The Effect of the Finite deBroglie Wavelength in the Theory of Beta-Decay*

M. E. Rose and C. L. Perry
Oak Ridge National Laboratory, Oak Ridge, Tennessee
(Received January 19, 1953)

An accurate numerical evaluation of the Z and energy dependent parts of the correction factors for allowed and forbidden beta-transitions has been carried out. The range of parameters considered is: 88 values of Z (from 10 to 96 inclusive in steps of 2 and for positrons ($Z<0$) as well as electrons ($Z>0$)); 33 values of p, the electron momentum, in mc units ($0.1\leqslant p\leqslant 25$) for $|Z|$ through 24 and 31 values of p ($0.1\leqslant p\leqslant 15$) for $|Z|>24$; four values of j (the electron momentum ($\frac{1}{2}\leqslant j\leqslant 7/2$). For each value of Z, p, and j the six wave-function combinations (L_ν, M_ν, N_ν, P_ν, Q_ν, R_ν) needed for the general mixed beta-interaction are obtained. For the given range of j, analysis of beta-spectra can be carried out through third forbidden transitions. The results given here show that the theoretically allowed spectra should be modified by an energy-dependent factor. For not-too-low maximum energies, this modification in the shape of the allowed beta-spectrum may be observable with presently attainable precision.

A NUMBER of recent developments in the investigation of beta-spectra has made it advisable to make provision for accurate analyses of the experimental results. First, the development of experimental techniques whereby very precise measurements of shapes of beta-spectra become available and, at the same time, the fact that certain forbidden spectra deviate only slightly from the allowed shape, makes it almost imperative that the calculated spectra be highly accurate. For allowed spectra the requisite numerical information has been provided by Feister[1] and others.[2] The second fact which emerges from recent work is simply that the determination of the nature of the beta-interaction will very likely depend, in part, on the analysis of forbidden unfavorable parity change transitions.[3] The analysis of such spectra is, of course, greatly facilitated by the availability of the numerical results described below. Finally, there is now strong evidence that the beta-interaction is mixed (linear combination of invariants). Hence, analyses of beta-spectra should, in general, take into account the cross terms which arise.[4] In the numerical results discussed here these are included.

As is well-known, the beta-spectrum is represented in terms of a correction factor. Following the customary definition,[5] the correction factor C_{nX} is defined so that C_{nX}/C_{0X} is the ratio of nth forbidden to allowed energy distributions. For pure interactions ($X=S, V, T, A$, or P) the correction factors have been given by Greuling,[5] while for the cross terms arising from mixed interactions the corresponding factors have been given by Pursey[4] and independently by Smith.[4] In each case these correction factors depend bilinearly on nuclear matrix elements and each matrix element product is multiplied by a sum of functions of the form

$$S_n(p,Z)=q^{2n+\epsilon}\sum_{\nu=0}^{n}a_{n\nu}q^{-2\nu}E_\nu(p,Z). \tag{1}$$

Here ϵ and $a_{n\nu}$ are constants, $q=W_0-W$ is the neutrino energy, and p the electron momentum. The $E_\nu(p, Z)$ are essentially bilinear combinations of electron wave functions evaluated at the nuclear radius. Finally $\nu=j-\frac{1}{2}$, where j is the electron angular momentum. For pure interactions there are, for each ν, three different combinations which are denoted by L_ν, M_ν, and N_ν, to use the customary notation.[5] For mixed interactions three additional combinations occur: P_ν, Q_ν, and R_ν in Pursey's notation.[4] All these quantities are defined explicitly in the references given above.

It is clear that by far the most convenient procedure is to obtain numerical values for the various E_ν (L_ν, M_ν, N_ν, P_ν, Q_ν, R_ν) since these depend only on three parameters. From these it is a comparatively simple matter to obtain the requisite S_n. Analytical expressions, obtained from the first term or two in the power series expansion of the radial wave functions have been given for the E_ν.[4,5] The validity of this procedure rests on the fact that the electron (or positron) deBroglie wavelength is large compared to the nuclear radius.[6] Exami-

* This paper is based on work performed for the U. S. Atomic Energy Commission at the Oak Ridge National Laboratory.

[1] I. Feister, Phys. Rev. **78**, 375 (1950). See also *Tables for the Analysis of Beta Spectra*, National Bureau of Standards, Applied Mathematics Series, No. 13 (1952).

[2] Dismuke, Rose, Perry, and Bell, Oak Ridge National Laboratory Report No. 1222 (unpublished).

[3] C. S. Wu, Amsterdam Conference on Beta- and Gamma-Radioactivity (1952), to appear in Physica.

[4] D. L. Pursey, Phil. Mag. **42**, 1193 (1951); A. M. Smith, Phys. Rev. **82**, 955 (1951).

[5] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941); E. Greuling, Phys. Rev. **61**, 568 (1942).

[6] The precise statement for the validity of the expansions referred to is $p\rho\ll 1$ (where ρ is the nuclear radius, in units $\hbar/mc$) but $\alpha ZW/p$ is not large compared to unity. Here $W=(p^2+1)^{\frac{1}{2}}$ is the electron energy. Even for $p\to 0$ one cannot replace the confluent hypergeometric functions which occur in the radial wave functions [see M. E. Rose, Phys. Rev. **51**, 484 (1937)], by unity—that is, by the first term in their series representation. In this limit the radial wave functions can be expressed in terms of Bessel functions (of irrational index) and argument $2(2\alpha Z\rho)^{\frac{1}{2}}$. For $\rho\simeq\frac{1}{2}\alpha A^{\frac{1}{3}}$ and $\alpha Z\ll 1$ this argument is small enough so that the Bessel functions can be replaced by the first term in their series expansion. This procedure corresponds to the expansions given by Greuling (reference 5) and Pursey (reference 4) for $p\to 0$. In view of these remarks it is understandable that an accurate calculation will deviate considerably from the approximate ones not only for large p but if αZ is not small, for small p as well. This is borne out by the results for L_0, for example, see Fig. 1 below, where the ap-

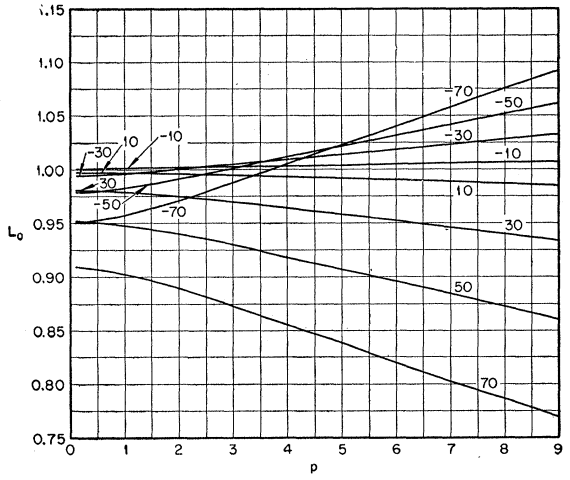

FIG. 1. L_0, on a linear scale, *versus* the momentum p. The numbers attached to the curves designate the values of Z (>0 for electrons, <0 for positrons). When L_0 is plotted *versus* energy W, the curves are very nearly, but not exactly, linear.

nation of these approximate expressions has shown that they are fairly good (at least for $\nu \leqslant 3$) in most practical cases. Usually an error of, at most, 2 percent is incurred, the error being determined simply by comparison with the accurate numerical results discussed below. However, there are exceptional cases. For example, for positrons and small Z ($Z \sim 10$) the M_ν, for all ν considered, as given by the approximate expressions were totally unreliable for momenta in the range $p \gtrsim 9$. In fact, for $p \gtrsim 10$ the M_ν as calculated in this way turn

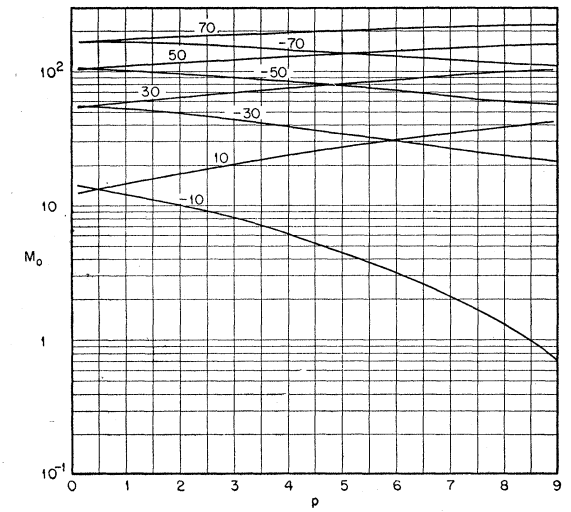

FIG. 2. M_0, on a logarithmic scale, *versus* p. The energy dependence of M_0, even for large Z, is evident.

proximate expressions would give $L_0=\frac{1}{2}[1+(1-\alpha^2Z^2)^{\frac{1}{2}}]$ independent of energy and independent of the sign of Z. A more appropriate designation of the effect considered here is to describe it as an effect of the nonvanishing nuclear radius. We have adopted the designation "finite wavelength effect" in order to avoid confusion with an entirely separate effect arising from the nonvanishing nuclear radius [see M. E. Rose and D. K. Holmes, Phys. Rev. **83**, 190 (1951)].

out to be negative, over a range of p, whereas these quantities are positive definite by definition. Another exception, in which the error is not so great but is nevertheless of special importance, is discussed below.

This fact led us to the conclusion that the only safe way to obtain the required numerical results was to calculate the radial functions with an essentially exact procedure. The procedure involved the use of the series representation of the wave functions, which converges for all finite values of the argument. Representing the series by

$$\sum_{n=0}^{\infty} (\mathrm{Re}t_n + i\mathrm{Im}t_n)$$

the terms of this series were computed up to a value $n=N$ for which $|\mathrm{Re}\, t_N| + |\mathrm{Im}\, t_N| < 2\times 10^{-7}$. We estimate the error in the confluent hypergeometric series to be of order 10^{-8}. The work was carried out on the

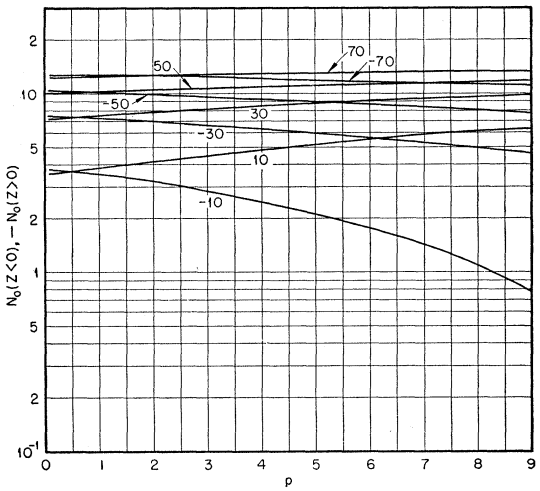

FIG. 3. N_0, on a logarithmic scale, *versus* p. For $Z<0$, N_0 is given and for $Z>0$, the ordinate is $-N_0$.

Whirlwind I, the general purpose digital computer at the Digital Computer Laboratory of Massachusetts Institute of Technology.

The range of parameters for which the six E_ν functions were computed was $|Z|=10$ to 96 in steps of 2; 33 values of p from 0.1 to 25 for $|Z| \leqslant 24$ and 31 values from 0.1 to 15 for $|Z|>24$, and finally $\nu=0$, 1, 2, and 3. The nuclear radius is taken to be $\frac{1}{2}\alpha A^{\frac{1}{3}}$, and the mass number A is chosen for each Z so that it represents an average of known electron or positron emitters. Space limitations prevent anything but a fragmentary presentation of the results here.[7] Figures 1–6, wherein the six functions are given for $\nu=0$ and four values of Z, show some of the results. We have here given results for $p \leqslant 9$, but the tables extend beyond this point.

It will be noted that in the beta-decay theory as it

[7] Tables of complete numerical results will be found in Rose, Perry, and Dismuke, Oak Ridge National Laboratory Report ORNL 1459 (unpublished).

has hitherto been used the correction factor for allowed transitions, which is L_0 for any interaction or mixture, is given by $L_0=\frac{1}{2}[1+(1-\alpha^2Z^2)^{\frac{1}{2}}]$ which is, of course, energy independent. (This also applies for first forbidden transitions which have the allowed shape due to predominance of matrix elements $\int\alpha$, $\int\gamma_5$.) However, as Fig. 1 shows L_0 is not quite constant. Therefore some deviation from the usual theoretical allowed shape should be expected, the size of this deviation depending, of course, on the endpoint energy.

To take an extreme case, for $Z=70$ and a maximum momentum of 9 the factor by which the spectrum should be multiplied changes by 15.5 percent of its zero momentum value over the range $0\leqslant p\leqslant 9$. For electron emission, the effect of the present correction is to increase the relative number of slow electrons, and for positron emission the effect goes in the opposite direction. This, as Fig. 1 shows, is valid for all allowed-shape transitions. The effect is somewhat reduced by the fact that one cannot measure the spectrum at low energies because of finite source thickness and backing. However, this is a trivial influence for thin sources for which one can measure down to about 150 kev or less and does not appreciably modify the above result for thicker sources. A more important limitation arises when the spectrum is complex and only the upper end of the spectrum can be seen. Thus, for the example cited above between $p=5$ and $p=9$ say, the over-all change is 8.4 percent of the low energy counting rate. Considering now the effect on the Kurie-Fermi plot, what one plots with the customary procedure is proportional to $L_0^{\frac{1}{2}}(W_0-W)$ rather than to W_0-W. Let us write $L_0=1-\delta$ by normalizing L_0 to unity at the lowest energy for which one can expect a straight Kurie-Fermi plot. That is, $\delta=0$ at $p=p_{\min}$. Also write $G=L_0^{\frac{1}{2}}(W_0-W)$ and $G_0=W_0-W$. Then we find that the maximum relative deviation $(G-G_0)/G_0$, between the ordinary Kurie-Fermi plot (that is, the one based on the assumption $L_0=$constant) and the true one, with the L_0 given here, is of order $\frac{1}{4}\delta_{\max}$. Hence in the above examples, the effect discussed here is of order 4 percent when most of the spectrum can be measured and of order 2 percent when about half of the spectrum can be measured. Of course, for lower maximum energies and/or lower values of Z the effect decreases.

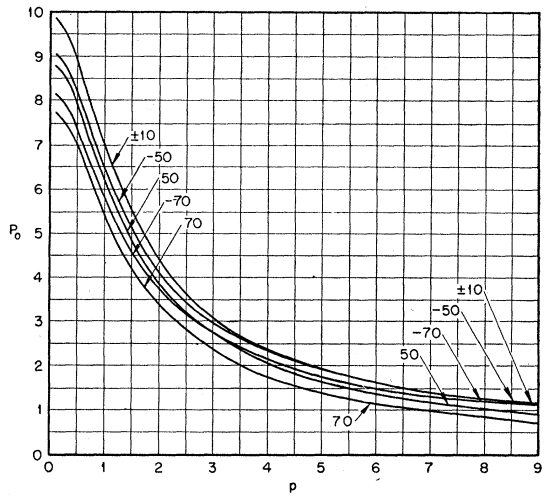

FIG. 4. P_0, on a linear scale, as function of p.

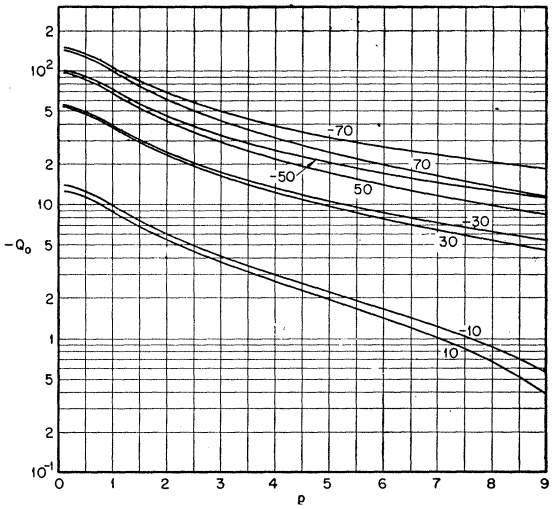

FIG. 5. $-Q_0$, on a logarithmic scale, as function of p.

At low energies there is the additional effect of screening to consider and it is noteworthy that the present correction is in the opposite direction to the screening correction for both electrons and positrons.[8] For low energies the screening correction varies more rapidly with energy than does the present correction. However, even after the screening correction is made, the experimental Kurie-Fermi plots show too many

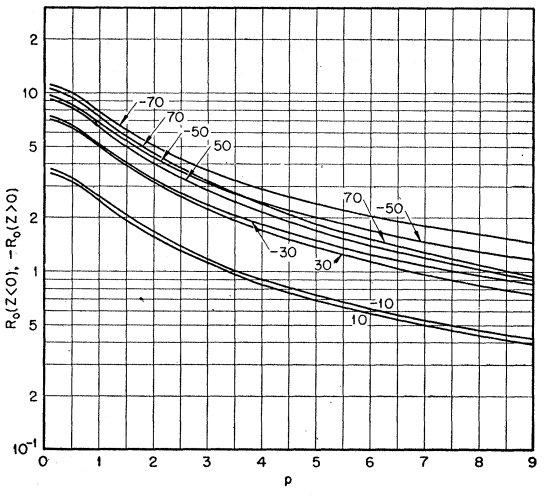

FIG. 6. R_0, on a logarithmic scale, as function of p. The ordinate is $-R_0$ for $Z>0$ and R_0 for $Z<0$.

[8] See, for example, J. R. Reitz, Phys. Rev. **77**, 10 (1950).

slow electrons.[9] The present effect is in the right direction to explain part of this but the major part must still be attributed to source and backing-distortion. This is abundantly clear if we consider positrons where the finite wavelength effect depresses the number of slow positrons while the experiments show too many slow positrons even after screening corrections are made. Of course, for high energies the screening effects are much less important than those due to finite wavelength.

From this discussion it appears that the present results are not in disagreement with any known spectra which have been classified as allowed transitions (or as first forbidden with allowed shape). There are no known well-investigated cases of allowed shape spectra with large Z and reasonably high end point to provide a comparison of experiment and these calculations. However, it is entirely reasonable to expect that the effects discussed here can be detected with present experimental technique and they would clearly be of decisive importance for the classification of transitions involving large Z and W_0.

Turning now to the case of favorable parity change transitions, that is, those with spin change equal to $n+1$ (GT selection rules) the correction factor in these cases involves L_0. We have therefore examined the modification of the Kurie-Fermi plot for $Y^{90}(W_0\approx 5.50)$ which corresponds to about as high a Z and W_0 as is encountered in this type of transition. Moreover, reasonable thin sources were used in the measurements of this spectrum. In this case the correction involved here produces a linear plot over a somewhat greater energy range and does not change the conclusion that the transition is first forbidden. In the case of other known transitions with a unique matrix element similar conclusions should result.

Finally, it is fairly evident that the approximation which has hitherto been inherent in the beta-decay theory could not have led to any appreciable falsification of the endpoint energies as determined by extrapolation of the "straight" Kurie-Fermi plots.

It is a pleasure to acknowledge the courtesy of J. W. Forrester, J. W. Carr, III, and C. W. Adams of the Digital Computer Laboratory, who authorized and expedited the computations. We are also indebted to D. Combelic of the Digital Computer Laboratory and to members of the Mathematics Panel of the Oak Ridge National Laboratory for coding and checking the computations.

[9] For example, C. S. Wu and R. D. Albert, Phys. Rev. **75**, 1107 (1949).

PHYSICAL REVIEW VOLUME 90, NUMBER 3 MAY 1, 1953

Effect of a Short-Range Repulsion on Low Energy Singlet Nucleon-Nucleon Scattering*

M. H. Hull, Jr., and A. Herschman
Yale University, New Haven, Connecticut

(Received January 15, 1953)

Modifications needed in order that low energy singlet scattering be described when a repulsive core is introduced in a phenomenological nucleon-nucleon interaction are investigated. It is found that if a square well is used to represent the attractive part of the interaction, a range of potential parameters allows a description of the data and also satisfies the charge independence hypothesis. When the Yukawa potential is investigated, the assumption of charge independence is more restrictive, and a single set of parameters is found.

I. INTRODUCTION

Jastrow[1] introduced a short range repulsive core into a phenomenological nucleon-nucleon interaction in order to allow a charge independent description of high energy scattering experiments. At low energies, the theoretical interpretation of scattering experiments had indicated charge independence of nuclear forces at an early date[2] without a repulsive core in the assumed interaction. The question arises as to what modification of the phenomenological interaction used to describe low energy scattering is necessary when a core is introduced. It is the purpose of this note to report on an investigation of this point.

The low energy singlet nucleon-nucleon scattering has been described by the f function of Breit, Condon and Present,[2] where

$$f=(C_0^2/\eta)\cot K_0-2\ln\eta+q_0/\eta$$

in the notation of BCP. The quantity K_0 is the phase shift due to the specifically nuclear interaction. For the limit of the charge e going to zero, one has[3]

$$\lim_{e\to 0}(f/a)=k\cot K_0,$$

* Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.

[1] R. Jastrow, Phys. Rev. **81**, 165 (1951).

[2] Breit, Condon, and Present, Phys. Rev. **50**, 825 (1936) (referred to as BCP); G. Breit and J. R. Stehn, Phys. Rev. **52**, 396 (1937). For other early evidence not depending exclusively on scattering, see G. Breit and E. Feenberg, Phys. Rev. **50**, 850 (1936) and references cited by them.

[3] G. Breit and M. H. Hull, Jr., Am. J. Phys. **21**, 184 (1953).