Radiative Correction for the Collision Loss of Heavy Particles

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The correction to the Bethe-Bloch formula for the stopping power of fast heavy particles, due to virtual photons and the emission of real photons, has been computed using the Born approximation for the extreme relativistic case. The fractional correction increases with increasing energy of the incident particle. It is approximately 1 percent when the kinetic energy of the particle is 100 times its rest energy.

RECENTLY Jauch[1] has given an estimate for the correction, due to radiation, to the energy loss of heavy particles passing through matter. This estimate was based upon Schwinger's[2] correction to the elastic cross section due to the emission and re-absorption of virtual photons and the emission of soft real photons.

The purpose of this paper is to present a more complete calculation of the radiative correction which includes also the emission of real photons without restriction as to their energy. The results show that this is a positive correction which increases as the incident energy increases, in the relativistic region, and is quite small even at very high energy. Because of this, the more complicated low energy region is not treated here, and the calculation assumes from the start that we have heavy particles bombarding matter with a velocity $v=\beta c$ very close to that of light.

We shall find the collision loss per unit path length by taking the energy loss per collision, multiplying it by the probability of such a collision, and integrating over all possible collisions. The probability per unit path length of a collision is $n\sigma$, where n is the number of electrons per unit volume and σ is the differential scattering cross section. Since the Bethe-Heitler formula gives the differential bremsstrahlung cross section under the assumption that the scattering center is very massive compared to the particle that is scattered, it is convenient to express the energy loss per collision (in the laboratory system) in terms of quantities proper to the Lorentz frame in which the heavy particle is at rest. A Lorentz transformation will then give the energy loss in the laboratory frame:

$$\begin{aligned}\Delta\mathcal{E}' &= -\gamma[(\mathcal{E}-Mc^2)-\beta q\cos\psi] \\ &\approx \gamma q[\beta\cos\psi-\tfrac{1}{2}q/Mc^2].\end{aligned} \tag{1}$$

Here, $\mathcal{E}$, q, and ψ are the recoil energy, momentum, and direction, respectively, in the rest frame of the incident heavy particle of mass M, and $\gamma^2=1/(1-\beta^2)$.

Let $E_0=E+k$, and

$$\begin{aligned}q^2=|\mathbf{p}_0-\mathbf{p}-\mathbf{k}|^2 &= p_0^2+p^2+k^2-2p_0k\cos\theta_0+2pk\cos\theta \\ &\quad -2p_0p(\cos\theta_0\cos\theta+\sin\theta_0\sin\theta\cos\phi),\end{aligned}$$

where E_0, p_0 are the energy and momentum (expressed in energy units) of the incident electron, E, p those of the scattered electron, k the energy of the emitted photon, θ_0 the angle between photon and incident electron, θ the angle between photon and scattered electron, and ϕ the angle between planes in which θ_0 and θ lie. Then we can write the energy loss,

$$\Delta\mathcal{E}'=\frac{\gamma}{p_0}\left[k(E-p\cos\theta)+\frac{q^2}{2}\left(1-\frac{m\gamma}{M}\right)\right]. \tag{2}$$

We can shorten our integration by splitting the energy loss into two parts:

$$\frac{\gamma}{p_0}k(E-p\cos\theta) \quad\text{and}\quad \frac{\gamma}{p_0}\frac{q^2}{2}\left(1-\frac{m\gamma}{M}\right).$$

For the first part we use Sommerfeld's[3] integration, which gives the differential cross section in terms of k and q, and for the second part we use Bethe's[4] integration.

Sommerfeld[3] gives the differential bremsstrahlung cross section for Coulomb scattering in terms of k and θ_0. However, because of the symmetry of the Bethe-Heitler formula, we can write the differential cross section in terms of k and θ merely by exchanging the symbols with and without subscript zero. Doing this and using our notation, we obtain

$$\begin{aligned}\frac{Z^2e^4}{137}\frac{dk}{k}\frac{p}{p_0}\sin\theta d\theta\Bigg\{&\frac{4\mu^2\sin^2\theta}{(E-p\cos\theta)^4}-\frac{(E_0+E)^2}{E_0E(E-p\cos\theta)^2} \\ &+\left[-\frac{\mu^2\sin^2\theta}{(E-p\cos\theta)^4}+\frac{E_0^2+E^2}{E_0E(E-p\cos\theta)^2}\right] \\ &\times\log\left(\frac{4E_0^2E^2}{\mu^2k^2}\right)\Bigg\}.\end{aligned} \tag{3}$$

If Eq. (3) is multiplied by $(\gamma/p_0)k(E-p\cos\theta)$, the integral of the expression thus obtained converges. After the somewhat tedious term-by-term integration, we obtain as the highest term in the development in

[1] J. M. Jauch, Phys. Rev. 85, 951 (1952).
[2] J. Schwinger, Phys. Rev. 75, 899 (1949).
[3] A. Sommerfeld, *Atombau und Spektrallinien* (Friedr. Vieweg and Son, Braunschweig, 1939), p. 551.
[4] H. A. Bethe, Proc. Cambridge Phil. Soc. 30, 524 (1934).

powers of γ

$$\frac{Z^2e^4}{137\mu}\left[\frac{4}{3}\log^3 2\gamma+\log^2 2\gamma-\left(3+\frac{\pi^2}{2}\right)\log 2\gamma +9+\frac{\pi^2}{3}-\frac{7}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}-\sum_{n=1}^{\infty}\frac{1}{2^n n^3}\right]. \quad (4)$$

To evaluate the contribution of the first term of (2) we can use Bethe's[4] expression for the differential cross section. For $p_0-p-k\ll q\ll E_0$ this is

$$\frac{16Z^2e^4}{137}\frac{dk}{k}\frac{p}{p_0}\frac{dq}{q^3}\left\{\left(\frac{E_0}{E}+\frac{E}{E_0}\right)\frac{q}{2\mu}\frac{\sinh^{-1}(q/2\mu)}{[1+(q/2\mu)^2]^{\frac{1}{2}}} +\frac{\sinh^{-1}(q/2\mu)}{(q/2\mu)[1+(q/2\mu)^2]^{\frac{1}{2}}}-1\right\}. \quad (5)$$

Multiplying (5) by $(\gamma q^2)/(2p_0)$ and integrating over k from $k=k_{\min}$ to $k\approx E_0$ and over q from $q=0$ to $q=Q\gg\mu$ (since values of q of the order p_0-p-k do not contribute in this approximation), we get

$$\frac{8Z^2e^4}{137\mu}\left[\log\frac{E_0}{k_{\min}}\left(\log^2\frac{Q}{\mu}-\log\frac{Q}{\mu}+1\right) -\frac{3}{4}\log^2\frac{Q}{\mu}+\log\frac{Q}{\mu}-1\right]. \quad (6)$$

However, Bethe's formula is not valid for q comparable in magnitude to E_0. For those values of q we are going to use formulas derived by Schiff.[5] He shows that for large recoil momenta we can divide the differential cross section into two parts: one due to the emission of photons close to the direction of the incident electron, another due to the emission of photons close to the direction of the scattered electron. If we take his formulas (2) and (3), make the correction substituting $[\log(2E_0\cos\gamma/\mu)-\frac{1}{2}]$ for $\log(E_0/\mu)$, and then substitute $q/2E$ and $q/2E_0$ for $\cos\gamma$ in (2) and (3), respectively, we find that the differential cross section for large q is the sum of

$$\frac{8Z^2e^4}{137}\frac{dk}{k}\frac{dq}{q^3}\left(1+\frac{E^2}{E_0^2}\right)\left(1-\frac{q^2}{4E^2}\right)\left(\log\frac{q}{\mu}-\frac{1}{2}\log\frac{E_0}{E}\right)$$

and (7)

$$\frac{8Z^2e^4}{137}\frac{dk}{k}\frac{dq}{q^3}\left(1+\frac{E^2}{E_0^2}\right)\left(1-\frac{q^2}{4E_0^2}\right)\left(\log\frac{q}{\mu}-\frac{1}{2}\right),$$

where the first expression is to be taken only if $q<2E$, and the second is valid up to $q=2E_0$. Multiplying (7) by $\gamma q^2/2p_0$, integrating over q and over $k(k_{\min}<k<E_0)$, and neglecting terms that vanish as $k_{\min}/E_0$ or Q/E_0 vanish, we get

$$\frac{8Z^2e^4}{137\mu}\left[\log\frac{E_0}{k_{\min}}\left(\log^2 2\gamma-2\log 2\gamma+2-\log^2\frac{Q}{\mu}+\log\frac{Q}{\mu}-1\right) +\frac{3}{4}\log^2\frac{Q}{\mu}-\left(\frac{1}{8}+\frac{\pi^2}{6}\right)\log\frac{Q}{\mu}-\sum_{n=1}^{\infty}\frac{1}{n^3}+\frac{1}{4}+\frac{\pi^2}{12}\right]. \quad (8)$$

Table I. Fractional radiative correction to the Bethe-Block formula.

γ	10	20	50	100	
$\Delta(\%)$	0.27	0.45	0.75	0.95	if $Q=\mu$
$\Delta(\%)$	0.19	0.37	0.66	0.88	if $Q=2E_0$

We have to account also for energy loss connected with the emission of virtual photons and of soft real photons. In this case the energy loss is essentially that of the elastic collision

$$\Delta\mathcal{E}'=2\mu\beta^2\gamma^2\sin^2\tfrac{1}{2}\vartheta, \quad (9)$$

where ϑ is the angle of the scattered electron (measured in the meson rest system).

The cross section of such a collision is given by Schwinger.[2] In the extremely relativistic case his formula (1) becomes

$$\delta=\frac{4}{137\pi}\left\{\log\frac{E_0}{k_{\min}}\left[\frac{\lambda^2+\frac{1}{2}}{\lambda(1+\lambda^2)^{\frac{1}{2}}}\sinh^{-1}\lambda-\frac{1}{2}\right] +\frac{-13\lambda^4-8\lambda^2+2}{12\lambda^3(1+\lambda^2)^{\frac{1}{2}}}\sinh^{-1}\lambda-\frac{1}{6}\frac{1}{\lambda^2}+\frac{7}{9}+\phi(\vartheta)\right\}, \quad (10)$$

where $\lambda=(p_0/mc^2)\sin\frac{1}{2}\vartheta\approx\gamma\sin\frac{1}{2}\vartheta$. $\phi(\vartheta)$ is given in the form of a definite integral in reference 2. The additive correction to the differential scattering cross section is $-\delta$ multiplied by the elastic cross section due to Coulomb scattering,[6]

$$\sigma_0(\vartheta)=(\pi Z^2e^4/\gamma^2\mu^2)\cot^3\tfrac{1}{2}\vartheta d\vartheta. \quad (11)$$

Multiplying $-\delta$ with σ_0 and $\Delta\mathcal{E}'$ and integrating over ϑ, we obtain the energy loss due to nearly elastic collisions,

$$\frac{8Z^2e^4}{137\mu}\left[-\log\frac{E_0}{k_{\min}}(\log^2 2\gamma-2\log 2\gamma+2) +\frac{13}{12}\log^2 2\gamma-\frac{95}{36}\log 2\gamma+\frac{829}{216}-\frac{\pi^2}{12}\right]. \quad (12)$$

In performing this integration the most troublesome term is that involving ϕ; it is evaluated exactly by changing the order of the integrations over ϑ and over the variable x of reference 2.

Adding (4), (6), (8), and (12), the division $k_{\min}$ between soft and hard photons cancels out, as it should,

[5] L. I. Schiff, Phys. Rev. **87**, 750 (1952); in Eq. (4) and the two preceding expressions of this paper, a factor E_0^2 is missing in the denominator.

[6] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1949), second edition, p. 80.

and we get for the energy loss due to bremsstrahlung

$$\frac{4Z^2e^4}{137\mu}\left[\frac{1}{3}\log^3 2\gamma+\frac{29}{12}\log^2 2\gamma - \left(\frac{217}{36}+\frac{\pi^2}{8}\right)\log 2\gamma-\left(\frac{\pi^2}{3}-\frac{7}{4}\right)\log\frac{Q}{\mu} +\frac{899}{108}+\frac{\pi^2}{12}-\frac{39}{16}\sum_{n=1}^{\infty}\frac{1}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{2^n n^3}\right]. \quad (13)$$

Since energy lost in elastic collisions is given by the Bethe-Bloch formula,[7]

$$-\frac{dw}{dx}=n\frac{4\pi Z^2e^4}{\mu\beta^2}\log\frac{2\mu\beta^2\gamma^2}{I}, \quad (14)$$

we get for the fractional radiative correction for an infinitely heavy particle

$$\Delta=\frac{1}{137\pi\log(2\mu\beta^2\gamma^2/I)}[0.333\log^3 2\gamma+2.42\log^2 2\gamma -7.26\log 2\gamma-1.54\log(Q/\mu)+6.18]. \quad (15)$$

Unfortunately, the division Q between small and large momenta does not quite cancel out of the final formula, but Δ depends only logarithmically on Q so that it seems to be reasonable to expect that a reliable result would be obtained by choosing Q somewhere between the extremes $\mu<Q<2E_0$. However, since the Bethe-Heitler formula does not take into account the recoil of the scattering center, formula (15) may be in error when $m\gamma$ is larger than or of the order of M.

Taking collisions of pi-mesons ($M=270m$) with argon atoms as an example, we obtain the values in Table I for the percent radiative correction. The correction is positive, as one would expect, since mesons are losing some additional energy to radiation. Also, the correction is small for mesons of available energy.

Radiation observed in the laboratory should not show a marked anisotropy and the maximum energy of the emitted photons should be of the order of $\frac{1}{2}$ Mev, except for a few hard photons close to the direction of the incident heavy particle.

The author is indebted to Professor L. I. Schiff for suggesting this problem and for his continued interest and advice throughout the solution.

[7] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 218.

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Radiations of Pu^{243}

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The radiations of Pu^{243} were studied with beta- and gamma-scintillation spectrometers alone and in coincidence. An incomplete disintegration scheme is deduced which leads to a total beta-disintegration energy of 560 kev. A half-life of 4.98±0.02 hours was observed.

INTRODUCTION

THE radiations of Pu^{243} were first studied[1] by absorption measurements which indicated a maximum beta-energy of 0.5 Mev. O'Kelley and Orth[2] reported a preliminary value of 0.39 Mev for the maximum beta-energy and gammas of 0.095 and 0.12 Mev. The purpose of this research was to determine the total decay energy of Pu^{243}. Therefore, in addition to examining the beta- and gamma-spectra, beta-gamma and gamma-gamma coincidence measurements were undertaken.

SAMPLE PREPARATION

Samples of a nitrate solution of plutonium enriched in Pu^{242} were evaporated to dryness in a quartz tube and irradiated in the thimble of the Argonne Heavy Water Reactor for approximately 15 hours.

Immediately after irradiation the plutonium was purified from all extraneous activity with a resin column, a series of precipitations, and solvent extractions. Several irradiations were made to complete the experiments reported.

APPARATUS

Scintillation spectrometers were employed for the measurement of the beta- and gamma-ray spectra. Thallium-activated sodium iodide crystals $1\frac{1}{4}$ inches in diameter and $\frac{1}{2}$ inch thick were used for gamma-detection, and an anthracene crystal of the same diameter and $\frac{1}{4}$ inch thick was used for the beta-counter. The sodium iodide crystals were sealed in cylindrical 17 ST aluminum cups turned to a thickness of 0.013 inch on the end facing the sample and closed on the other end with a Pyrex window. A similar assembly was used for

[1] Sullivan, Pyle, Studier, Fields, and Manning, Phys. Rev. **83**, 1267 (1951).
[2] G. D. O'Kelley and D. A. Orth, quoted by Thompson, Street, Ghiorso, and Reynolds, Phys. Rev. **84**, 165 (1951).