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The Spins of Li^6 and B^{10} in the Shell Model

NISSAN ZELDES
Department of Physics, Hebrew University, Jerusalem, Israel
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The intermediate coupling approximation succeeds in explaining the different spins of Li^6 and B^{10}, where the LS coupling and the jj coupling fail.

INTRODUCTION

THE calculations of the energy levels of Li^6 and B^{10} in the p^n shell model have shown that in LS coupling the ground states of these nuclei belong to $J=1$, and in jj coupling to $J=3$.[1] The experimental spins are 1 for Li^6 and 3 for B^{10}. It seemed probable, therefore, that an intermediate coupling approximation would succeed,[2] where the extreme couplings failed.

In order to see this, the matrices of the binding energy of these two nuclei for $J=1$ and for $J=3$ had to be calculated in this approximation, and the levels with highest binding energy compared.

In the present model and approximation the energy, apart from the central field energy which is common to all the configuration and which can be overlooked, therefore, when comparing the levels, consists of two parts considered together as a perturbation on the central field energy: the energy of the nuclear forces proper, which is taken as the sum of the interaction energies between all pairs of the nucleons, and the spin-orbit interaction which is written analogously to the atomic case. The elements of the energy matrix will therefore be linear combinations of three parameters, F_0, F_2, and ζ_p, where the F's are Slater's generalized parameters and ζ_p is the spin-orbit parameter. Therefore, the energy values will be, in appropriate units, functions of two ratios only.

If the binding energies are described by surfaces in 3-dimensional space, the experimental spins will show that for ζ_p very large as compared with the F's the surface for $J=3$ will be higher than that for $J=1$, or $E_3>E_1$, while for ζ_p small the contrary happens. There is, therefore, a line in the finite part of the plane, where $E_1=E_3$. And in order that both spins will be accounted for by the theory, the line for Li^6 should be on the side of the higher ζ with respect to that for B^{10}, at least for some values of F_2/F_0.

In the case of Li^6 the matrix is of order three for $J=1$ and reduces to one element for $J=3$; therefore, one can find the equation of the curve in terms of the above ratios of the parameters explicitly. In B^{10} the matrices are of the tenth order, and so an explicit equation cannot be found; but from the equation for Li^6, the value of the critical ζ_0, for which $E_1=E_3$ (in terms of F_0 and F_2), has to be substituted for ζ in the matrices of B^{10}, the eigenvalues of which may be expressed, in arbitrary units, as functions of the ratio F_2/F_0. Varying this ratio one obtains each time two numerical matrices the eigenvalues of which are to be compared.

Between which values is the ratio to be varied? Owing to the rarity of experimental material, very little is known of Slater's nuclear parameters. They are known to be positive by definition, and the ratio F_2/F_0 is known to depend upon the range of the nuclear forces, being small for long-range forces, and reaching the value 0.2 for δ-interaction. As the range of the nuclear forces is only little known, we explored all the interval between 0 and 0.2.

THE INTERMEDIATE COUPLING APPROXIMATION

1. The Energy Matrices

Rosenfeld shows[3] that, if the interaction between two nucleons is given by a linear combination of the interactions of Wigner, Majorana, Bartlett, and Heisenberg, with the same distance dependence in each, the coefficients of the combination are approximately determined by the saturation requirements of the nuclear forces. The combination which results is

$$V=-\frac{2}{15}V_W+\frac{14}{15}V_M+\frac{7}{15}V_B-\frac{4}{15}V_H, \tag{1}$$

and we shall suppose this form for the nuclear potential. The part of the energy due to the nuclear forces is then found from the tables of Racah.[4]

The spin-orbit interaction has the form

$$\zeta_p\Sigma_i(\mathbf{l}_i\cdot\mathbf{s}_i), \tag{2}$$

and the calculation reduces to that of $\Sigma_i(\mathbf{l}_i\cdot\mathbf{s}_i)$. A general formula for the matrices of such operators has been given by Racah.[5] In the practical utilization of it we used further the property of the coefficients of fractional parentage to decompose into factors, each of which depends only on part of the quantum numbers specifying the states.[6] These factors had already been calculated by Racah,[7] and were used by us for the

[1] E. Feenberg, Phys. Rev. **76**, 1275 (1949).
[2] G. Racah and N. Zeldes, Phys. Rev. **79**, 1012 (1950).
[3] L. Rosenfeld, *Nuclear Forces* (Interscience Publishers, New York, 1948), p. 234.
[4] G. Racah, Helv. Phys. Acta **23**, 229 (1950).
[5] G. Racah, Phys. Rev. **63**, 367 (1943), Eq. (23).
[6] G. Racah, Phys. Rev. **76**, 1352 (1949); Princeton Notes, 1951, p. 66 (unpublished).
[7] G. Racah (private communication). Mean while extensive tables of these coefficients were published by H. A. Jahn in Proc. Roy. Soc. (London) **205**, 192 (1951).

TABLE I. The matrix of $\Sigma_i(\mathbf{l}_i\cdot\mathbf{s}_i)$ of p^6 in the LS scheme.

(a) $60\ \Sigma_i(\mathbf{l}_i\cdot\mathbf{s}_i)$ for $J=1$

ΣTSL	(222) ^{13}S	(321) ^{15}P	(321) ^{15}D	(321) ^{13}P	(321) ^{13}D	A^{a} ^{11}P	B^{a} ^{11}P	(420) ^{13}S	(420) ^{13}D	(420) $^{13}D'$
(222) ^{13}S	0	$30(3)^{\frac12}$	0	$50(3)^{\frac12}$	0	0	0	0	0	0
(321) ^{15}P	$30(3)^{\frac12}$	0	$15(3)^{\frac12}$	0	$-3(15)^{\frac12}$	0	0	-50	$2(35)^{\frac12}$	0
(321) ^{15}D	0	$15(3)^{\frac12}$	0	$45(3)^{\frac12}$	0	0	0	0	0	-30
(321) ^{13}P	$50(3)^{\frac12}$	0	$45(3)^{\frac12}$	0	$-(15)^{\frac12}$	$-60(2)^{\frac12}$	0	-30	$-6(35)^{\frac12}$	0
(321) ^{13}D	0	$-3(15)^{\frac12}$	0	$-(15)^{\frac12}$	0	0	$12(30)^{\frac12}$	0	0	$18(5)^{\frac12}$
A ^{11}P	0	0	0	$-60(2)^{\frac12}$	0	0	0	0	0	$30(6)^{\frac12}$
B ^{11}P	0	0	0	0	$12(30)^{\frac12}$	0	0	$40(2)^{\frac12}$	$2(70)^{\frac12}$	0
(420) ^{13}S	0	-50	0	-30	0	0	$40(2)^{\frac12}$	0	0	0
(420) ^{13}D	0	$2(35)^{\frac12}$	0	$-6(35)^{\frac12}$	0	0	$2(70)^{\frac12}$	0	0	$6(105)^{\frac12}$
(420) $^{13}D'$	0	0	-30	0	$18(5)^{\frac12}$	$30(6)^{\frac12}$	0	0	$6(105)^{\frac12}$	0

(b) $210\ \Sigma_i(\mathbf{l}_i\cdot\mathbf{s}_i)$ for $J=3$

ΣTSL	(222) ^{17}S	(321) ^{15}P	(321) ^{15}D	(321) ^{13}D	A ^{11}F	B ^{11}F	(420) ^{13}D	(420) $^{13}D'$	(420) ^{13}F	(420) ^{13}G
(222) ^{17}S	0	$210(2)^{\frac12}$	0	0	0	0	0	0	0	0
(321) ^{15}P	$210(2)^{\frac12}$	0	$105(2)^{\frac12}$	$-63(15)^{\frac12}$	0	0	$42(35)^{\frac12}$	0	0	0
(321) ^{15}D	0	$105(2)^{\frac12}$	0	0	0	0	0	$-70(6)^{\frac12}$	$70(3)^{\frac12}$	0
(321) ^{13}D	0	$-63(15)^{\frac12}$	0	0	0	$-42(30)^{\frac12}$	0	$-42(5)^{\frac12}$	$-42(10)^{\frac12}$	0
A ^{11}F	0	0	0	0	0	0	0	$70(6)^{\frac12}$	$-70(3)^{\frac12}$	0
B ^{11}F	0	0	0	$-42(30)^{\frac12}$	0	0	$18(70)^{\frac12}$	0	0	$-90(7)^{\frac12}$
(420) ^{13}D	0	$42(35)^{\frac12}$	0	0	0	$18(70)^{\frac12}$	0	$-14(105)^{\frac12}$	$16(210)^{\frac12}$	0
(420) $^{13}D'$	0	0	$-70(6)^{\frac12}$	$-42(5)^{\frac12}$	$70(6)^{\frac12}$	0	$-14(105)^{\frac12}$	0	0	0
(420) ^{13}F	0	0	$70(3)^{\frac12}$	$-42(10)^{\frac12}$	$-70(3)^{\frac12}$	0	$16(210)^{\frac12}$	0	0	$-45(21)^{\frac12}$
(420) ^{13}G	0	0	0	0	0	$-90(7)^{\frac12}$	0	0	$-45(21)^{\frac12}$	0

[a] $\psi(A)=[\psi(411)+\psi(330)]/(2)^{\frac12}$, $\psi(B)=[\psi(411)-\psi(330)]/(2)^{\frac12}$.

calculation of the spin-orbit matrices for the charge singlets of p^6 with $J=1$ and $J=3$, which are given in Tables I(a) and I(b). The states are labeled by Σ, the partition specifying Wigner's supermultiplet to which the state belongs, and by $2T+1$, $2S+1$ and L, T and S being the isotopic and ordinary spin numbers and L the orbital quantum number. $\alpha=\frac{1}{2}\zeta_p$ is the positive spin-orbit parameter. For p^2 the matrices are well known.[8]

2. The Curve for Li^6

The energy matrix for the charge singlet with $J=1$ is

$$\begin{matrix} ^{13}D \\ ^{11}P \\ ^{13}S \end{matrix}\left\|\begin{matrix} E_D-3\alpha & -(10/3)^{\frac12}\alpha & 0 \\ -(10/3)^{\frac12}\alpha & E_P & 2(2/3)^{\frac12}\alpha \\ 0 & 2(2/3)^{\frac12}\alpha & E_S \end{matrix}\right\|, \quad (3)$$

where the E_L stands for the nuclear binding energy of the corresponding L term without spin-orbit interaction. For $J=3$ the matrix has only one element:

$$^{13}D=E_D+2\alpha. \quad (3')$$

Subtracting this element from the diagonal elements of the matrix for $J=1$ one obtains a matrix

$$\left\|\begin{matrix} -5\alpha & -(10/3)^{\frac12}\alpha & 0 \\ -(10/3)^{\frac12}\alpha & E_P-E_D-2\alpha & 2(2/3)^{\frac12}\alpha \\ 0 & 2(2/3)^{\frac12}\alpha & E_S-E_D-2\alpha \end{matrix}\right\|, \quad (4)$$

all the eigenvalues of which are negative when α is vary large, while for a small α there is also a positive one. The transition occurs at the critical value α_0 for which there is a zero eigenvalue, and therefore the determinant (4) vanishes. Developing this determinant we obtain for α_0 the value

$$\alpha_0=\frac{3(E_D-E_P)(E_S-E_D)}{10E_D-6E_P-4E_S}, \quad (5)$$

and inserting the values of the terms in Rosenfeld's mixture (1),

$$E_D=F_0+F_2,\ E_P=-(9/5)(F_0-5F_2),\ E_S=F_0+10F_2, \quad (6)$$

one obtains finally

$$\alpha_0=\frac{9}{14}\cdot F_2\cdot\frac{7F_0-20F_2}{F_0-5F_2}. \quad (7)$$

3. Limiting Cases

The value of α_0 from Eq. (7) was now substituted for α in the energy matrices of B^{10}, and the highest binding energies of $J=1$ and $J=3$ were compared.

The end points of the interval of variation of the ratio F_2/F_0 could be dealt with easier than the inner part of the interval, and so will be considered first:

A. Long-Range Limit: $F_2 \ll F_0$

Neglecting higher powers of F_2/F_0 we get, from (7),

$$\alpha_0=(9/2)F_2, \quad (8)$$

and we shall regard F_2 as a perturbation. In this approximation it is sufficient to diagonalize that part of

[8] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935), p. 268.

TABLE II. The nonvanishing elements of the nuclear forces binding energy matrix of p^6 in the LS scheme.

Row and column	The element
(a) $J=1$	
Diagonal elements	
(222) ^{13}S	$-(5/3)F_0+(275/6)F_2$
(321) ^{15}P	$3\ F_0+36\ F_2$
(321) ^{15}D	$3\ F_0+(168/5)F_2$
(321) ^{13}P	$(17/15)F_0+(644/15)F_2$
(321) ^{13}D	$(17/15)F_0+(2968/75)F_2$
A ^{11}P	$3\ F_0+(231/5)F_2$
B ^{11}P	$3\ F_0+(243/5)F_2$
(420) ^{13}S	$(29/5)F_0+(487/10)F_2$
(420) ^{13}D	$(29/5)F_2+(979/25)F_2$
(420) $^{13}D'$	$(29/5)F_0+(203/5)F_2$
Nondiagonal elements	
(222) $^{13}S-$(420)^{13}S	$(3/2)(3)^{\frac{1}{2}}\ F_2$
(321)$^{13}D-$(420)^{13}D	$(12/25)(21)^{\frac{1}{2}}\ F_2$
(b) $J=3$	
Diagonal elements	
(222) ^{17}S	$3\ F_0+30\ F_2$
(321) ^{15}P	$3\ F_0+36\ F_2$
(321) ^{15}D	$3\ F_0+(168/5)F_2$
(321) ^{13}D	$(17/15)F_0+(2968/75)F_2$
A ^{11}F	$3\ F_0+(186/5)F_2$
B ^{11}F	$3\ F_0+(168/5)F_2$
(420) ^{13}D	$(29/5)F_0+(979/25)F_2$
(420) $^{13}D'$	$(29/5)F_0+(203/5)F_2$
(420) ^{13}F	$(29/5)F_0+(158/5)F_2$
(420) ^{13}G	$(29/5)F_0+(116/5)F_2$
Nondiagonal elements	
(321)$^{13}D-$(420)^{13}D	$(12/25)(21)^{\frac{1}{2}}\ F_2$

the energy matrix which belongs to the maximal unperturbed eigenvalue, and from Tables II(a) and (b) it follows that this unperturbed value is that of the super-multiplet (420), both for $J=1$ and for $J=3$.

The secular equations for this supermultiplet are of order 3 and 4 for $J=1$ and $J=3$, respectively. When the value (8) for α_0 is inserted in them and the calculation performed, one obtains

$$\begin{aligned} E_1&=(29/5)F_0+49.13F_2,\\ E_3&=(29/5)F_0+49.77F_2, \end{aligned} \tag{9}$$

and we see that the binding energy for $J=3$ is higher than for $J=1$.

B. Short-Range Limit: $F_0-5F_2 \ll F_0$

Here α_0 becomes infinite of the first order, and we need to go over to the jj scheme. The matrix of α becomes diagonal, and the matrices of the coefficients of F_0 and F_2 are given in Tables III(a), (b) and IV(a), (b), where the states are labeled by the numbers of nucleons having $j=\frac{1}{2}$ and $j=\frac{3}{2}$, though this specification is not complete.

Here we consider F_0 and F_2 as a perturbation to α. Neglecting F_0-5F_2, we obtain for the eigenvalues in the first approximation of the perturbation theory

$$E_1=E_3=6\alpha_0+54F_2, \tag{10}$$

and we have to pass to the second approximation. The usual formula gives

$$\begin{aligned} E_1&=6\alpha_0+54F_2+(107/45)(F_0-5F_2)\\ &\qquad+(146/5)(F_2^2/\alpha_0),\\ E_3&=6\alpha_0+54F_2+(59/15)(F_0-5F_2)\\ &\qquad+(79/5)(F_2^2/\alpha_0). \end{aligned} \tag{11}$$

TABLE III. The matrix of the coefficients of $F_0/45$ of p^6 in the jj scheme.

(a) $J=1$

	$p_{3/2}^6$	$(p_{3/2}^5 p_{1/2})_A$	$(p_{3/2}^5 p_{1/2})_B$	$(p_{3/2}^4 p_{1/2}^2)_A$	$(p_{3/2}^4 p_{1/2}^2)_B$	$(p_{3/2}^4 p_{1/2}^2)_C$	$(p_{3/2}^4 p_{1/2}^2)_D$	$(p_{3/2}^3 p_{1/2}^3)_A$	$(p_{3/2}^3 p_{1/2}^3)_B$	$p_{3/2}^2 p_{1/2}^4$
$p_{3/2}^6$	107	$14(6)^{\frac{1}{2}}$	56	$28(2)^{\frac{1}{2}}$	$14(3)^{\frac{1}{2}}$	$14(14)^{\frac{1}{2}}$	$14(3)^{\frac{1}{2}}$	0	0	0
$(p_{3/2}^5 p_{1/2})_A$	$14(6)^{\frac{1}{2}}$	177	0	$14(3)^{\frac{1}{2}}$	$-28(2)^{\frac{1}{2}}$	$-14(21)^{\frac{1}{2}}$	$14(2)^{\frac{1}{2}}$	0	$14(6)^{\frac{1}{2}}$	0
$(p_{3/2}^5 p_{1/2})_B$	56	0	107	$7(2)^{\frac{1}{2}}$	$14(3)^{\frac{1}{2}}$	$-7(14)^{\frac{1}{2}}$	$28(3)^{\frac{1}{2}}$	$14(6)^{\frac{1}{2}}$	42	0
$(p_{3/2}^4 p_{1/2}^2)_A$	$28(2)^{\frac{1}{2}}$	$14(3)^{\frac{1}{2}}$	$7(2)^{\frac{1}{2}}$	149	0	0	0	$14(3)^{\frac{1}{2}}$	$7(2)^{\frac{1}{2}}$	$28(2)^{\frac{1}{2}}$
$(p_{3/2}^4 p_{1/2}^2)_B$	$14(3)^{\frac{1}{2}}$	$-28(2)^{\frac{1}{2}}$	$14(3)^{\frac{1}{2}}$	0	149	0	0	$28(2)^{\frac{1}{2}}$	$-14(3)^{\frac{1}{2}}$	$-14(3)^{\frac{1}{2}}$
$(p_{3/2}^4 p_{1/2}^2)_C$	$14(14)^{\frac{1}{2}}$	$-14(21)^{\frac{1}{2}}$	$-7(14)^{\frac{1}{2}}$	0	0	65	0	$14(21)^{\frac{1}{2}}$	$7(14)^{\frac{1}{2}}$	$-14(14)^{\frac{1}{2}}$
$(p_{3/2}^4 p_{1/2}^2)_D$	$14(3)^{\frac{1}{2}}$	$14(2)^{\frac{1}{2}}$	$28(3)^{\frac{1}{2}}$	0	0	0	205	$-14(2)^{\frac{1}{2}}$	$-28(3)^{\frac{1}{2}}$	$-14(3)^{\frac{1}{2}}$
$(p_{3/2}^3 p_{1/2}^3)_A$	0	0	$14(6)^{\frac{1}{2}}$	$14(3)^{\frac{1}{2}}$	$28(2)^{\frac{1}{2}}$	$14(21)^{\frac{1}{2}}$	$-14(2)^{\frac{1}{2}}$	177	0	$14(6)^{\frac{1}{2}}$
$(p_{3/2}^3 p_{1/2}^3)_B$	0	$14(6)^{\frac{1}{2}}$	42	$7(2)^{\frac{1}{2}}$	$-14(3)^{\frac{1}{2}}$	$7(14)^{\frac{1}{2}}$	$-28(3)^{\frac{1}{2}}$	0	107	56
$p_{3/2}^2 p_{1/2}^4$	0	0	0	$28(2)^{\frac{1}{2}}$	$-14(3)^{\frac{1}{2}}$	$-14(14)^{\frac{1}{2}}$	$-14(3)^{\frac{1}{2}}$	$14(6)^{\frac{1}{2}}$	56	107

(b) $J=3$

	$p_{3/2}^6$	$(p_{3/2}^5 p_{1/2})_A$	$(p_{3/2}^5 p_{1/2})_B$	$(p_{3/2}^4 p_{1/2}^2)_A$	$(p_{3/2}^4 p_{1/2}^2)_B$	$(p_{3/2}^4 p_{1/2}^2)_C$	$(p_{3/2}^4 p_{1/2}^2)_D$	$(p_{3/2}^3 p_{1/2}^3)_A$	$(p_{3/2}^3 p_{1/2}^3)_B$	$p_{3/2}^2 p_{1/2}^4$
$p_{3/2}^6$	177	$-12(14)^{\frac{1}{2}}$	$-4(21)^{\frac{1}{2}}$	$14(3)^{\frac{1}{2}}$	$12(7)^{\frac{1}{2}}$	$8(21)^{\frac{1}{2}}$	$14(3)^{\frac{1}{2}}$	0	0	0
$(p_{3/2}^5 p_{1/2})_A$	$-12(14)^{\frac{1}{2}}$	177	0	$-2(42)^{\frac{1}{2}}$	$-6(2)^{\frac{1}{2}}$	$-4(6)^{\frac{1}{2}}$	$2(42)^{\frac{1}{2}}$	12	$16(6)^{\frac{1}{2}}$	0
$(p_{3/2}^5 p_{1/2})_B$	$-4(21)^{\frac{1}{2}}$	0	177	$12(7)^{\frac{1}{2}}$	$12(3)^{\frac{1}{2}}$	24	$-12(7)^{\frac{1}{2}}$	$16(6)^{\frac{1}{2}}$	30	0
$(p_{3/2}^4 p_{1/2}^2)_A$	$14(3)^{\frac{1}{2}}$	$-2(42)^{\frac{1}{2}}$	$12(7)^{\frac{1}{2}}$	219	0	0	0	$-2(42)^{\frac{1}{2}}$	$12(7)^{\frac{1}{2}}$	$14(3)^{\frac{1}{2}}$
$(p_{3/2}^4 p_{1/2}^2)_B$	$12(7)^{\frac{1}{2}}$	$-6(2)^{\frac{1}{2}}$	$12(3)^{\frac{1}{2}}$	0	135	0	0	$6(2)^{\frac{1}{2}}$	$-12(3)^{\frac{1}{2}}$	$-12(7)^{\frac{1}{2}}$
$(p_{3/2}^4 p_{1/2}^2)_C$	$8(21)^{\frac{1}{2}}$	$-4(6)^{\frac{1}{2}}$	24	0	0	135	0	$4(6)^{\frac{1}{2}}$	-24	$-8(21)^{\frac{1}{2}}$
$(p_{3/2}^4 p_{1/2}^2)_D$	$14(3)^{\frac{1}{2}}$	$2(42)^{\frac{1}{2}}$	$-12(7)^{\frac{1}{2}}$	0	0	0	219	$-2(42)^{\frac{1}{2}}$	$12(7)^{\frac{1}{2}}$	$-14(3)^{\frac{1}{2}}$
$(p_{3/2}^3 p_{1/2}^3)_A$	0	12	$16(6)^{\frac{1}{2}}$	$-2(42)^{\frac{1}{2}}$	$6(2)^{\frac{1}{2}}$	$4(6)^{\frac{1}{2}}$	$-2(42)^{\frac{1}{2}}$	177	0	$-12(14)^{\frac{1}{2}}$
$(p_{3/2}^3 p_{1/2}^3)_B$	0	$16(6)^{\frac{1}{2}}$	30	$12(7)^{\frac{1}{2}}$	$-12(3)^{\frac{1}{2}}$	-24	$12(7)^{\frac{1}{2}}$	0	177	$-4(21)^{\frac{1}{2}}$
$p_{3/2}^2 p_{1/2}^4$	0	0	0	$14(3)^{\frac{1}{2}}$	$-12(7)^{\frac{1}{2}}$	$-8(21)^{\frac{1}{2}}$	$-14(3)^{\frac{1}{2}}$	$-12(14)^{\frac{1}{2}}$	$-4(21)^{\frac{1}{2}}$	177

TABLE IV. The matrix of coefficients of F_2 of p^6 in the jj scheme.

(a) Coefficients of $F_2/45$ for $J=1$

	$p_{3/2}^6$	$(p_{3/2}^5 p_{1/2})_A$	$(p_{3/2}^5 p_{1/2})_B$	$(p_{3/2}^4 p_{1/2}^2)_A$	$(p_{3/2}^4 p_{1/2}^2)_B$	$(p_{3/2}^4 p_{1/2}^2)_C$	$(p_{3/2}^4 p_{1/2}^2)_D$	$(p_{3/2}^3 p_{1/2}^3)_A$	$(p_{3/2}^3 p_{1/2}^3)_B$	$p_{3/2}^2 p_{1/2}^4$
$p_{3/2}^6$	1895	$2(6)^{1/2}$	26	$-50(2)^{1/2}$	$65(3)^{1/2}$	$-25(14)^{1/2}$	$-25(3)^{1/2}$	0	0	0
$(p_{3/2}^5 p_{1/2})_A$	$2(6)^{1/2}$	1905	$-36(6)^{1/2}$	$65(3)^{1/2}$	$-31(2)^{1/2}$	$-11(21)^{1/2}$	$-70(2)^{1/2}$	-90	$-25(6)^{1/2}$	0
$(p_{3/2}^5 p_{1/2})_B$	26	$-36(6)^{1/2}$	1994	$-35(2)^{1/2}$	$20(3)^{1/2}$	$35(14)^{1/2}$	$22(3)^{1/2}$	$-25(6)^{1/2}$	60	0
$(p_{3/2}^4 p_{1/2}^2)_A$	$-50(2)^{1/2}$	$65(3)^{1/2}$	$-35(2)^{1/2}$	1802	0	0	0	$65(3)^{1/2}$	$-35(2)^{1/2}$	$-50(2)^{1/2}$
$(p_{3/2}^4 p_{1/2}^2)_B$	$65(3)^{1/2}$	$-31(2)^{1/2}$	$20(3)^{1/2}$	0	1874	$36(42)^{1/2}$	-72	$31(2)^{1/2}$	$-20(3)^{1/2}$	$-65(3)^{1/2}$
$(p_{3/2}^4 p_{1/2}^2)_C$	$-25(14)^{1/2}$	$-11(21)^{1/2}$	$35(14)^{1/2}$	0	$36(42)^{1/2}$	1898	0	$11(21)^{1/2}$	$-35(14)^{1/2}$	$25(14)^{1/2}$
$(p_{3/2}^4 p_{1/2}^2)_D$	$-25(3)^{1/2}$	$-70(2)^{1/2}$	$22(3)^{1/2}$	0	-72	0	1792	$70(2)^{1/2}$	$-22(3)^{1/2}$	$25(3)^{1/2}$
$(p_{3/2}^3 p_{1/2}^3)_A$	0	-90	$-25(6)^{1/2}$	$65(3)^{1/2}$	$31(2)^{1/2}$	$11(21)^{1/2}$	$70(2)^{1/2}$	1905	$-36(6)^{1/2}$	$2(6)^{1/2}$
$(p_{3/2}^3 p_{1/2}^3)_B$	0	$-25(6)^{1/2}$	60	$-35(2)^{1/2}$	$-20(3)^{1/2}$	$-35(14)^{1/2}$	$-22(3)^{1/2}$	$-36(6)^{1/2}$	1994	26
$p_{3/2}^2 p_{1/2}^4$	0	0	0	$-50(2)^{1/2}$	$-65(3)^{1/2}$	$25(14)^{1/2}$	$25(3)^{1/2}$	$2(6)^{1/2}$	26	379

(b) Coefficients of $F_2/315$ for $J=3$

	$p_{3/2}^6$	$(p_{3/2}^5 p_{1/2})_A$	$(p_{3/2}^5 p_{1/2})_B$	$(p_{3/2}^4 p_{1/2}^2)_A$	$(p_{3/2}^4 p_{1/2}^2)_B$	$(p_{3/2}^4 p_{1/2}^2)_C$	$(p_{3/2}^4 p_{1/2}^2)_D$	$(p_{3/2}^3 p_{1/2}^3)_A$	$(p_{3/2}^3 p_{1/2}^3)_B$	$p_{3/2}^2 p_{1/2}^4$
$p_{3/2}^6$	10.815	$96(14)^{1/2}$	$-94(21)^{1/2}$	$-175(3)^{1/2}$	$-150(7)^{1/2}$	$-100(21)^{1/2}$	$455(3)^{1/2}$	0	0	0
$(p_{3/2}^5 p_{1/2})_A$	$96(14)^{1/2}$	10.383	$432(6)^{1/2}$	$70(42)^{1/2}$	$-168(2)^{1/2}$	$140(6)^{1/2}$	$-70(42)^{1/2}$	795	$-200(6)^{1/2}$	0
$(p_{3/2}^5 p_{1/2})_B$	$-94(21)^{1/2}$	$432(6)^{1/2}$	11.310	$210(7)^{1/2}$	$-420(3)^{1/2}$	-588	$42(7)^{1/2}$	$-200(6)^{1/2}$	570	0
$(p_{3/2}^4 p_{1/2}^2)_A$	$-175(3)^{1/2}$	$70(42)^{1/2}$	$210(7)^{1/2}$	10.794	0	0	0	$70(42)^{1/2}$	$210(7)^{1/2}$	$-175(3)^{1/2}$
$(p_{3/2}^4 p_{1/2}^2)_B$	$-150(7)^{1/2}$	$-168(2)^{1/2}$	$-420(3)^{1/2}$	0	10.584	0	$216(21)^{1/2}$	$168(2)^{1/2}$	$420(3)^{1/2}$	$150(7)^{1/2}$
$(p_{3/2}^4 p_{1/2}^2)_C$	$-100(21)^{1/2}$	$140(6)^{1/2}$	-588	0	0	11.466	$432(7)^{1/2}$	$-140(6)^{1/2}$	588	$100(21)^{1/2}$
$(p_{3/2}^4 p_{1/2}^2)_D$	$455(3)^{1/2}$	$-70(42)^{1/2}$	$42(7)^{1/2}$	0	$216(21)^{1/2}$	$432(7)^{1/2}$	10.668	$70(42)^{1/2}$	$-42(7)^{1/2}$	$-455(3)^{1/2}$
$(p_{3/2}^3 p_{1/2}^3)_A$	0	795	$-200(6)^{1/2}$	$70(42)^{1/2}$	$168(2)^{1/2}$	$-140(6)^{1/2}$	$70(42)^{1/2}$	10.383	$432(6)^{1/2}$	$96(14)^{1/2}$
$(p_{3/2}^3 p_{1/2}^3)_B$	0	$-200(6)^{1/2}$	570	$210(7)^{1/2}$	$420(3)^{1/2}$	588	$-42(7)^{1/2}$	$432(6)^{1/2}$	11.310	$-94(21)^{1/2}$
$p_{3/2}^2 p_{1/2}^4$	0	0	0	$-175(3)^{1/2}$	$150(7)^{1/2}$	$100(21)^{1/2}$	$-455(3)^{1/2}$	$96(14)^{1/2}$	$-94(21)^{1/2}$	10.815

After inserting for α_0 in the last terms its value in this approximation, which is seen from (7) to be

$$\alpha_0=\frac{135}{14}\cdot\frac{F_2^2}{F_0-5F_2}, \tag{12}$$

the final values obtained are

$$E_1=6\alpha_0+54F_2+(3649/675)(F_0-5F_2),$$
$$E_3=6\alpha_0+54F_2+(3761/675)(F_0-5F_2), \tag{13}$$

and we see that here too $E_3>E_1$.

4. Numerical Calculations

To see what happens for intermediate ranges of the nuclear forces, we had to diagonalize numerically the matrices of order 10, after inserting for α the critical value α_0. We took a set of values of F_2/F_0 between 1/35 and 0.198. For $F_2/F_0\leq 0.16$, the matrices in LS scheme were used, and the diagonalization was made with the electrical network of the Physical Department of the University.[9] In the last five cases, $F_2/F_0\geq 0.17$, the matrices in the jj scheme were used, and the eigenvalues were found by the perturbation formulas up to and including the third order, which sufficed for our purposes, as the rapid convergence of the first three approximations had shown.

TABLE V. Values of E_3/E_1 for $0<F_2/F_0<0.2$.

F_2/F_0	E_1/F_0	E_3/F_0	E_3/E_1
1/35	7.2585	7.2959	1.0052
0.05	8.4645	8.5380	1.0087
0.08	10.412	10.520	1.0104
0.10	11.994	12.104	1.0092
0.11	12.920	13.025	1.0081
0.12	13.977	14.077	1.0072
0.14	16.741	16.819	1.0047
0.16	21.438	21.488	1.0023
0.17	25.738	25.764	1.00101
0.18	33.869	33.887	1.00053
0.19	57.433	57.443	1.00017
0.195	103.931	103.935	1.000038
0.198	242.915	242.917	1.0000082

The results are given in Table V and are seen to behave quite regularly. It is seen that the theory succeeds in explaining the facts for all the ranges of the nuclear forces.

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[9] A. Many, Rev. Sci. Instr. **21**, 972 (1950); thesis, Jerusalem, 1950 (unpublished).