Fredholm Theory of Scattering in a Given Time-Dependent Field

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It is shown that Feynman's relativistic solution for the scattering of an electron (or pair creation) by a given external field is the Fredholm resolvent of the related integral equation and is thus the unique and absolutely convergent solution for any strength of field.

INTRODUCTION

THE Fredholm theory of integral equations has been applied to the nonrelativistic theory of scattering by Jost and Pais.[1] We here consider the extension of this theory to the interaction of the quantized electron-positron field with a prescribed external electromagnetic field. This problem has been considered by Feynman.[2] Feynman's solution is most simply derived from the S matrix in the form given by Dyson.[3] The appropriate matrix element for electron scattering or pair creation is obtained as an expansion in the external field and is normalized by multiplying by the vacuum expectation value of the S matrix. We show that this is identical with the Fredholm resolvent of a related integral equation and is thus absolutely convergent for any strength of the external field, for which the cross section has any meaning.

In the first section the Fredholm theory is stated in a form given by Plemelj,[4] which exhibits the Fredholm solution in terms of the iterations of the kernel and its traces. These quantities have the advantage over the usual form of the theory[5] that they are either the same as, or closely related to, expressions occurring in the S matrix and can be written down directly by Feynman's graphical methods. The relation of the Fredholm solution to the solution by iteration is discussed. The problem of scattering in a pure external field is then treated in Secs. 2 and 3, with the result stated above. The case of a static field is related to the work of Jost and Pais.[1]

1. FREDHOLM THEORY

Consider Fredholm's integral equation

$$x(s)=y(s)+\lambda\int K(s,t)x(t)dt, \tag{1.1}$$

$$(\text{or } x=y+\lambda Kx),$$

where the integration may be over a fixed interval, finite or infinite. Smithies[6] has shown that, if $K(s,t)$ is a measurable function of s and t, and

$$C_1\equiv\int\int|K(s,t)|^2dsdt<\infty, \tag{1.2}$$

then (1.1) has the unique solution

$$\begin{aligned}x(s)&=d^{-1}(\lambda)\int D(\lambda,s,t)y(t)dt,\\ &=d^{-1}(\lambda)\Delta(\lambda,s),\end{aligned} \tag{1.3}$$

for all λ for which $d(\lambda)\neq 0$. Here

$$d(\lambda)=\sum_{n=0}^{\infty}d_n\lambda^n, \tag{1.4}$$

$$D(\lambda,s,t)=\sum_{n=0}^{\infty}D_n(s,t)\lambda^n \tag{1.5}$$

[$D(\lambda,s,t)$ is called the Fredholm resolvent], where $d_0=1$,

$$d_n=\frac{(-1)^n}{n!}\begin{vmatrix}\sigma_1 & n-1 & 0 & \cdots & 0\\ \sigma_2 & \sigma_1 & n-2 & \cdots & 0\\ \vdots & & & & \\ \sigma_n & \sigma_{n-1} & & \cdots & \sigma_1\end{vmatrix}, \tag{1.6}$$

$$D_n(s,t)=\frac{(-1)^n}{n!}\begin{vmatrix}\delta(s-t) & n & 0 & 0 & \cdots & 0\\ K(s,t) & \sigma_1 & n-1 & 0 & \cdots & 0\\ K^2(s,t) & \sigma_2 & \sigma_1 & n-2 & \cdots & 0\\ \vdots & & & & & \\ K^n(s,t) & \sigma_n & \sigma_{n-1} & & \cdots & \sigma_1\end{vmatrix} \tag{1.7}$$

$$\begin{aligned}K^n(s,t)&=\int K^{n-1}(s,u)K(u,t)du\\ &=\int K(s,u)K^{n-1}(u,t)du,\end{aligned} \tag{1.8}$$

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[1] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

[2] R. P. Feynman, Phys. Rev. **76**, 749 (1949).

[3] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

[4] J. Plemelj, Monatsch. Math. **15**, 93 (1904).

[5] See, for example, E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1940), fourth edition, Chapter XI.

[6] F. Smithies, Duke Math. J. **8**, 107 (1941).

and

$$\sigma_n = \mathrm{Trace} K^n = \int K^n(s, s) ds. \quad (1.9)$$

If $x(s)$ is a spinor function, the trace in (1.9) is also taken over the spinor suffixes of K. The function $d(\lambda)$ is proved to be convergent for all complex values of λ, so that it is an integral function of λ; the resolvent, $D(\lambda, s, t)$, is also uniformly convergent for all λ, so the solution (1.3) is valid for the whole complex λ-plane (except for λ satisfying $d(\lambda)=0$). This formulation is shown to be identical with the usual formulation by Smithies.[6]

As already stated in the introduction, an advantage of this form is that for any field theoretic equations, the expressions $K^n y$ are just those appearing in the solution by iteration[7] and can be calculated directly by the Feynman-Dyson[2,3] method. The σ_n are closely related to the vacuum graphs of the S matrix, and Feynman's methods are again immediately applicable.

From the definition (1.6),

$$d_n = -\frac{1}{n} \sum_{m=1}^{n-1} \sigma_m d_{n-m}, \quad (1.10)$$

so that

$$\sum_{n=0}^{\infty} (n+1) d_{n+1} \lambda^n = -\left(\sum_{n=0}^{\infty} d_n \lambda^n\right)\left(\sum_{q=0}^{\infty} \sigma_{q+1} \lambda^q\right). \quad (1.11)$$

Thus

$$d^{-1} \frac{\partial d}{\partial \lambda} = -\sum_{q=0}^{\infty} \sigma_{q+1} \lambda^q.$$

Hence

$$d(\lambda) = \exp\left[-\sum_{n=1}^{\infty} \sigma_n \lambda^n / n\right]. \quad (1.12)$$

From the definition (1.7),

$$D_n = \sum_{m=0}^{n} K^m d_{n-m} (= I d_n + K D_{n-1}). \quad (1.13)$$

Hence

$$D(\lambda) = d(\lambda)(1 + \lambda K + \lambda^2 K^2 + \cdots). \quad (1.14)$$

The relation between the Fredholm solution and the solution by iteration is brought out clearly by substituting (1.14) into (1.3). The Fredholm solution then reduces to

$$x = \frac{D(\lambda) y}{d(\lambda)} = \frac{d(\lambda)(1 + \lambda K + \lambda^2 K^2 + \cdots) y}{d(\lambda)}, \quad (1.15)$$

which is just the solution by iteration if the $d(\lambda)$ term is canceled. The Fredholm solution is exhibited as an analytic continuation of the iterated solution; the function $d(\lambda)$ is an integral function such that its zeros cancel the poles of the solution by iteration,[8] $(1+\lambda K+\lambda^2 K^2+\cdots)y$, thus making $D(\lambda)y$ also an integral function. Fredholm's work shows not only that such a function exists but provides a method for actually calculating it.

The proof of the convergence of the Fredholm solution is based on the inequality

$$|\sigma_n| \leqslant |C_1|^{n/2} \quad (n = 2, 3, \cdots). \quad (1.16)$$

The condition (1.2) does not require that σ_1 be finite, and in fact the solution is unaltered by replacing σ_1 by zero in (1.6) and (1.7), whatever its actual value. However, if σ_2 diverges the condition (1.2) cannot be satisfied.

It may happen that $C_1, \cdots, C_{m-1}$ are not bounded, but that C_m is, where, by definition

$$C_m \equiv \int\int |K^m(s, t)|^2 ds dt. \quad (1.17)$$

One can still obtain a convergent solution to the equation by first iterating m times. Thus, in operator notation,

$$x = y + \lambda K x$$
$$= (1 + \lambda K + \lambda^2 K^2 + \cdots + \lambda^{m-1} K^{m-1}) y + \lambda^m K^m x. \quad (1.18)$$

This equation has a solution, since

$$C_m < \infty,$$

namely,

$$x = \frac{D_m(\lambda) y}{d_m(\lambda^m)} = \frac{d_m(\lambda^m)}{d_m(\lambda^m)} \cdot (1 - \lambda^m K^m)^{-1} \times (1 + \lambda K + \cdots + \lambda^{m-1} K^{m-1}) y$$
$$= \frac{d_m(\lambda^m)}{d_m(\lambda^m)} \cdot (1 + \lambda K + \lambda^2 K^2 + \cdots) y, \quad (1.19)$$

where

$$d_m(\lambda^m) = \exp\left\{-\left[\lambda^m \sigma_m + \frac{\lambda^{2m}}{2} \sigma_{2m} + \cdots\right]\right\}. \quad (1.20)$$

[7] The solution by iteration is known as the Neumann-Liouville solution. This is the Born approximation in nonrelativistic scattering theory and is the "weak coupling" expansion of relativistic field theory.

[8] This result has been established quite rigorously in the literature. Just to clarify the structure of the Fredholm solution, we can cast $d(\lambda)$ in a different form by the following nonrigorous argument. This was pointed out to the authors by Professor R. E. Peierls.

The solution by iteration is the binomial expansion of

$$x = \frac{1}{1 - \lambda K} y.$$

The poles of this solution are given by the eigenvalues of the homogeneous equation

$$(1 - \lambda_\alpha K) x_\alpha = 0.$$

The eigenvalues of K^n are $(1/\lambda_\alpha)^n$, and the trace σ_n is thus

$$\sigma_n = \Sigma_\alpha (1/\lambda_\alpha)^n.$$

Hence, by Eq. (1.12),

$$d(\lambda) = \exp\left[-\sum_{n=1}^{\infty} \sum_\alpha \frac{1}{n}\left(\frac{\lambda}{\lambda_\alpha}\right)^n\right]$$
$$= \exp[\Sigma_\alpha \log(1 - \lambda/\lambda_\alpha)]$$
$$= \Pi_\alpha (1 - \lambda/\lambda_\alpha),$$

which is just the form required for $d(\lambda)$ to cancel the poles of the solution by iteration.

This solution is reducible, and one can factor an integral function from the numerator and the denominator. Poincare[9] has shown the irreducible solution is given by Eqs. (1.3)–(1.9) with $\sigma_1, \sigma_2, \cdots \sigma_{m-1}$ replaced by zero.

2. THE EXTERNAL FIELD

Consider the problem of the quantized Dirac field in interaction with a given external field, which is time dependent and may create pairs. The field produced by the electrons themselves is neglected. The S matrix for the system is

$$S=\sum_n \frac{(-i)^n}{n!}\int\cdots\int dx_1\cdots dx_n \times P(H^e(x_1),\cdots H^e(x_n)), \quad (2.1)$$

where

$$H^e(x)=-i\lambda\bar{\psi}(x)A^e(x)\psi(x), \quad (2.2)$$

$$A^e(x)=A^e{}_\mu(x)\gamma_\mu, \quad (2.3)$$

and λ is a constant which determines the strength of the field.

We consider the cross section for scattering of an electron from a state $u(q)$ to a state $u(p)$, where both momentum and spin are specified. The transition amplitude obtained from the S matrix is written as $(p|S|q)$. From Dyson's theory[3] applied to (2.1) it follows that

$$(p|S|q)=(0|S|0)(p|R|q); \quad (2.4)$$

where $(0|S|0)$ is the vacuum expectation value of the S matrix, represented by vacuum graphs, while $(p|R|q)$ is, by definition, represented by the connected graphs for the scattering process itself.[10] Because we are dealing with a time-dependent external field the vacuum-to-vacuum transition probability $|(0|S|0)|^2$ is not unity, and so $(0|S|0)$ is not a mere phase factor and has to be expressly considered.

To calculate $(p|R|q)$ define $M(p,q)$ such that

$$(p|R|q)=\bar{u}(p)M(p,q)u(q). \quad (2.5)$$

From the S matrix an integral equation for $M(p,q)$ can readily be derived. For weak external fields (λ small), where an expansion in field strength is permissible we have, by Feynman-Dyson methods,

$$M(p,q)=\lambda A^e(p-q)-2\pi\lambda^2\int A^e(p-k)S_F(k) \times A^e(k-q)d^4k, +\cdots, \quad (2.6)$$

where

$$S_F(k)=\frac{1}{2\pi i}\frac{i\gamma_\mu k_\mu-\kappa}{k^2+\kappa^2-i\epsilon}, \quad \epsilon>0. \quad (2.7)$$

[9] H. Poincare, Acta Math. **33**, 57 (1910).

[10] Dyson's treatment is a restatement in more conventional terms of Feynman's solution. $(p|S|q)$ is called the "absolute" probability amplitude and $(p|R|q)$ the "relative" probability amplitude by Feynman.

Write $M_I(p,q)$ for the right-hand side of (2.6) and define

$$(p|R_I|q)=\bar{u}(p)M_I(p,q)u(q). \quad (2.8)$$

Now $M_I(p,q)$ is the solution by iteration (hence the suffix I) of the integral equation[11]

$$M(p,q)=\lambda A^e(p-q)-2\pi\lambda\int A^e(p-k)S_F(k) \times M(k,q)d^4k, \quad (2.9)$$

which defines $M(p,q)$ for any strength of field. This is an equation for $M(p,q)$, (q constant) of the Fredholm type and we can consider the application of the theory of the previous section.[12]

The kernel of the equation is

$$K(p,k)=A^e(p-k)S_F(k)(-2\pi). \quad (2.10)$$

This is a measurable function for reasonable $A^e(p-k)$ provided the limit $\epsilon\to 0$ is not taken until the end of the calculation.[13]

The Fredholm traces σ_n can be looked at graphically, as stated in the introduction. For this particular integral equation, the σ's happen to be precisely the expressions represented by the vacuum graphs of the S matrix theory. Define

$$L=\sum_{n=1}^{\infty}\frac{\sigma_{2n}\lambda^{2n}}{2n} \quad (2.11)$$

so that the Fredholm determinant [Eq. (1.12)] is

$$d(\lambda)=\exp(-L). \quad (2.12)$$

From (2.11), L is the sum of all vacuum graphs taken singly.[14] With this definition of L, Feynman has shown that

$$(0|S|0)=\exp(-L), \quad (2.13)$$

[11] This equation has already been considered by M. Neuman, Phys. Rev. **83**, 1258 (1951); **85**, 129 (1952).

[12] Since $S_F{}^A(x,y)$, the electron propagator in a given field, satisfies

$$\left[\gamma_\mu\frac{\partial}{\partial x_\mu}-\kappa-ieA(x)\right]S_F{}^A(x,y)=-2i\delta(x-y),$$

$$S_F{}^A(x,y)=S_F(x-y)-\frac{e}{2}\int S_F(x-z)A(z)S_F{}^A(z,y)dz.$$

This is the same equation and can be solved by the same methods.

[13] We refer here to the standard procedure for calculating Feynman integrals for σ_n and K^n, in terms of which the answer is expressed [formulas (1.3)–(1.9)]. This procedure has been demonstrated for particular examples by R. P. Feynman [Phys. Rev. **76**, 769 (1949); see particularly the penultimate paragraph of Sec. 7] and discussed generally by R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952).

For $\epsilon>0$ the momentum integrals are all taken along the real axis. After completing these integrals, it is, in general, necessary to keep $\epsilon>0$ in order to define the analytic continuation of these functions round the branch points which occur at creation thresholds. Only after this can the limit $\epsilon\to 0$ be taken, to give the physical answer.

[14] Note that $\sigma_{2n+1}=0$ by Furry's theorem. The factor $(1/2n)$ in the nth term comes from $(2n!)^{-1}$ in (2.1) multiplied by $(2n-1)!$, the number of permutations of the $2n$ points round the closed loop of the graph for σ_{2n}.

and, therefore,[15]

$$(0|S|0)=d(\lambda). \tag{2.14}$$

Let us now consider the σ's for convergence. The expression for σ_2 is

$$\sigma_2=(2\pi)^2\int\int dpdk \times\mathrm{Tr}[A^e(p-k)S_F(k)A^e(k-p)S_F(p)]. \tag{2.15}$$

By a change of variable,

$$\sigma_2=(2\pi)^2\int dtA_\mu^e(t)A_\nu^e(-t)\int dk \times\mathrm{Tr}[\gamma_\mu S_F(k)\gamma_\nu S_F(t-k)]. \tag{2.15'}$$

The integral over k is the expression which occurs in the vacuum polarization and is divergent. (This is obvious from the graph of σ_2, which is the well-known loop of two electron lines.) Thus, σ_2 diverges for any choice of A^e. By (1.16) it follows that

$$C_1>\infty, \tag{2.16}$$

and Fredholm theory is not immediately applicable to the equation in this form. This difficulty can be overcome by the device of Poincare,[8] stated at the end of the previous section. Fredholm theory can be applied in a slightly modified form to our equation, provided[16]

$$C_2<\infty. \tag{2.17}$$

For the moment we will assume this condition to be satisfied and will show in the next section how all fields of physical interest can be included.

Assuming (2.17), the unique convergent solution for any strength of field is given by replacing σ_2 by zero in the general formulas of Sec. 1. By (1.3), (2.8), and (2.12) this is

$$(p|R|q)=\{\bar{u}(p)\Delta_q(\lambda,p)u(q)\}/\exp(-L'), \tag{2.18}$$

where

$$L'=\sum_{n=2}^{\infty}[\sigma_{2n}\lambda^{2n}/2n], \tag{2.19}$$

and Δ is defined by (1.7) applied to (2.10). The suffix q on Δ_q denotes that it is also a function of q. By (2.11) and (2.13)

$$(p|S|q)=\exp(-L)\{\bar{u}(p)\Delta_q(\lambda,p)u(q)\}/\exp(-L') =\{\bar{u}(p)\Delta_q(\lambda,p)u(q)\}\exp[-\sigma_2\lambda^2/2]. \tag{2.20}$$

The factor in curly brackets is essentially the resolvent of the Fredholm solution of (2.9) and is thus an integral function of λ. The σ_2 which occurs in the other term is infinite, but its real part is finite,[17] and it thus contributes a finite numerical factor to the probability $|(p|S|q)|^2$. Our final expression for $|(p|S|q)|^2$ is thus an infinite series in λ, each term of which is finite, and which is absolutely and uniformly convergent for any value of λ, that is, for any strength of field.

But according to the form of $\Delta(\lambda)$ given by (1.15),

$$\Delta_q(\lambda,p)=\exp(-L')M_I(p,q). \tag{2.21}$$

Thus, the Fredholm solution, Eq. (2.20), can be written

$$(p|S|q)=(p|R_I|q)\exp(-L). \tag{2.22}$$

This is precisely the solution by iteration, properly normalized, which is derivable from (2.1) by standard methods and was originally given by Feynman.[2] This is a rather surprising result, since the iteration $(p|R_I|q)$ has, in general, a finite radius of convergence, and from Feynman's point of view the term $\exp(-L)$ is introduced simply as a normalizing factor. This identity of the normalized iterated solution and the Fredholm solution is a special property of the system under discussion and is a direct consequence of the relation (2.14) between the Fredholm determinant and the vacuum expectation value of the S matrix. The identity is not true for static external fields, which, for reasons which will be made clear in the next section, cannot be regarded as a particular case of the time dependent field.

We remark finally that the above theory can equally well be applied to the calculation of pair creation. Also, since electrons interact with each other explicitly only through the exclusion principle on the initial and final states, we can consider any number of electrons or positrons. This has been shown by Feynman.[2]

3. FREDHOLM CONDITION AND THE STATIC FIELD

A. The Fredholm Condition

We have yet to establish the condition (2.17), which justifies the application of the Fredholm theory to (2.9). From (1.17) and (2.10),

$$C_2=\int dk_1dk_2dldm \times|A(l-k_1)S_F(k_1)A(k_1-m)S_F(m) \times A(l-k_2)S_F(k_2)A(k_2-m)S_F(m)|. \tag{3.1}$$

[15] This result was obtained previously by M. Neuman. See reference 11.

[16] We are actually employing a generalization of Poincare's result, which has been proved by numerous authors [F. Smithies (private communication)].

[17] This has been calculated directly by J. Schwinger and others. The result in our notation is quoted by R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950). That the real part is finite can be inferred, however, from the unitarity of the S matrix. Let

$$S=1+eS_1+e^2S_2+\cdots.$$

Since S is unitary,

$$S_2+S_2^*=-S_1S_1^*.$$

Therefore,

$$\mathrm{Rl}[2\sigma_2]=(0|S_2+S_2^*|0)=-\sum_{p,q}|(0|S_1|p,q)|^2,$$

where the right-hand side is the total probability of creating one pair, calculated to lowest order. This is finite.

It must be shown that this is convergent. It is not true of C_2 as it was of C_1, that by a change of variables one can produce a divergent factor, independent of $A(k)$. Dyson has shown that the convergence of such integrals can be estimated by counting powers. On this basis it can be seen that C_2 converges, provided $A(k)$ falls off faster than k^{-3} for large k.

Consider the expression

$$E=\int J_\mu(k)A_\mu(k)d^4k=\int k^2A_\mu(k)A_\mu(-k)d^4k, \quad (3.2)$$

where J_μ is the current producing the external field. By a Fourier transformation this is equal to the energy density of the field, integrated over all space and time.

Now the transition probability

$$\omega_\infty=|(p|S|q)|^2 \quad (3.3)$$

is the probability for the whole of time and the whole of space. For purely general fields this is infinite and there is, in fact, no physically significant cross section that one can consider for such a field. We consider now the space extension only. In most cases the field will be defined throughout space, but will be effectively nonzero only over a finite region. In such a case the probability for the whole of space gives the probability for the whole field, which is the physically significant quantity. Alternatively the field may have some general periodicity in a lattice with some cell volume v. One may then ask for the cross section per cell. This may be calculated by considering the probability for all space for a field equal to the given field over a large but finite volume $V(\gg v)$ and zero elsewhere. The boundary should be made smooth and taken large enough for surface effects to be negligible.

The treatment of the time is exactly analogous. Either the field is defined for all time in such a way that it is effectively nonzero only for a finite period (in this case ω_∞ is, directly, the physically significant expression) or it has a periodicity τ. In this latter case the probability per period τ can be calculated from ω_∞, for the field switched on for a long, but finite time $T(\gg\tau)$.

In any significant case, the field considered is nonzero only over a finite region and for a finite time. For such fields the energy integral E is finite. But this integral is only convergent if $A(k)$ falls off faster than k^{-3} for large k.[18a] Therefore, for any time-dependent field for which a physically reasonable cross section can be considered, we have

$$C_2<\infty, \quad (3.4)$$

and Fredholm theory is applicable.

B. The Static Field

For a static field the expression of physical significance is not ω_∞, but ω, the probability per unit time. To derive this exactly from the S matrix, Eq. (2.1), it is necessary to consider the field to be on for an infinite time. Since the argument above depended essentially on the fact that the field was only "on" for a finite time, the static field requires special consideration.

A static field is

$$A_\mu(k)=A_\mu(\mathbf{k})\delta(k_0), \quad (3.5)$$

where $\mathbf{k}$ is the space part of k. Energy is conserved in any real process, so that

$$|(0|S|0)|^2=1.$$

Thus, $(0|S|0)$ is simply an (infinite) phase factor and probability amplitudes are given directly by $(p|R|q)$. Define $M'(\mathbf{p},\mathbf{q})$ by

$$M(p,q)=M'(\mathbf{p},\mathbf{q})\delta(p_0-q_0). \quad (3.6)$$

The solution by iteration for $M'(\mathbf{p},\mathbf{q})$ and hence $(p|R_I|q)$ is obtained by substituting (3.5) and (3.6) into (2.6).[18] The integral equation for $M'(\mathbf{p},\mathbf{q})$ is, from (2.9),

$$M'(\mathbf{p},\mathbf{q})=\lambda A^e(\mathbf{p}-\mathbf{q})-2\pi\lambda^2\int A^e(\mathbf{p}-\mathbf{k})S_0(\mathbf{k}) \times M'(\mathbf{k},\mathbf{q})d\mathbf{k}, \quad (3.7)$$

where

$$S_0(\mathbf{k})=S_F(\mathbf{k},p_0) \quad (3.8)$$

is given by (2.7) with k_0 replaced by p_0. The Fredholm solution of (3.7) can be performed analogously to the solution of (2.10). The condition for the applicability of Fredholm theory is the convergence of the three-dimensional analog of (3.1). This can be deduced from the convergence of

$$\int|\mathbf{k}|^2A_\mu(\mathbf{k})A_\mu(-\mathbf{k})d\mathbf{k}. \quad (3.9)$$

This is the integral of the energy density over all space (but not time), which must converge for any physically reasonable field, as in the previous section.

From (3.6),

$$(p|R|q)=\bar{u}(\mathbf{p})M'(\mathbf{p},\mathbf{q})u(\mathbf{q})\delta(p_0-q_0)=(\mathbf{p}|\mathbf{R}|\mathbf{q})\delta(p_0-q_0). \quad (3.10)$$

Thus,

$$\omega_\infty=|(\mathbf{p}|\mathbf{R}|\mathbf{q})|^2\delta(p_0-q_0)\delta(0). \quad (3.11)$$

This is, of course, infinite, but the final factor is

$$\delta(0)=\lim_{T\to\infty}\lim_{x\to0}\frac{1}{2\pi}\int_{-T/2}^{T/2}e^{ixt}dt=\lim_{T\to\infty}(T/2\pi), \quad (3.12)$$

where T is the time for which the field is on. The probability per unit time is

$$\omega=\lim_{T\to\infty}(\omega_\infty/T)=\frac{1}{2\pi}|(\mathbf{p}|\mathbf{R}|\mathbf{q})|^2\delta(p_0-q_0). \quad (3.13)$$

[18a] This is not true if $A(x)$ is singular at the origin, but such cases may be included by introducing a suitable factor to smooth out the singularity.

[18] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).

Note that owing to the extraction of a δ-function factor in the derivation of (3.7) from (2.10), it is *not* true that the Fredholm determinant $d(\lambda)$ of (3.7) is equal to $(0|S|0)$. Consequently, for a static field, the iterated solution, $(\mathbf{p}|\mathbf{R}_I|\mathbf{q})$, (Born approximation) is *not* the same as the Fredholm solution. This is in agreement with the conclusion of Jost and Pais.[1]

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Spherically Symmetric Solutions in Nonsymmetrical Field Theories. I. The Skew Symmetric Tensor*

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The spherically symmetric form of the skew tensor $g_{\check{ik}}$, given by Papapetrou, is not sufficiently general. A more general form is found. This necessitates a reconsideration of the program of spherically symmetric solutions in nonsymmetrical field theories initiated by Papapetrou. The present paper makes a beginning in this direction.

The new form of the spherically symmetric tensor $g_{\check{ik}}$ is derived from a consideration of the infinitesimal rotation of a sphere about a diameter. It is hoped to use this form to obtain nonstatic solutions in nonsymmetrical field theories which will correspond to solutions of a radiating star in general relativity.

I. INTRODUCTION

Papapetrou[1] initiated the study of rigorous (nonapproximate) solutions in the various unified field theories by showing that the skew symmetric tensor $g_{\check{ik}}$ with only $g_{\check{14}}=w(r,t)$ and $g_{\check{23}}=v(r,t)\sin\theta$ as the surviving components, is spherically symmetric. This form of the tensor was the starting point for a number of investigations in this direction. Papapetrou himself worked out the solutions of the field equations of Schrödinger.[2] Rigorous solutions of the field equations of Einstein and Strauss[3] were given by Wyman.[4] Bandyopadhyaya[5] gave a simple solution of the latest unified field theory of Einstein.[6] Recently Bonner[7] has satisfactorily solved the problem of finding static spherically symmetric solutions in Einstein's unified field theory. All these investigations began with the form of $g_{\check{ik}}$ found by Papapetrou.

A nonsymmetrical tensor field can be split up into its symmetrical and skew symmetrical parts. We write

$$g_{ik}=g_{\underline{ik}}+g_{\check{ik}},$$

the bar or the hook below the suffixes distinguishing the two parts, respectively.

For spherically symmetric solution the form of $g_{\underline{ik}}$ is well known from general relativity. In order to find the spherically symmetric form of $g_{\check{ik}}$, Papapetrou considered the rotation of a sphere about a diameter POP' and compared the values of the various components of $g_{\check{ik}}$ before and after the rotation at the point P on the sphere and the axis of rotation. Since a rotation through a right angle will interchange the components perpendicular to OP, these would vanish at a point on the axis of rotation. Hence Papapetrou's method will naturally give the components of $g_{\check{ik}}$ along the radial direction only. That is why only $g_{\check{14}}$ and $g_{\check{23}}$ (which correspond to the radial components of magnetic and electric field, respectively) survived in his tensor. In what follows we consider an infinitesimal rotation of a sphere and compare the values of $g_{\check{ik}}$ at a point not on the axis of rotation. We shall, of course, recover Papapetrou's components; but we shall also find that there are some other components of $g_{\check{ik}}$ which are nonzero.

II. INFINITESIMAL ROTATIONS OF A SPHERE

In this section we shall be interested in the two-dimensional geometry of the surface of a sphere of radius a. The fundamental quadratic form ψ on this surface is given by

$$\psi=a^2d\theta^2+a^2\sin^2\theta d\varphi^2=g_{22}(dx^2)^2+g_{33}(dx^3)^2. \quad (2.1)$$

The contravariant components ξ^μ, $(\mu=2,3)$ of an infinitesimal transformation which would represent a motion of the sphere into itself, satisfy the following

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