_{\underset{\vee}{ik}}$ which is spherically symmetric and which exhibits polarization about the z axis.

[9] See V. V. Narlikar and P. C. Vaidya, Nature **159**, 642 (1947); Proc. Natl. Inst. Sci. (India) **14**, 53 (1948).

V. THE SYMMETRIC TENSOR $g_{\underline{ik}}$

In Eq. (3.7) we have written down the spherically symmetric form of $g_{\underline{ik}}$ which is used in general relativity. From the discussion of the last section on the polarization of tensor-fields, we can now see that the general relativity form (3.7) of $g_{\underline{ik}}$ is its general spherically symmetric unpolarized form. The question now arises: What is its general polarized form? This question never arose in general relativity, because there $g_{\underline{ik}}$ is the metric tensor of space-time and polarization of a metric tensor has no physical meaning. In nonsymmetrical field theories, $g_{\underline{ik}}$ need not be the metric tensor of space-time. Though all the previous authors (Papapetrou,[1] Wyman,[4] Bonner[7]) have tried to interpret their static spherically symmetric solutions on the assumption that $g_{\underline{ik}}$ is the metric tensor, doubts have been raised by many (including Wyman) as regards this assumption. Wyman has actually put forward alternative suggestions for the metric tensor. In his letter Infeld states that Einstein's theory gives no interaction of a moving charged mass-particle with the electromagnetic field.[10] This may be taken to mean that the identification of the metric tensor with $g_{\underline{ik}}$ cannot be true even at sufficiently large distances from the center of the particle. As a matter of fact, this is the view taken by Kurşunoğlu,[11] who has suggested an alternative set of field equations to replace those given by Einstein, which explicitly recognize a metric tensor $b_{\underline{ik}}$ distinct from the field-tensor $g_{\underline{ik}}$.

It is obvious that the tensor $g_{\underline{ik}}$ must be connected in some way with the metric tensor $b_{\underline{ik}}$. However, at the present stage of development of nonsymmetric field theories, it does not seem possible to say definitely that $g_{\underline{ik}}$ is identical with the metric tensor $b_{\underline{ik}}$. We are, therefore, specifying here the general form of $g_{\underline{ik}}$ which is spherically symmetric with a polarization. By working systematically with the equations obtained from the criterion (3.6), it will be found that this general form of $g_{\underline{ik}}$ can be expressed in terms of the functions u, v, and w defined earlier, in the following way:

$$g_{11}=-\alpha(r,t)F(w),\quad g_{\underline{14}}=a(r,t)F(w),\quad g_{44}=\gamma(r,t)F(w),$$
$$g_{22}=-[\beta(r,t)+B(r,t)v^2]G(w),$$
$$g_{\underline{23}}=+B(r,t)uvG(w)\sin\theta,$$
$$g_{33}=-[\beta(r,t)+B(r,t)u^2]G(w)\sin^2\theta,\quad g_{\underline{12}}=\lambda(r,t)v,$$
$$g_{\underline{13}}=-\lambda(r,t)u\sin\theta,\quad g_{\underline{24}}=\mu(r,t)v,\quad g_{\underline{34}}=-\mu(r,t)u\sin\theta.$$

F and G are arbitrary functions of w. In putting down the same function $F(w)$ as a factor in each of the three components g_{11}, $g_{\underline{14}}$, and g_{44}, we have used the fact that these components are interconnected by arbitrary transformations of the coordinate r and t.

[10] L. Infeld, Nature **166**, 1075 (1950).

[11] B. Kursunoğlu, Proc. Phys. Soc. (London) **A65**, 81 (1952).