Geometrical Characterization of Nuclear States and the Theory of Angular Correlations

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Degenerate states of nuclei, atoms, or radiations are identified by their tensor properties, that is, by the mean values of a set of multipole operators. A treatment is outlined which serves to construct angular correlation functions on a geometrical basis and provides a geometrical interpretation of known correlation formulas; the dynamics of specific nuclear reactions influences only a final averaging procedure.

THE various degenerate states of a nuclear level with nonzero spin differ in geometrical properties loosely called "spin orientation" or "spin polarization." This note points out a geometrical characterization of the degenerate states of nuclei, atoms, or radiations, which serves to keep the geometrical elements of angular correlations separate from the dynamical elements.

The observation of the direction in which a nucleus has emitted a radiation provides *some* information about the "spin polarization" of the nucleus following the emission. This information, in turn, determines the directional distribution of radiations subsequently emitted by the same nucleus. Thus the data on the geometry of the intermediate nuclear state constitutes the central element of the angular correlation of successive nuclear radiations.

The correlation effect is but one example of problems in which the information available on a state amounts to less than the maximum afforded by quantum mechanics. States identified by less-than-maximum information are not described by a wave function but by a density matrix or, in an equivalent but more immediately operational manner, by the mean values $\langle U_\alpha \rangle$ of a complete set of operators U_α.[1]

The observable characteristics of a degenerate nuclear state that depend on its "spin polarization" include its coupling with anisotropic external fields. The coupling energies are the mean values of irreducible 2^k-pole tensor operators T_{kq} which transform under space rotations like the spherical harmonics Y_{kq}. The matrix elements of such operators in the (jm) scheme, $(j'm'|T_{kq}|jm)$, are known to consist of two factors. One factor depends on the angular momentum quantum numbers j', j and on the multipole order k as well as on the "spin orientation" and "multipole orientation" numbers m', m, and q. This factor may be expressed as a standard Wigner coefficient[2] $(kjj'm'|kjqm)$ or $(-1)^{j-m}(j'jm'-m|j'jkq)$. Being the same function of the quantum numbers for all different nuclei, it constitutes a purely geometrical element. The remaining factor, independent of the "orientation" quantum numbers m', m, and q, indicated by $(j'\vdots T_k \vdots j)$ or $(j'\|T^k\|j)$,[3] depends on the kind and strength of the interaction and on the nuclear structure, i.e., is different for different nuclei with equal j's.

A tensor operator, for which the second factor of the matrix element equals 1 for a specified pair of quantum numbers $(J'J)$ and zero otherwise, is a universal function of quantum numbers and therefore a purely geometrical element. Consider now a degenerate energy level of a nucleus, atom or radiation, which may include states with different angular momentum quantum numbers J, $J'\cdots$. Define the tensor operators $U_{kq}^{(J'J)}$ by the matrices

$$(j'm'|U_{kq}^{(J'J)}|jm) = (-1)^{j-m}(j'jm'-m|j'jkq)\delta_{j'J'}\delta_{jJ}, \quad (1)$$

where k runs from $|J-J'|$ to $J+J'$, q from $-k$ to k, and $(J'J)$ take all pairs of values pertaining to the energy level. This set of operators is "complete" in the sense that any matrix $(j'm'|T|jm)$ can be represented as a linear combination of matrices (1) and, particularly, that the density matrix of any state A of the energy level is completely determined by the set of mean values $\langle U_{kq}^{(J'J)}\rangle^{(A)}$. The irreducible tensors $\langle \mathbf{U}_k^{(J'J)}\rangle^{(A)} \equiv (\langle U_{kk}^{(J'J)}\rangle^{(A)}, \langle U_{kk-1}^{(J'J)}\rangle^{(A)} \cdots)$ may be called "state multipoles," owing to their relationship with multipole moments and interactions. For example, the state of spin j with fully random spin orientation ("natural spin polarization") is identified by the multipoles $\langle U_{kq}^{(J'J)}\rangle = (2j+1)^{-\frac{1}{2}}\delta_{jJ'}\delta_{jJ}\delta_{k0}\delta_{q0}$. The factor δ_{k0} indicates that all multipoles vanish in this unpolarized state (except the scalar $\langle U_{00}^{(JJ)}\rangle$, whose magnitude represents a trivial normalization constant). For a state of paramagnetic polarization, at least the dipole $\langle \mathbf{U}_1^{(JJ)}\rangle$ must be $\neq 0$.[4]

A technique based on the concept of "state multipoles" has been applied to the theory of angular correlations,[5] but its detailed publication has lagged pending

[1] See, for example, the application by L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952), and especially the description of optical polarization by means of Stokes parameters, G. G. Stokes, Proc. Cambridge Phil. Soc. **9**, 399 (1852); U. Fano, J. Opt. Soc. Am. **39**, 859 (1949); D. L. Falkoff and J. E. Macdonald, J. Opt. Soc. Am. **41**, 861 (1951), which led to the ideas presented here.

[2] See, for example, E. U. Condon and G. Shortley, *Theory of Atomic Spectra* (Princeton University Press, Princeton, 1935), p. 71.

[3] G. Racah, Phys. Rev. **62**, 438 (1942), Sec. 3.

[4] For relativistic electrons and neutrinos a complete set of quantum numbers includes not only j and m but also $k=\pm(j+\frac{1}{2})$. Similarly for γ-rays one must add a quantum number of "electric" or "magnetic" multipolarity. Such quantum numbers must be treated like j's and one should actually write them alongside the j's, for example, $\langle \mathbf{U}_k^{(J'k'\,Jk)}\rangle$.

[5] National Bureau of Standards Report 1214 (1951).

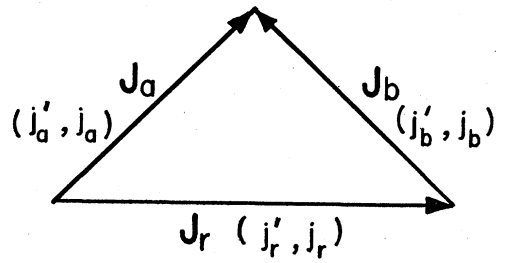

FIG. 1. Diagram of conservation of momentum and j quantum numbers.

improvements of the tensor operator algebra formalism in collaboration with G. Racah. An outline of the technique follows.

Consider the emission (or absorption) of radiation in the transition from a nuclear level a to a level b. The conservation of momentum in the transition is represented by the triangular diagram in Fig. 1. Each side of the triangle corresponds to one "reactant" (level a, level b, radiation). We also label it with a pair of quantum numbers $(j'j)$ which correspond to those of a set of state multipoles $\langle \mathbf{U}_k^{(j'j)} \rangle$. (When there is no degeneracy in j, i.e., when one reactant has a definite angular momentum j_0, there is only one pair of j's, namely, $(j_0 j_0)$.) The geometries of the three reactants are interdependent and the technique aims at expressing the "state multipoles" of any one of the three in terms of those of the other two. The usual situation is one in which (1) no information is available on the orientation of $\mathbf{J}_a$ so that

$$\langle \mathbf{U}_k^{(j_a' j_a)} \rangle^{(a)} = (2j_a+1)^{-\frac{1}{2}} \delta_{j_a' j_a} \delta_{k0},$$

and (2) the emitted radiation has been detected by a polarization analyzer P in a direction $\mathbf{q}$, which determines the state multipoles $\langle \mathbf{U}_k^{(j_r' j_r)} \rangle^{(qP)}$. Owing to the isotropic geometry of the state a, the state multipoles of b, $\langle \mathbf{U}_k^{(j_b' j_b)} \rangle^{(b)}$, are bound to be proportional to the radiation multipoles $\langle \mathbf{U}_k^{(j_r' j_r)} \rangle^{(qP)}$. However, the radiation geometry would impress itself in full detail onto the geometry of b only if the angular momenta $\mathbf{J}_r$ and $\mathbf{J}_b$ were parallel, i.e., if $j_a=0$. Otherwise the coupling through the randomly oriented $\mathbf{J}_a$ has a smearing effect which results in a "depolarization ratio"

$$\langle \mathbf{U}_k^{(j_b' j_b)} \rangle^{(b)} / \langle \mathbf{U}_k^{(j_r' j_r)} \rangle^{(qP)} \leqslant 1. \qquad (2)$$

This ratio may be described as the transformation coefficient from a coupling scheme $\{[(j_a j_a)0, (j_r' j_r)k]k\}$ to $\{[(j_a j_r')j_b', (j_a j_r)j_b]k\}$ and is equal, to within a normalization constant, to the Racah coefficient $W(j_r' j_r j_b' j_b, k j_a)$.[3]

In a process of emission of two or more successive radiations, the diagram of conservation of momentum (Fig. 2) can be broken down into triangles. The "reactants" at the outer contour of the diagram are amenable to geometrical observation, in principle, but usually a correlation is established between the observations on two radiations only. Starting from the triangle containing the first observed radiation, one can "resolve" each triangle in succession, as for the triangle in Fig. 1 above. In this manner one constructs an expression for the expected multipoles $\langle \mathbf{U}_k^{(j_2 j_2)} \rangle^{(\exp)}$ of the second radiation. These multipoles equal the multipoles $\langle \mathbf{U}_k^{(j_1 j_1)} \rangle^{(q_1 P_1)}$ of the first radiation, reduced by a depolarization ratio S_k which depends on the j's of all other "reactants." The detection of the second radiation by a detector $\mathbf{q}_2 P_2$ defines a different set of multipoles, $\langle \mathbf{U}_k^{(j_2 j_2)} \rangle^{(q_2 P_2)}$. The predicted probability of detection by this detector is essentially the projection of the "expected" multipoles onto the detector's multipoles, that is, the sum of multipole scalar products

$$\Sigma_k \langle \mathbf{U}_k^{(j_2 j_2)} \rangle^{(\exp)} \cdot \langle \mathbf{U}_k^{(j_2 j_2)} \rangle^{(q_2 P_2)} = \Sigma_k S_k \langle \mathbf{U}_k^{(j_1 j_1)} \rangle^{(q_1 P_1)} \cdot \langle \mathbf{U}_k^{(j_2 j_2)} \rangle^{(q_2 P_2)}. \qquad (3)$$

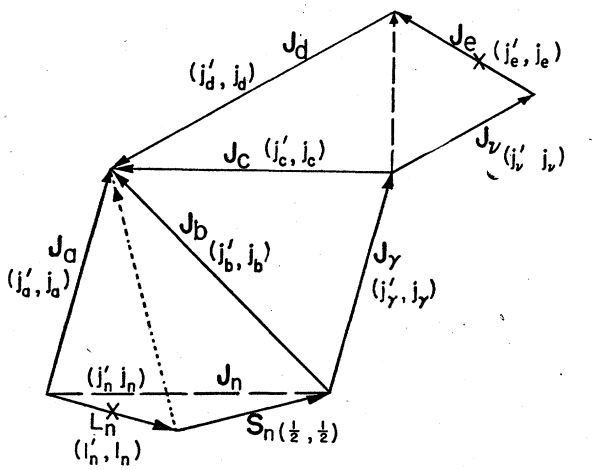

FIG. 2. Angular momentum diagram for the emission of a neutron in the transition from a to b, followed by an unobserved γ, from b to c, and by a β-process from c to d. "Reactants" whose geometry is observed are marked by a cross. The $\mathbf{J}_n$ vector is to be considered under conditions of strong spin-orbit coupling of the neutron, the dotted vector under conditions of strong spin-spin coupling with the nucleus. The spin-orbit coupling of electrons, neutrinos, and photons cannot be resolved, because of relativity (see reference 4).

This is a basic angular correlation function between $(\mathbf{q}_1 P_1)$ and $(\mathbf{q}_2 P_2)$, which depends only on quantum numbers j.[4] Therefore, it is a purely *geometrical* entity. This function pertains to a specified set of pairs of numbers $(j_a' j_a, j_1' j_1, j_b' j_b, \cdots)$ for each nuclear level and each radiation.

If various sets of j's have to be considered, a grand average must be performed over the corresponding correlation functions. Here is where the *dynamics* of the radiation emission by a specific nucleus enters into play; it determines the "weights" of the various elements in the average.

Formulas derived according to this procedure[5] are quite similar to formulas derived independently by F.

Coester and J. M. Jauch[6] and fully equivalent to those given by various authors, especially by Racah.[7]

The author is greatly indebted to G. Racah for a long series of discussions.

[6] F. Coester, Helv. Phys. Acta, **26**, 3 (1953). Personal communication in advance of publication is gratefully acknowledged.

[7] G. Racah, Phys. Rev. **84**, 910 (1951). The scalar product $\langle U_k^{(j_1 j_2)}\rangle^{(q_1 P_1)} \cdot \langle U_k^{(j_1 j_2)}\rangle^{(q_2 P_2)}$ corresponds to the $\Sigma_{\rho\sigma} D^{(k)}{}_{\rho\sigma} C_{k\rho}{}^* C_{k\sigma}$ in Racah's Eq. (8), the depolarization ratio S_k to the product of W's. See also the general treatments of the angular correlations by L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. (to be published); of the angular distribution of scattering and reaction cross sections by J. Blatt and L. C. Biedenharn, Revs. Modern Phys. **24**, 258 (1952); of the production of polarized particles in nuclear reactions by A. Simon and T. A. Welton, Phys. Rev. **89**, 886 (1953). The results of all these papers may be derived by the procedure outlined above.

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Nuclear Levels Associated with Zirconium 95 and Niobium 95†

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Using sources of zirconium 95, both as a fission product and as derived from neutron capture in enriched Zr^{94}, in magnetic spectrometers, a study has been made of the beta- and gamma-energies and of the half-lives associated with the radioactive decay. Zirconium 95 emits three beta-rays of energy about 910, 405, and 360 kev, each followed by a gamma-transition leading to a radioactive daughter product, niobium 95. The gamma-energies are 758, 725, and 235.2 kev. The niobium 95 decays by beta-emission, of energy 165 kev, to molybdenum 95 with accompanying gamma-energies of 753 and 768 kev. The observed transitions are found to fit a level scheme that is not incompatible with shell theory.

A LONG-LIVED radioactivity in zirconium, obtained as a fission product from uranium, was first observed[1] in 1940 by Grosse and Booth. Contemporary studies determined[2] the half-life of the activity to be 63 days and associated it with the isotope of mass 93. Subsequent investigations have shown that the activity is more likely in zirconium 95, which decays to radioactive daughter products in niobium 95. Many measurements have been made of the beta- and gamma-energies in the decay processes, with a rather wide divergence in the expressed values, as shown in Table I.

TABLE I. Previous data relative to Zr^{95} and Nb^{95}.

Item		Half-life	Energy in Mev PE[a]	RW[b]	MSK[c]	HL[d]	N[e]	F[f]	Z[g]
Zr^{95}	β_1	65 days	0.80			0.887	1.00		
	β_2		0.29			0.400	0.39		
	γ_1		0.85		0.91		0.92		
	γ_2					0.708	0.73		0.73
Nb^{95}	γ_3	90 hr				0.216	0.23		
Nb^{95}	β_1	35 days	0.14		0.14	0.146	0.15	0.163	
	γ_1		0.78	0.75	0.92	0.758	0.77	0.771	0.76

[a] M. Pool and J. Edwards, Phys. Rev. **67**, 60 (1945).
[b] W. Rall and R. Wilkinson, Phys. Rev. **71**, 321 (1947).
[c] Mandeville, Scherb, and Keighton, Phys. Rev. **74**, 888 (1948).
[d] J. Hugdens and W. Lyon, Phys. Rev. **75**, 206 (1949); *Radiochemical Studies: The Fission Products* (McGraw-Hill Book Company, Inc., New York, 1950), Paper No. 90.
[e] V. Nedzel, National Nuclear Energy Series, Plutonium Project Record, Vol. 9, Div. IV and earlier papers in Plutonium Project Reports.
[f] C. Fan, Phys. Rev. **81**, 300 (1951).
[g] Zucker, Mandeville, Shapiro, Mendenhall, and Conklin, Bull. Am. Phys. Soc. **28**, No. 1, 58 (1953).

† This investigation received the joint support of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.

[1] A. V. Grosse and F. Booth, Phys. Rev. **57**, 664 (1940).
[2] Sagane, Kojima, Miyamoto, and Ikawa, Phys. Rev. **57**, 1179 (1940).

In the present investigation specimens of Zr^{95} have been obtained both as a fission product and as produced in the pile by neutron capture in enriched (92 percent) zirconium 94. The gamma-energies have been determined from electron lines derived from internal conversion and from photoemission in lead radiators. Photographic magnetic spectrometers of high resolution have been employed and it is believed that the reported results are accurate to plus or minus 0.2 percent. The decay of Zr^{95} has been followed for more than a year and its half-life appears to be 65.2±1 days. A metastable state of niobium 95 with a half-life of 90

TABLE II. Electron energies associated with Zr^{95} and Nb^{95}.

Electron energy, kev	Interpretation	Energy sum, kev
216.1	*K* (41)	235.1
232.6	*L* (41)	235.3
235.0	*M* (41)	235.5
706.3	*K* (41)	725.3
722.7	*L* (41)	725.4
733.0	*K* (42)	753.0
739.6	*K* (41)	758.6
748.6	*K* (42)	768.6
765.8	*L* (42)	768.7
678	*K* (Pb)	766

TABLE III. Gamma energies due to transitions in Nb^{95} and Mo^{95}.

Nucleus	Energy, kev	*K*/*L* ratio
Nb^{95}	235.2	4.5±0.6
	725	5.0±1.0
	758	—
Mo^{95}	753	—
	768	7.6±0.6