The First Excited States of Even-Even Nuclei

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The collective model of the nucleus, as expressed in A. Bohr's strong coupling approximation, is applied to the low levels of even-even nuclei. It yields the correct level order of the first few states, and predicts the qualitative regularities of the first excited energy surface which are observed experimentally. The approximation is shown to be much better for two or more extra nucleons than for one, but the first excited energy spacing is sensitive to second-order corrections even for many extra nucleons. Predicted nuclear distortions are larger than is reasonable (a) in the rare earth group, and (b) near doubly magic Pb^{208}. An empirical way to correct for this discrepancy is to diminish the particle-to-surface coupling coefficient.

A simple formula is given for computing an upper limit to the nuclear distortion from the first excited energy of even nuclei. After correction by a single adjustable parameter, this formula yields a fair correlation with quadrupole moments and a better correlation with isotope shifts in the region $50<N<126$. The energy level behavior beyond Pb gives a prediction of quadrupole moment and isotope shift behavior for $N>126$. Certain regularities in the levels of odd-even nuclei are also predicted.

I. INTRODUCTION

THE first excited levels of even-even nuclei (hereafter called simply even nuclei) are known to exhibit regularities in spin and parity,[1,2] and also in energy.[2] A large majority of these levels where spin and parity are known have spin 2, even parity. (There is also a tendency, less marked, for the second excited states of even nuclei to have spin 4, even parity.[1,2]) As a function of mass number A, the energy of the first excited states of even nuclei shows a generally decreasing trend with A, but with large variations around the mean. These variations are strongly correlated with the magic numbers, and show a remarkable regularity as a function of N and Z.[2] Closing a shell in either N or Z brings about a sharp increase in the energy of the first excited state, and closing shells in both N and Z (Ca^{40}, Pb^{208}) a still sharper increase. Those nuclei farthest from any magic number, the rare earths, exhibit anomalously low energies of the first excited state, as do the heaviest elements, around uranium. If the data available for the energy of first excited states of even nuclei are plotted as a function of both N and Z, an energy surface is suggested with sharp spikes at the doubly magic nuclei, sharp ridges along the magic numbers, and broad valleys between the magic numbers.

The predominance of spin-two, even parity first excited levels of even nuclei has received theoretical explanation in terms of the shell model (independent nucleons in spherical well with particle interaction added).[3–5] It is assumed that the first excited level is due to the excitation of a single pair of nucleons. Taking a central attractive force of reasonable range between the nucleons, it is found that the ground state is 0+, as observed; the first excited state is 2+, as usually observed; and the second excited state is 4+, as sometimes observed. The 0–2 energy spacing for the configuration $(j)^2$ is proportional to $(2j+1)$ and equal to about 1.5 Mev for $j=\frac{5}{2}$, a value large compared to most experimental values. The 2–4 spacing for the $(j)^2$ configuration is found to be much smaller than the 0–2 spacing. Experimentally the 0–2 and 2–4 spacings are comparable. De Shalit and Goldhaber[5] argue that the observed regularities in the first excited energies can be explained qualitatively in terms of mixtures of states of excitation of different neutron and proton pairs. The farther one is from a magic number, the more states will be mixed in the first excited level, and the more will that level be lowered due to their interaction, other factors being equal.

The preponderance of spin-two, even parity excited levels receives a simple explanation also in terms of the pure liquid droplet model. On this model, the lowest excited state of collective oscillation is the P_2 ellipsoidal mode, with spin two, even parity. Because of the collective description, however, no explanation of shell structure effects can be given by this model. It gives for the lowest excitation of vibrational energy,

$$E_1=\hbar(C/B)^{\frac{1}{2}}, \tag{1}$$

where

$$C=C_{\text{surface}}+C_{\text{coulomb}}=4r_0^2A^{\frac{2}{3}}\mathcal{O}-(3/10\pi)(Z^2e^2/r_0A^{\frac{1}{3}}) \tag{2}$$

and

$$B=\tfrac{1}{2}\rho_0 r_0^5 A^{5/3}=\text{mass coefficient}. \tag{3}$$

Here $\mathcal{O}$ is the nuclear surface tension, given empirically by $4\pi r_0^2\mathcal{O}\cong 15$ Mev. ρ_0 is the density of nuclear matter, and the nuclear radius is assumed to be equal to $r_0A^{\frac{1}{3}}$. For medium and heavy nuclei, where the liquid droplet model is expected to have validity, $C\cong 60$ Mev and does not vary much with A. In this range, therefore,

$$E_1\cong(100/A^{5/6})\text{ Mev}. \tag{4}$$

No such simple law is obeyed. Sharp variations in the empirical E_1 due to shell structure are observed, and the average E_1, as nearly as it can be approximated,

[1] M. Goldhaber and A. W. Sunyar, Phys. Rev. **83**, 906 (1951).
[2] G. Scharff-Goldhaber, Phys. Rev. **87**, 218 (1952).
[3] B. H. Flowers, Phys. Rev. **86**, 254 (1952).
[4] D. Kurath, Phys. Rev. **87**, 218 (1952).
[5] A. De Shalit and M. Goldhaber (to be published).

seems to decrease more rapidly with A than the negative 5/6 power in Eq. (4).

Both the independent particle model of the nucleus and the liquid droplet model are inadequate to explain the observed regularities of the first excited states of even nuclei. Because of the marked shell structure effects, the pure liquid droplet model is especially inappropriate.

The purpose of the present article is to apply the theory of the combined liquid droplet-free particle model, hereafter called the *collective model*, to the problem of the low states of even nuclei. This model has been developed principally by Bohr[6] and by Hill and Wheeler.[7] According to the collective model, the rapid variation of potential near the edge of the nucleus creates a membrane-like surface capable of undergoing oscillation (conventional liquid droplet model) and in addition capable of being distorted by the action of the "free" nucleons within the nucleus into a permanently nonspherical shape. The nonsphericity is important for quadrupole moments. In addition, the nuclear distortion acts back on the particles and strongly affects the interparticle coupling. The nonspherical shape of the nucleus, therefore, will affect a large number of nuclear properties. In particular, magnetic moments[8] and the order of nuclear levels will require an understanding of the nuclear surface effect for their explanation.

In the following sections, Bohr's theory of the collective model is generalized to several extra nucleons and applied in a straightforward way to even nuclei. It is found that the model explains the qualitative regularities of the first excited states but requires modification to give reasonable quantitative values for the energies in the calculable cases near closed shells. The model suggests a relation between the energy of the first excited state of even nuclei and the quadrupole moments of neighboring odd-even nuclei. The regularities in these two quantities are in good qualitative agreement for the heavy nuclei, but the nuclear distortion calculated from excitation energy of even nuclei is considerably larger than the distortion calculated from quadrupole moments.

II. SPIN, PARITY, AND ENERGY OF FIRST EXCITED STATES

A. Bohr Theory for Single Extra Nucleon

Our starting point is the theory of the interaction of extra nucleons with the nuclear surface, as developed by Bohr, especially his strong coupling (of particle to surface) approximation. The direct interaction among the nucleons is neglected. In his treatment of a single extra nucleon outside closed shells, this effect did not enter. We neglect the direct interparticle interaction because it is the aim here to show that the coupling of particles to the nuclear surface—and thereby indirectly to each other—is alone sufficient to account for the general pattern of the low states of even nuclei. Since the interparticle interaction tends to produce the same order of levels as is derived here, its inclusion would serve to reinforce the conclusions on level order derived from the collective model alone. The inclusion of the direct particle interaction in a calculation of the energy of first excited states would greatly complicate the problem and prevent the remarkable result of the collective model—the sensitive dependence of excitation energy on the permanent distortion of the nucleus—from emerging with full clarity.

For ease in treating the case of strong coupling of particle to surface, Bohr develops a Hamiltonian for the nucleus in terms of particle coordinates relative to nuclear axes ($\mathbf{r}'$, σ'), coordinates of distortion relative to nuclear axes (β, γ), and coordinates of the nuclear axes relative to space axes (Euler angles $\theta_1\theta_2\theta_3$). A principal underlying assumption is that the particle motion is so rapid compared to the motion of the nucleus as a whole, that it is a good approximation to treat the particle motion as if the nuclear axes are fixed in space. It is also assumed for convenience that very strong spin-orbit interaction exists, and that j for each particle remains a good quantum number. In fact, the strong surface interaction will break down j-j coupling, and it will be important to consider a finite spin-orbit interaction in problems involving an odd number of particles (e.g., ground-state magnetic moments). For even nuclei, however, the surface-induced pairing effect is so strong that the magnitude of the spin-orbit coupling is probably less important for the behavior of the ground and first excited state than in odd-even nuclei. This point is being investigated in greater detail, but the assumption that j is a good quantum number is retained in the present paper.

The nuclear Hamiltonian is written as the sum of three terms: H_p, particle Hamiltonian; H_s, surface Hamiltonian; and H_{int}, interaction energy averaged over the particle motion. The particle term is

$$H_p = T_p' + V'_{\text{square well}} \text{ (prime denotes variable } r' \text{ relative to nuclear axes).} \quad (5)$$

Taking the nuclear surface relative to the nuclear axes to be given by

$$R = R_0[1+\textstyle\sum_\mu \alpha_\mu' Y_{2\mu}(\theta', \varphi')], \quad (6)$$

where

$$\alpha_2' = \alpha_{-2}' = (1/\sqrt{2})\beta \sin\gamma, \quad \alpha_1' = \alpha_{-1}' = 0, \quad \alpha_0' = \beta\cos\gamma,$$

we obtain the surface energy in terms of the distortion coordinates β, γ, and the Euler angles defining the nuclear orientation in space:

$$H_s = T_{\text{vib}} + T_{\text{rot}} + V_s. \quad (7)$$

[6] A. Bohr, Kgl. Danske Videnskab, Selskab. Mat. fys. Medd. **26**, No. 14 (1952).
[7] D. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).
[8] A. Bohr, Phys. Rev. **81**, 134 (1951).

Surface vibration energy:

$$T_{\text{vib}}=-\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{\beta^2}\frac{1}{\sin3\gamma}\frac{\partial}{\partial\gamma}\sin3\gamma\frac{\partial}{\partial\gamma}\right]. \quad (8)$$

Surface rotational energy:

$$T_{\text{rot}}=\sum_{\kappa=1}^{3}(\hbar^2Q_\kappa^2/2\mathcal{I}_\kappa). \quad (9)$$

Surface potential energy:

$$V_s=\tfrac{1}{2}C\beta^2. \quad (10)$$

B and C are the quantities defined by Eqs. (3) and (2). $\mathcal{I}_\kappa$ is the effective moment of inertia of the nuclear surface about the κ axis and is given by

$$\mathcal{I}_\kappa=4B\beta^2\sin^2[\gamma-(2\pi/3)\kappa]. \quad (11)$$

The component of the rotational angular momentum of the nuclear surface, Q_κ, is given simply by

$$Q_\kappa=I_\kappa-j_\kappa, \quad (12)$$

where I_κ and j_κ are components of the total angular momentum and of the angular momentum of the particle *along the nuclear axes*. The interaction energy, taken to first order in $R-R_0$, is given by

$$H_{\text{int}}=-V_0R_0\delta(r'-R_0)\textstyle\sum_\mu\alpha_\mu'Y_{2\mu}(\theta',\varphi'), \quad (13)$$

where V_0 is the depth of the nuclear potential well. Consistent with the assumption of rapid particle motion relative to nuclear vibration and rotation, this expression is averaged over particle coordinates $r'\theta'\varphi'$ before insertion into the Hamiltonian. It yields

$$\bar{H}^{(1)}{}_{\text{int}}=(nljm|H_{\text{int}}|nljm)=-\beta\cos\gamma(5/\pi)^{\frac{1}{2}}\bar{T}\frac{j(j+1)-3m^2}{4j(j+1)}, \quad (14)$$

where $\bar{T}$ is an energy of the order of magnitude of the particle kinetic energy, which depends on n and l but is here assumed to be a constant. (In the limit of infinite nuclear potential well depth, $\bar{T}$ is exactly the particle kinetic energy.) The matrix elements of Eq. (13) which are diagonal in nlj but off-diagonal in m,

$$\bar{H}^{(2)}{}_{\text{int}}=(nljm_1|H_{\text{int}}|nljm_2)=\beta\sin\gamma(15/\pi)^{\frac{1}{2}}\bar{T}[(j\pm m_1)(j\pm m_1-1)\times(j\mp m_1+1)(j\mp m_1+2)]^{\frac{1}{2}}\,\delta(m_1,m_2\pm2), \quad (15)$$

must also be added to the Hamiltonian to account for the mixing of different m's.

Bohr then splits the Hamiltonian into a zero-order part H_0, which commutes with I_3 and j_3, and a perturbing part U, with no diagonal matrix elements in an I_3j_3 representation. These are[9]

$$H_0=H_p+T_{\text{vib}}+\left\{V_s+\left(\frac{\hbar^2}{4\mathcal{I}_1}+\frac{\hbar^2}{4\mathcal{I}_2}\right)[I(I+1)-I_3^2+j(j+1)-j_3^2-(j+\tfrac{1}{2})(I+\tfrac{1}{2})(-1)^{I-j}\delta(\Omega,\tfrac{1}{2})\delta(K,\tfrac{1}{2})]+(\hbar^2/2\mathcal{I}_3)(I_3-j_3)^2+\beta\cos\gamma(5/\pi)^{\frac{1}{2}}\bar{T}\frac{3\Omega^2-j(j+1)}{4j(j+1)}\right\}, \quad (16)$$

and

$$U=U_1+U_2+U_3,$$

$$U_1=(1-D)\left(-\frac{\hbar^2}{\mathcal{I}_1}I_1j_1-\frac{\hbar^2}{\mathcal{I}_2}I_2j_2\right), \quad (17a)$$

where D is the projection operator onto the $j_3I_3(\Omega,K)$ representation,

$$U_2=\left(\frac{\hbar^2}{4\mathcal{I}_1}-\frac{\hbar^2}{4\mathcal{I}_2}+(15/\pi)^{\frac{1}{2}}\frac{\bar{T}\beta\sin\gamma}{4j(j+1)}\right)(j_1^2-j_2^2), \quad (17b)$$

$$U_3=\left(\frac{\hbar^2}{4\mathcal{I}_1}-\frac{\hbar^2}{4\mathcal{I}_2}\right)(I_1^2-I_2^2). \quad (17c)$$

The part of H_0 enclosed in curly brackets, since it contains no derivatives in β, γ, may be regarded as an effective surface potential. It is denoted by $W(\beta,\gamma)$, and the values of β, γ for which W is minimum, by β_1, γ_1. W contains three parts in the following historical order: (1) $V_s=\frac{1}{2}C\beta^2$ = classical liquid droplet surface potential, minimum at $\beta=0$, appropriate for spherically symmetric nuclei; (2) the interaction term H_{int}, linear in β, which displaces the minimum of W to β_1 different from zero, as considered, e.g., by Rainwater[10] and by Feenberg and Hammack;[11] (3) the rotational terms, proportional to $1/\beta^2$, introduced by Bohr to take proper account of the total nuclear dynamics. For extremely strong particle to surface coupling, the rotational terms become negligible and one may revert to the simpler theory which treats the nucleus as stationary in space. Such a limit is not attained in actual nuclei. For rather strong coupling, the rotational terms are smaller than the surface potential V_s or the interaction term H_{int}, but are not negligible. This appears to be the situation for the low levels of most heavy nuclei ($A\gtrsim150$). The strong coupling approximation is then valid (particle motion rapid relative to nuclear rotation) and the perturbation terms U are found to be small. For larger rotation terms, the strong coupling approximation begins to fail, the amplitude of nuclear oscillation

[9] These expressions are not exactly as given by Bohr, because of the diagonal contribution of Bohr's U_1, pointed out by Davidson and Feenberg (see reference 14).

[10] J. Rainwater, Phys. Rev. **79**, 432 (1950).

[11] E. Feenberg and K. C. Hammack, Phys. Rev. **81**, 285 (1951).

becomes comparable to the mean distortion, and the perturbation terms in U become large. This appears to be the situation for $A \lesssim 75$. In the range $75<A<150$, the validity of the strong coupling approximation depends on the number of extra nucleons participating in the distortion. Adding extra nucleons increases the values of H_{int} and V_s, increasing β_1, and thereby decreasing the magnitude of the rotational kinetic energy and of most of the terms in U.

B. Extension to Several Extra Nucleons

The extension of the theory to several extra nucleons is straightforward. The interaction term becomes a sum over the extra particles, and the components of rotational angular momentum are given by

$$Q_\kappa = I_\kappa - j_{a\kappa} - j_{b\kappa} - \cdots - j_{n\kappa}.$$

Insisting on I_3, j_{i3}, all i, as good quantum numbers of the zero-order Hamiltonian, we obtain

$$\begin{aligned} H_0 = \textstyle\sum_i H_p{}^{(i)} + T_{\text{vib}} + \Big\{ & \tfrac{1}{2}C\beta^2 + (\hbar^2/2\mathcal{I}_3)(I_3 - \textstyle\sum_i j_{i3})^2 \\ & + \frac{1}{2}\left(\frac{\hbar^2}{2\mathcal{I}_1} + \frac{\hbar^2}{2\mathcal{I}_2}\right)[I(I+1) - I_3{}^2 + \textstyle\sum_i j_i(j_i+1) - j_{i3}{}^2 \\ & - \sum_{i\neq m} (j_m - \Omega_m)(j_m + \Omega_m + 1)\delta(j_i, j_m)\delta(\Omega_m, \Omega_i - 1) \\ & - (I+\tfrac{1}{2})(-1)^{I - \Sigma_i j_i + \frac{1}{2}(n-1)}\delta(K, \tfrac{1}{2}) \\ & \times \textstyle\sum_i (j_i + \tfrac{1}{2})\delta(\Omega_i, \tfrac{1}{2})\delta([\Omega_k], [-\Omega_k])] \\ & + (5/\pi)^{\frac{1}{2}}\bar{T}\beta \cos\gamma \textstyle\sum_i \dfrac{3\Omega_i{}^2 - j_i(j_i+1)}{4j_i(j_i+1)} \Big\}, \end{aligned} \tag{18}$$

where n is the number of extra nucleons, $\delta(j_i, j_m) = 0$ unless the quantum numbers $(nlj)_i = (nlj)_m$, and $\delta([\Omega_k], [-\Omega_k])$ vanishes unless for every particle with quantum numbers $(nlj\Omega)$, except the ith particle, there is another particle with quantum numbers $(nlj, -\Omega)$. Note that for a closed subshell all of the sums over particle quantum numbers vanish. The perturbing Hamiltonian is

$$U = U_0 + U_1 + U_2 + U_3 + U_4.$$

$$U_0 = (1-D)(\hbar^2/4\mathcal{I}_1 + \hbar^2/4\mathcal{I}_2)\Big(\sum_{i\neq m} j_{i1}j_{m1} + j_{i2}j_{m2}\Big), \tag{19a}$$

where D is the projection operator onto the $I_3 j_{i3}$ representation;

$$U_1 = (1-D)[-(\hbar^2/\mathcal{I}_1)I_1 \textstyle\sum_i j_{i1} - (\hbar^2/\mathcal{I}_2)I_2 \sum_i j_{i2}]; \tag{19b}$$

$$\begin{aligned} U_2 = \textstyle\sum_i \Bigg(& \frac{\hbar^2}{4\mathcal{I}_1} - \frac{\hbar^2}{4\mathcal{I}_2} \\ & + (15/\pi)^{\frac{1}{2}} \frac{\bar{T}\beta \sin\gamma}{4j_i(j_i+1)} \Bigg)(j_{i1}{}^2 - j_{i2}{}^2); \end{aligned} \tag{19c}$$

$$U_3 = \tfrac{1}{4}(\hbar^2/\mathcal{I}_1 - \hbar^2/\mathcal{I}_2)(I_1{}^2 - I_2{}^2); \tag{19d}$$

$$U_4 = \tfrac{1}{4}(\hbar^2/\mathcal{I}_1 - \hbar^2/\mathcal{I}_2)\Big(\sum_{i\neq m} j_{i1}j_{m1} - j_{i2}j_{m2}\Big). \tag{19e}$$

As before U has no diagonal elements in the I_3, j_{i3} representation. The terms $U_0, \cdots U_4$ are arranged more or less in order of decreasing importance. Matrix elements of angular momenta appearing in U_1, U_2, and U_3 have been given by Bohr.[6] Matrix elements needed for evaluating the effect of U_0 and U_4 are

$$\begin{aligned} &(\Omega_i\Omega_m | \, j_{i1}j_{m1} + j_{i2}j_{m2} | \, \Omega_i \pm 1, \Omega_m \mp 1) \\ &\quad = \tfrac{1}{2}[(j_i \pm \Omega_i + 1)(j_i \mp \Omega_i)(j_m \mp \Omega_m + 1)(j_m \pm \Omega_m)]^{\frac{1}{2}}, \\ &(\Omega_i\Omega_m | \, j_{i1}j_{m1} - j_{i2}j_{m2} | \, \Omega_i \pm 1, \Omega_m \pm 1) \\ &\quad = \tfrac{1}{2}[(j_i \pm \Omega_i + 1)(j_i \mp \Omega_i)(j_m \pm \Omega_m + 1)(j_m \mp \Omega_m)]^{\frac{1}{2}}. \end{aligned}$$

U_0 connects states for which two particles have Ω_i changed by one, and $\sum_i \Omega_i = \Omega$ is unchanged. U_1 connects states for which one particle has Ω_i changed by one, and K is changed by one. U_2 connects states for which one particle has Ω_i changed by two. U_3 connects states for which K is changed by two. U_4 connects states for which two particles have Ω_i changed by one, and $\sum_i \Omega_i = \Omega$ is changed by two.

In order to get numbers for the energy levels of nuclei on this theory, one further approximation is necessary. The energies will be given by

$$E = W(\beta_1\gamma_1) + E_{\text{particle}} + E_\beta + E_\gamma. \tag{20}$$

E_β and E_γ are the energies of the β and γ vibration oscillations about the equilibrium value β_1, γ_1. These can be found approximately by expanding W about β_1, γ_1 to order $(\beta-\beta_1)^2$ and $(\gamma-\gamma_1)^2$, giving a harmonic potential for the vibration levels. Introducing a corresponding approximation in T_{vib}, Bohr obtains

$$E_\beta = (2\hbar^2/B\beta_1{}^2) + \hbar(n_\beta + \tfrac{1}{2})[(\partial^2 W/\partial\beta^2)_{\beta_1\gamma_1}/B]^{\frac{1}{2}}, \tag{21}$$

$$E_\gamma = \hbar(n_\gamma + 1)[(\partial^2 W/\partial\gamma^2)_{\beta_1\gamma_1}/B\beta_1{}^2]^{\frac{1}{2}}, \tag{22}$$

for the case $\gamma_1 = 0$ or π. For other values of γ_1, to the same approximation, E_β is the same and

$$\begin{aligned} E_\gamma = \{ & (9/8)(\hbar^2/B\beta_1{}^2)\cot^2 3\gamma_1 \\ & + \hbar(n_\gamma + \tfrac{1}{2})[(\partial^2 W/\partial\gamma^2)_{\beta_1\gamma_1}/B\beta_1{}^2]^{\frac{1}{2}} \}. \end{aligned}$$

Here n_β and n_γ are integers equal to the number of excitation quanta in the β and γ vibrations. The level spacing of the β and γ vibrations is, in general, large compared to the level spacing of different rotational states. This gives the important result that the rotational energy of the nuclear surface contributes significantly to the ground and low excited states of nuclei, while the vibration levels do not. The previous conclusion that the liquid droplet model cannot take account of shell structure properties is thereby altered completely. According to the collective model, the important contribution of the nuclear surface to low-lying levels comes via rotational energy, but the rotational

energy depends sensitively on the nuclear distortion ($\sim\beta^{-2}$), which in turn is determined by the orbits of the extra nucleons. The collective distortions of the nucleus in this respect, therefore, enhance and multiply shell structure effects, rather than smooth over them. It is found also that the vibrational energies $E_\beta{}^{(0)}$ and $E_\gamma{}^{(0)}$ are insensitive to changes in β_1, γ_1, so that the level order and level spacing are given to fair approximation by the potential minimum $W(\beta_1\gamma_1)$ alone. In the calculations described in the next part, W alone was first minimized; then E_β and E_γ were calculated. A more consistent procedure would probably be to find β_1, γ_1 by minimizing the entire energy expression (20). This would yield slightly larger values of the nuclear distortion than minimization of W alone.

Assuming for a moment that the perturbing terms U have a small effect on the solution of the zero-order Hamiltonian H_0, as appears to be the case for heavy nuclei containing several extra nucleons outside closed shells, it is worth while to examine some of the basic qualitative differences between the free particle model alone and the collective model of the nucleus. (1) According to the collective model, a given nuclear state is characterized by more quantum numbers: I, the total angular momentum, and its component M along an axis in space; K, the component of I along the 3-axis of the distorted nucleus; $\Omega_1\cdots\Omega_n$, the components of the particle angular momenta j_i along the 3-axis of the nucleus, and n_β, n_γ, the degrees of excitation of the surface vibration. J, the total angular momentum of the particles, is *not* a good quantum number; nor is Q, the angular momentum of the nuclear surface. (The term nuclear surface rather than nuclear core is used because the rotation is not a rigid body rotation, but rather the rotation of a surface wave, whose amplitude depends on the distortion of the nucleus, vanishing for a spherical nucleus.) (2) The collective model predicts a larger number of nuclear levels than does the free particle model. (3) According to the collective model, the mechanism of coupling of different nucleons is entirely different from that predicted by the free particle model. The interaction of particles with the nuclear surface may dominate over the direct coupling of particles to each other. Even if j remains a good quantum number, j-j coupling is destroyed. For example, on the independent particle model a state of two particles each of spin 5/2, with net angular momentum zero, contains all m's from $-5/2$ to $+5/2$ in a prescribed combination. According to the collective model, the same two particles can combine to zero net angular momentum with m's (Ω's) $=\pm 5/2$ only. (Inclusion of perturbation terms U mixes in Ω's of $\pm\frac{3}{2}$, $\pm\frac{1}{2}$, but in a proportion unrelated to the unique specification of j-j coupling.) A more detailed discussion of new features of the collective model is given by Hill and Wheeler.[7]

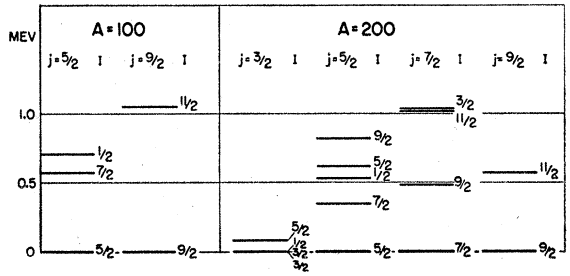

FIG. 1. Predicted energy levels for one extra nucleon outside closed shells. These comprise only those low levels in which the particle states are unchanged. Additional results of detailed calculations with Bohr strong coupling approximation are given in Table III.

C. Numerical Results for Several Examples

Using the Bohr strong coupling approximation, the low-lying levels have been computed for several hypothetical nuclei containing one, two, or four extra particles (of the same nucleon type) outside of a closed shell, with the other nucleon type considered to remain in a closed shell. Only those levels have been treated for which the extra nucleons are not excited to higher single particle levels. This is expected to represent the true situation for the low levels of most even nuclei; but it is not expected to yield the lowest excited states for odd-even nuclei, because of the close spacing of the free particle levels relative to the nuclear rotational levels.

The results for the case of one extra nucleon are given in Fig. 1. The ground state is seen to have spin $I=j$, as already pointed out by Bohr.[8] [For the case $j=\frac{3}{2}$, the ground-state degeneracy in first order will be important for magnetic moments (see reference 14).] The first excited states have spin $j+1$, the second excited excited states, spin $\frac{1}{2}$ or spin $j+2$, and the energy of the first excited state increases with j. This "first excited state" is the lowest excited state with the odd nucleon state unchanged. Lower excited states could appear due to a transition of the odd nucleon to another free particle state, which appears to be the case in most odd-even nuclei.

The Bohr strong coupling picture should give better results for two extra nucleons than for one: First, the high excitation energies of the magic nuclei suggest that pairs of like nucleons are strongly coupled to zero net angular momentum. Therefore, for even nuclei, states due to excitation of nuclear rotational degrees of freedom could lie below levels due to transitions of the extra nucleons among free particle states. Second, the approximation itself is better because of the larger nuclear distortions computed. The results of the computations for two and four extra nucleons are shown in Fig. 2. The computed level order agrees very well with available data. The computed spins for the ground, first excited, and second excited states are 0, 2, 4, with the third excited state being of spin 0, 2, or 6. In one

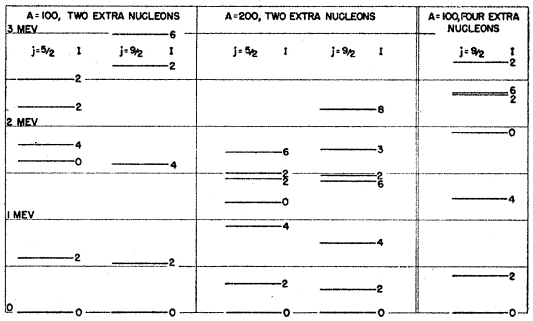

FIG. 2. Predicted energy levels for nuclei with two or four nucleons outside closed shells. Additional results of detailed calculations with Bohr strong coupling approximation are given in Table III.

of the computed cases the level order is 0, 2, 0, 4. The empirical data, which give 0; 2; 4 or 2 as the usual level order, is summarized in Table I, from the recent compilation by Scharff-Goldhaber.[2]

The computed level spacing appears to be of the right order of magnitude, but probably too small. The effect of inclusion of the off-diagonal terms in the Hamiltonian in the strong coupling approximation decreases the computed spacing, as discussed in part *D* of this section. Detailed comparison of theory and experiment for level spacing is possible only in the vicinity of doubly magic Pb^{208}, where the strong coupling approximation should be reasonably good, and where there exist simple examples of even nuclei with only two or four nucleons outside closed shells. Even here, however, the orbital assignments of the extra nucleons are not unambiguous. Table II gives a comparison of observed first excited energies with those calculated "from first principles," using the simplified formula derived in part III, and a nuclear distortion computed by minimizing the effective surface potential W neglecting the rotational terms. For the simplest example, Pb^{206} ($Z=82$, $N=124$), the neutron orbital assignment $(p_{\frac{1}{2}})^{-2}$ gives good agreement with experiment. The assignment $(i_{13/2})^{-2}$ gives considerably too small an energy spacing; the assignment $(p_{\frac{1}{2}})^{-2}$ gives a spherically symmetric nucleus with no low lying rotational state. For Pb^{204}, the neutron assignment $(p_{\frac{1}{2}})^{-2}(i_{13/2})^{-2}$ gives the best agreement with experiment. For Po^{212}, no reasonable assignment gives agreement.

It can be concluded from Table II that the computed nuclear distortions are unreasonably large, and the computed energies therefore too small, on the strong coupling approximation. Halving the particle-to-surface interaction, for example, would lower the values of β to more reasonable magnitude, would bring the energies of Pb^{206} and Pb^{204} into good agreement with the configurations $(i_{13/2})_N^{-2}$ and $(i_{13/2})_N^{-4}$, and would bring the predicted energy for Po^{212} more nearly into agreement with experiment.

Also for many extra nucleons, distortions calculated from first principles are too large. Assuming half-filled shells of $i_{13/2}$ neutrons and $h_{11/2}$ protons, one finds the maximum possible distortion in the rare earth group to be $\beta=1.0$, or three times larger than the largest distortion computed from quadrupole moments. Again a reduction in the particle-to-surface coupling appears to be required. For this case of strong interaction, however, the effect of higher order terms in the interaction energy should also be investigated.

Figure 3 shows the variation in energy of the first excited state and in extra binding energy due to surface coupling for even nuclei filling a hypothetical shell of $j=9/2$ particles. The closed shell energy levels in Fig. 3 and in Table II are computed from Eq. (1). It should be noted that the large energy at double magic Pb^{208} is an important defect of the theory. Two of the three basic parameters of the collective model enter Eq. (1), which should provide an *upper limit* for the excitation energy of doubly magic nuclei. An increase by a factor 4 of the product $(C)(\hbar^2/B)$ is required to secure agreement with experiment for Pb^{208}. A recently determined level[12] of Ca^{40} at 3.8 Mev does agree with the prediction of Eq. (1), however.

TABLE I. Comparison of observed and predicted level order for even nuclei.

State	Predicted spin, collective model, assuming no nucleonic excitation	Observed spin[a]
Ground	0+	0+, no known exceptions
First excited	2+	2+, no known exceptions above $A=75$ among 37 examples
Second excited	4+ (sometimes 0+)	4+, ~40 percent 2+, ~40 percent 0, 1, 3, ~20 percent
Third excited	0+, 2+, or 6+ if second excited is 4+ (4+ if second excited is 0+)	

[a] See reference 2.

The numerical results of the detailed computations using the Bohr strong coupling approximation are given in Table III, for those levels diagrammed in Figs. 1 and 2, and also for higher levels. The higher levels are not included in the figure because they are incomplete, leaving out some levels, especially for the quantum numbers K and Ω unequal, and also because the strong coupling approximation breaks down at the higher excitations. This is indicated by the fact that the rotational energy term W_3 becomes comparable in magnitude to W_1 and $|W_2|$.

The allowed combinations of quantum numbers can be found from the symmetry requirements on the wave function discussed by Bohr. In particular, the wave function must be invariant under a rotation of π about the 2-axis (his R_1 operator) and under a rotation of $\pi/2$ about the 3-axis (his R_2 operator). These operators

[12] J. A. Harvey, Phys. Rev. 88, 162 (1952).

act on an n particle wave function in the $(n_i l_i j_i m_i)$ representation as follows:

$$R_1\chi(\cdots n_i l_i j_i m_i \cdots) = (-1)^{\Sigma_i(j_i+m_i)}\chi(\cdots n_i l_i j_i - m_i \cdots), \quad (23a)$$

$$R_2\chi(\cdots n_i l_i j_i m_i \cdots) = e^{-i(\pi/2)\Sigma_i m_i}\chi(\cdots n_i l_i j_i m_i \cdots). \quad (23b)$$

As a result the wave function must be of the form (replacing m_i by Ω_i),

$$\Psi_M{}^I = \Big(\sum_{\Omega>0,\,K} + \sum_{\Omega=0,\,K\geq 0}\Big)\varphi^I{}_{\Omega,K}(\beta,\gamma) \times[\chi(\cdots j_i, \Omega_i \cdots)D^I{}_{MK}(\theta_i) + (-1)^{I-\Sigma_i j_i}\chi(\cdots j_i, -\Omega_i \cdots)D^I{}_{M,-K}(\theta_i)]. \quad (24)$$

The values of $|K-\Omega|$ are restricted to even integers.

TABLE II. First excited states of selected even nuclei near doubly magic Pb^{208}.

Nucleus	Theoretical: Assumed configuration	Distortion, $\beta\cos\gamma$	Energy of first excited state (Mev)	Experimental energy of first excited state (Mev)
$_{82}Pb_{122}$	$[(p_{3/2})^{-4}]_N$	0.0	1.3	
	$[(p_{3/2})^{-2}(p_{1/2})^{-2}]_N$	0.17	0.86	
	$[(p_{3/2})^{-2}(i_{13/2})^{-2}]_N$	0.48	0.10	0.374
	$[(p_{1/2})^{-2}(i_{13/2})^{-2}]_N$	0.33	0.21	
	$[(i_{13/2})^{-4}]_N$	0.50	0.10	
$_{82}Pb_{124}$	$[(p_{1/2})^{-2}]_N$	0.0	1.3	
	$[(p_{3/2})^{-2}]_N$	0.17	0.86	0.803
	$[(i_{13/2})^{-2}]_N$	0.33	0.21	
$_{82}Pb_{126}$	closed shell	0.0	1.3	2.62
$_{82}Pb_{128}$	$[(g_{9/2})^2]_N$	−0.30	0.25	unknown
	$[(d_{5/2})^2]_N$	−0.24	0.40	
$_{82}Pb_{130}$	$[(g_{9/2})^4]_N$	−0.39	0.15	
	$[(g_{9/2})^2(d_{5/2})^2]_N$	−0.52	0.085	unknown
	$[(d_{5/2})^4]_N$	0.23	0.44	
$_{84}Po_{126}$	$[(h_{9/2})^2]_P$	−0.30	0.25	>1(?)
$_{84}Po_{128}$	$[(h_{9/2})^2]_P[(g_{9/2})^2]_N$	−0.58	0.068	0.719
	$[(h_{9/2})^2]_P[(d_{5/2})^2]_N$	−0.52	0.085	

If K and Ω are separately good quantum numbers, as in the first-order approximation used here, then there is only one term in Eq. (24) and further restrictions arise. For an even number of extra nucleons, all of the same n, l, j, and for $\Omega=K=0$, Ψ vanishes for all odd I. These two rules were used to determine the allowed levels in Table III.

D. Validity of the Strong Coupling Approximation

The collective model of the nucleus has two appealing features. First, it contains only a few physical parameters, and none of these are arbitrary. Second, the application of the model through the strong coupling approximation developed by Bohr is relatively easy in first order. The question of the validity of the assumptions on which the whole model is based are therefore most easily approached by means of applications of the model to various examples and comparison with experiment. The more fundamental approach of analysis of the initial assumptions of the model from first principles has been begun by Hill and Wheeler.[7] It is intended here mainly to discuss the question of the validity of the first-order strong coupling approximation relative to the complete Hamiltonian developed by Bohr. Some of the limitations on the complete Hamiltonian are the following: (a) Only the ellipsoidal mode of deformation and vibration is considered. (b) The extra nucleons are taken to have j as a good quantum number. (c) The particle motion is taken to be rapid relative to the motion of the core, and the interaction term is averaged over the particle motion.

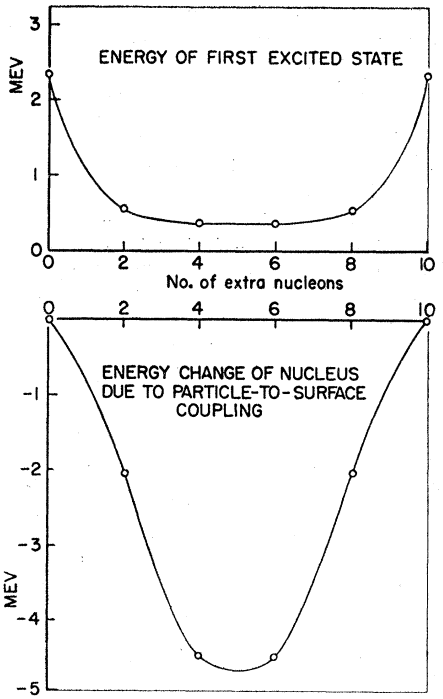

FIG. 3. Energy of first excited state and energy change of nucleus due to surface interaction for highly idealized nuclei. A shell of $j=9/2$ nucleons is assumed to fill while the other group of nucleons remains closed shell. The mass number is taken to be 100. The results at 2 and 4 (same as at 8 and 6) extra nucleons are taken from the strong coupling calculations of Table III and Fig. 2. The results at zero (and at 10) extra nucleons are the weak coupling limit with excitation energy given by Eq. (1) in the text and with no particle-to-surface interaction.

The three parameters which enter the theory are coefficients in the three terms in the effective surface potential W [Eq. (18) and caption of Table III]. The parameter C [Eq. (2)] is the coefficient of the nuclear distortion term, $W_1=\frac{1}{2}C\beta^2$. Using a surface tension energy $4\pi r_0{}^2\mathcal{O}=15.4$ Mev, values of C are obtained as given in Table IV. In the calculations performed at mass numbers 100 and 200, a value $C=65$ Mev was used. It is not certain that the dynamic surface tension is the same as the static surface tension,[7] but approxi-

TABLE III. Summarized results of calculations using Bohr strong coupling approximation.

I = Total angular momentum of nucleus.
Ω_i = Component of nucleon angular momentum along 3-axis of distorted nucleus.
$\Omega = \Sigma_i \Omega_i$.
K = Component of total angular momentum along 3-axis of distorted nucleus.
β_1 = Equilibrium value of distortion parameter.
γ_1 = Equilibrium value of shape parameter.
W_1 = Potential energy of surface deformation $= \frac{1}{2}C\beta^2$.
W_2 = Interaction energy of extra nucleons with nuclear surface ("Rainwater interaction")

$$= (5/\pi)^{\frac{1}{4}} \bar{T} \beta \cos\gamma \Sigma_i \frac{3\Omega_i^2 - j_i(j_i+1)}{4j_i(j_i+1)}.$$

W_3 = Kinetic energy of surface rotation (diagonal part)

$$= (1/4)(\hbar^2/\mathcal{I}_1 + \hbar^2/\mathcal{I}_2)[I(I+1) - K^2 + \Sigma_i j_i(j_i+1) - \Omega_i^2 - (I+\tfrac{1}{2})(-1)^{I-\Sigma_i j_i + \frac{1}{2}(n-1)} \delta(K, \tfrac{1}{2}) \times \Sigma_i (j_i+\tfrac{1}{2}) \delta(\Omega_i, \tfrac{1}{2}) \delta([\Omega_k], [-\Omega_k]) - \sum_{i \neq m} \delta(j_i, j_m) \delta(\Omega_m, \Omega_i - 1)(j_m - \Omega_m)(j_m + \Omega_m + 1)].$$

$W = W_1 + W_2 + W_3$ = Net effective surface potential energy.
E_β = Zero-point energy of β oscillation.
E_γ = Zero-point energy of γ oscillation.
$E - E_p = W + E_\beta + E_\gamma$ = (net nuclear energy) − (unperturbed particle energy) = diagonal matrix element of Hamiltonian in Bohr strong coupling approximation.
E_{rel} = Nuclear energy normalized to ground state energy = 0.
All energies in Mev.

I	Ω_i	Ω	K	β_1	$\cos\gamma_1$	W_1	W_2	W_3	W	E_β	E_γ	$E-E_p$	E_{rel}
					$A=100$, one extra nucleon, $j=5/2$								
5/2		5/2	5/2	0.2466	−1.0000	1.977	−1.778	1.088	1.287	4.460	2.829	8.576	0.000
7/2		5/2	5/2	0.2971	−1.0000	2.870	−2.142	1.799	2.526	3.726	2.897	9.150	0.574
1/2		1/2	1/2	0.2483	1.0000	2.004	−1.432	1.288	1.860	4.519	2.912	9.291	0.715
5/2		1/2	1/2	0.2646	1.0000	2.275	−1.526	1.512	2.262	4.234	2.931	9.427	0.851
9/2		5/2	5/2	0.3361	−1.0000	3.671	−2.423	2.460	3.708	3.376	2.936	10.021	1.445
3/2		1/2	1/2	0.3176	1.0000	3.278	−1.831	2.362	3.808	3.594	2.980	10.382	1.806
9/2		1/2	1/2	0.3176	1.0000	3.278	−1.831	2.362	3.808	3.594	2.980	10.382	1.806
3/2		3/2	3/2	0.2446	1.0000	1.945	−0.353	1.769	3.361	4.846	3.139	11.347	2.771
5/2		3/2	3/2	0.2755	1.0000	2.466	−0.397	2.267	4.336	4.295	3.147	11.779	3.203
7/2		1/2	1/2	0.3724	1.0000	4.508	−2.148	3.434	5.795	3.204	3.015	12.014	3.438
5/2		1/2	5/2	0.3607	0.8869	4.228	−1.845	3.305	5.688	3.298	3.078	12.065	3.489
7/2		3/2	3/2	0.3061	1.0000	3.045	−0.441	2.825	5.429	3.904	3.154	12.487	3.911
9/2		3/2	3/2	0.3353	1.0000	3.655	−0.484	3.413	6.584	3.627	3.159	13.370	4.794
					$A=100$, one extra nucleon, $j=9/2$								
9/2		9/2	9/2	0.2905	−1.0000	2.744	−2.666	1.411	1.489	3.692	2.795	7.976	0.000
11/2		9/2	9/2	0.3430	−1.0000	3.823	−3.147	2.250	2.926	3.239	2.863	9.028	1.052
1/2		1/2	1/2	0.3270	1.0000	3.475	−2.000	2.475	3.950	3.497	2.973	10.420	2.444
					$A=200$, one extra nucleon, $j=3/2$								
3/2		3/2	3/2	0.1641	−1.0000	0.876	−0.828	0.462	0.509	2.869	1.572	4.951	0.000
3/2		1/2	1/2	0.1642	1.0000	0.876	−0.829	0.462	0.509	2.869	1.572	4.951	0.000
1/2		1/2	1/2	0.1898	1.0000	1.170	−0.958	0.691	0.904	2.442	1.605	4.951	0.000
5/2		3/2	3/2	0.2019	−1.0000	1.324	−1.019	0.815	1.120	2.295	1.617	5.033	0.082
7/2		1/2	1/2	0.2163	1.0000	1.521	−1.092	0.975	1.405	2.152	1.630	5.186	0.235
7/2		3/2	3/2	0.2317	−1.0000	1.744	−1.169	1.160	1.735	2.029	1.641	5.405	0.454
5/2		1/2	1/2	0.2413	1.0000	1.892	−1.218	1.283	1.958	1.964	1.648	5.569	0.618
9/2		3/2	3/2	0.2574	−1.0000	2.153	−1.299	1.503	2.358	1.871	1.658	5.886	0.935
9/2		1/2	1/2	0.2857	1.0000	2.653	−1.442	1.932	3.143	1.745	1.672	6.561	1.610
					$A=200$, one extra nucleon, $j=5/2$								
5/2		5/2	5/2	0.1959	−1.0000	1.247	−1.412	0.541	0.376	2.263	1.523	4.162	0.000
7/2		5/2	5/2	0.2326	−1.0000	1.759	−1.677	0.920	1.002	1.940	1.570	4.512	0.350
1/2		1/2	1/2	0.1939	1.0000	1.222	−1.118	0.662	0.766	2.356	1.580	4.702	0.540
5/2		1/2	1/2	0.2058	1.0000	1.377	−1.187	0.784	0.973	2.222	1.593	4.789	0.627
9/2		5/2	5/2	0.2613	−1.0000	2.218	−1.883	1.276	1.612	1.780	1.597	4.988	0.826
1/2		5/2	1/2	0.2432	−0.8453	1.922	−1.482	1.182	1.622	1.915	1.558	5.095	0.933
3/2		1/2	1/2	0.2450	1.0000	1.951	−1.413	1.244	1.782	1.915	1.628	5.326	1.164
9/2		1/2	1/2	0.2450	1.0000	1.951	−1.413	1.244	1.782	1.915	1.628	5.326	1.164
7/2		1/2	1/2	0.2857	1.0000	2.654	−1.648	1.830	2.835	1.724	1.654	6.213	2.051
3/2		3/2	3/2	0.1846	1.0000	1.107	−0.266	0.974	1.816	2.675	1.744	6.235	2.073
5/2		1/2	5/2	0.2762	0.8899	2.480	−1.418	1.771	2.833	1.780	1.669	6.283	2.121
5/2		3/2	3/2	0.2076	1.0000	1.401	−0.299	1.251	2.353	2.375	1.750	6.479	2.317
7/2		3/2	3/2	0.2306	1.0000	1.728	−0.332	1.561	2.957	2.162	1.755	6.874	2.712
9/2		3/2	3/2	0.2524	1.0000	2.071	−0.364	1.889	3.596	2.011	1.759	7.366	3.204
					$A=200$, one extra nucleon, $j=7/2$								
7/2		7/2	7/2	0.2169	−1.0000	1.530	−1.825	0.617	0.322	2.004	1.507	3.833	0.000
9/2		7/2	7/2	0.2540	−1.0000	2.097	−2.136	1.029	0.990	1.772	1.553	4.315	0.482
11/2		7/2	7/2	0.2824	−1.0000	2.592	−2.375	1.405	1.622	1.655	1.580	4.857	1.024
3/2		1/2	1/2	0.2216	1.0000	1.595	−1.331	0.930	1.194	2.069	1.601	4.864	1.031
3/2		7/2	3/2	0.2631	−0.8694	2.250	−1.924	1.288	1.614	1.768	1.524	4.906	1.073
7/2		1/2	1/2	0.2368	1.0000	1.822	−1.422	1.111	1.510	1.958	1.614	5.082	1.249

TABLE III.—(*Continued*).

I	Ω_i		Ω	K	β_1	$\cos\gamma_1$	W_1	W_2	W_3	W	E_β	E_γ	$E-E_p$	E_{rel}
						$A=200$, one extra nucleon, $j=7/2$—(*Continued*)								
1/2			1/2	1/2	0.2520	1.0000	2.064	−1.514	1.307	1.857	1.867	1.626	5.350	1.517
11/2			1/2	1/2	0.2692	1.0000	2.355	−1.617	1.546	2.284	1.784	1.637	5.705	1.872
3/2			3/2	3/2	0.2246	1.0000	1.639	−0.809	1.234	2.064	2.136	1.684	5.884	2.051
5/2			1/2	1/2	0.2870	1.0000	2.676	−1.724	1.814	2.767	1.713	1.648	6.127	2.294
5/2			3/2	3/2	0.2401	1.0000	1.873	−0.865	1.440	2.448	2.021	1.692	6.161	2.328
5/2			5/2	5/2	0.2026	−1.0000	1.335	−0.243	1.213	2.304	2.441	1.758	6.502	2.669
7/2			3/2	3/2	0.2576	1.0000	2.154	−0.928	1.690	2.917	1.916	1.699	6.532	2.699
7/2			5/2	5/2	0.2267	−1.0000	1.670	−0.272	1.534	2.932	2.201	1.762	6.896	3.063
9/2			3/2	3/2	0.2754	1.0000	2.466	−0.993	1.969	3.442	1.829	1.706	6.977	3.144
9/2			1/2	1/2	0.3220	1.0000	3.369	−1.934	2.401	3.837	1.607	1.665	7.109	3.276
9/2			5/2	5/2	0.2493	−1.0000	2.020	−0.299	1.870	3.590	2.038	1.765	7.393	3.560
11/2			3/2	3/2	0.2933	1.0000	2.796	−1.057	2.267	4.006	1.757	1.712	7.475	3.642
11/2			5/2	5/2	0.2704	−1.0000	2.376	−0.325	2.214	4.265	1.920	1.768	7.953	4.120
						$A=200$, one extra nucleon, $j=9/2$								
9/2			9/2	9/2	0.2325	−1.0000	1.757	−2.134	0.691	0.315	1.860	1.501	3.676	0.000
11/2			9/2	9/2	0.2704	−1.0000	2.376	−2.481	1.135	1.031	1.674	1.546	4.251	0.575
1/2			1/2	1/2	0.2526	1.0000	2.073	−1.545	1.301	1.829	1.861	1.623	5.313	1.637
						$A=100$, two extra nucleons, $j=5/2$								
0	5/2	−5/2	0	0	0.2984	−1.0000	2.894	−4.303	0.743	−0.665	3.294	2.547	5.176	0.000
2	5/2	−5/2	0	0	0.3379	−1.0000	3.711	−4.872	1.275	0.114	3.009	2.633	5.757	0.581
0	1/2	−1/2	0	0	0.2992	1.0000	2.908	−3.450	1.183	0.641	3.467	2.695	6.803	1.627
4	5/2	−5/2	0	0	0.3915	−1.0000	4.981	−5.644	2.159	1.495	2.759	2.720	6.974	1.798
2	1/2	−1/2	0	0	0.3325	1.0000	3.593	−3.835	1.676	1.434	3.196	2.751	7.381	2.205
2	5/2	−5/2	0	2	0.3839	−0.8893	4.789	−4.922	2.328	2.196	2.859	2.619	7.673	2.497
6	5/2	−5/2	0	0	0.4426	−1.0000	6.366	−6.381	3.175	3.160	2.605	2.781	8.547	3.371
4	1/2	−1/2	0	0	0.3820	1.0000	4.742	−4.406	2.539	2.876	2.922	2.815	8.613	3.437
2	3/2	1/2	2	2	0.2792	1.0000	2.534	−2.013	1.528	2.049	3.940	2.876	8.865	3.689
4	5/2	3/2	4	4	0.2646	−1.0000	2.275	−1.526	1.512	2.262	4.234	2.931	9.427	4.251
1	3/2	−1/2	1	1	0.3165	1.0000	3.255	−2.281	2.114	3.088	3.536	2.918	9.542	4.366
1	5/2	−3/2	1	1	0.2781	−1.0000	2.513	−1.604	1.711	2.621	4.035	2.945	9.601	4.425
2	3/2	−1/2	1	1	0.3325	1.0000	3.593	−2.397	2.394	3.590	3.404	2.933	9.926	4.750
2	5/2	−3/2	1	1	0.2999	−1.0000	2.924	−1.730	2.059	3.253	3.769	2.966	9.988	4.812
6	1/2	−1/2	0	0	0.4313	1.0000	6.045	−4.974	3.558	4.628	2.743	2.863	10.234	5.058
8	5/2	−5/2	0	0	0.4893	−1.0000	7.783	−7.055	4.255	4.982	2.509	2.825	10.317	5.141
5	5/2	3/2	4	4	0.3176	−1.0000	3.278	−1.831	2.362	3.808	3.594	2.980	10.382	5.206
0	3/2	−3/2	0	0	0.2815	1.0000	2.576	−0.812	2.170	3.935	4.137	3.084	11.155	5.979
2	3/2	−3/2	0	0	0.3083	1.0000	3.089	−0.889	2.645	4.845	3.816	3.095	11.757	6.581
2	5/2	−1/2	2	2	0.2755	−1.0000	2.466	−0.397	2.267	4.336	4.295	3.147	11.779	6.603
3	5/2	1/2	3	3	0.2805	−1.0000	2.557	−0.404	2.355	4.507	4.222	3.149	11.878	6.702
						$A=100$, two extra nucleons, $j=9/2$								
0	9/2	−9/2	0	0	0.3605	−1.0000	4.224	−6.615	0.916	−1.475	2.681	2.506	3.712	0.000
2	9/2	−9/2	0	0	0.3874	−1.0000	4.877	−7.108	1.323	−0.909	2.588	2.561	4.241	0.529
4	9/2	−9/2	0	0	0.4304	−1.0000	6.021	−7.898	2.071	0.194	2.477	2.634	5.305	1.593
2	9/2	−9/2	0	2	0.4306	−0.9114	6.027	−7.202	2.426	1.251	2.544	2.565	6.360	2.648
6	9/2	−9/2	0	0	0.4755	−1.0000	7.348	−8.725	2.985	1.608	2.394	2.694	6.696	2.984
3	9/2	−9/2	0	2	0.4527	−0.9215	6.659	−7.655	2.832	1.837	2.489	2.645	6.970	3.258
8	9/2	7/2	8	8	0.3458	−1.0000	3.886	−4.230	1.771	1.427	3.076	2.741	7.243	3.531
1	9/2	−7/2	1	1	0.3538	−1.0000	4.067	−4.328	1.903	1.643	3.030	2.752	7.426	3.714
0	1/2	−1/2	0	0	0.3744	1.0000	4.556	−4.580	2.266	2.241	2.926	2.780	7.947	4.235
2	1/2	−1/2	0	0	0.3916	1.0000	4.984	−4.791	2.589	2.782	2.852	2.800	8.435	4.723
4	1/2	−1/2	0	0	0.4237	1.0000	5.835	−5.183	3.243	3.895	2.740	2.834	9.468	5.756
2	9/2	−5/2	2	2	0.3517	−1.0000	4.019	−2.689	2.675	4.006	3.250	2.932	10.188	6.476
1	3/2	−1/2	1	1	0.4230	1.0000	5.814	−4.528	3.551	4.837	2.799	2.883	10.520	6.808
0	7/2	−7/2	0	0	0.3441	−1.0000	3.847	−2.104	2.795	4.538	3.367	2.985	10.889	7.177
						$A=200$, two extra nucleons, $j=5/2$								
0	5/2	−5/2	0	0	0.2587	−1.0000	2.175	−3.730	0.310	−1.245	1.504	1.360	1.619	0.000
2	5/2	−5/2	0	0	0.2835	−1.0000	2.612	−4.087	0.568	−0.907	1.437	1.404	1.934	0.315
4	5/2	−5/2	0	0	0.3196	−1.0000	3.320	−4.608	1.016	−0.273	1.368	1.454	2.550	0.931
0	1/2	−1/2	0	0	0.2460	1.0000	1.967	−2.838	0.548	−0.322	1.685	1.439	2.801	1.182
2	5/2	−5/2	0	2	0.3131	−0.9035	3.187	−4.079	1.147	0.255	1.425	1.377	3.057	1.438
2	1/2	−1/2	0	0	0.2691	1.0000	2.354	−3.104	0.802	0.052	1.592	1.473	3.117	1.498
6	5/2	−5/2	0	0	0.3554	−1.0000	4.106	−5.125	1.544	0.525	1.322	1.492	3.340	1.721
4	1/2	−1/2	0	0	0.3043	1.0000	3.010	−3.510	1.255	0.754	1.492	1.514	3.761	2.142
8	5/2	−5/2	0	0	0.3889	−1.0000	4.916	−5.608	2.112	1.421	1.292	1.521	4.234	2.615
2	3/2	1/2	2	2	0.2195	1.0000	1.566	−1.583	0.775	0.759	2.036	1.555	4.350	2.731
6	1/2	−1/2	0	0	0.3400	1.0000	3.756	−3.921	1.795	1.630	1.423	1.547	4.600	2.981
3	3/2	1/2	2	2	0.2435	1.0000	1.927	−1.756	1.050	1.222	1.872	1.581	4.675	3.056

TABLE III.—(*Continued*).

I	Ω_i				Ω	K	β_1	$\cos\gamma_1$	W_1	W_2	W_3	W	E_β	E_γ	$E-E_p$	E_{rel}
								$A=200$, two extra nucleons, $j=5/2$—(*Continued*)								
1	3/2	−1/2			1	1	0.2468	1.0000	1.980	−1.779	1.090	1.291	1.853	1.584	4.728	3.109
4	5/2	3/2			4	4	0.2058	−1.0000	1.377	−1.187	0.784	0.973	2.222	1.593	4.789	3.170
1	5/2	−3/2			1	1	0.2158	−1.0000	1.513	−1.245	0.891	1.160	2.127	1.604	4.891	3.272
2	3/2	−1/2			1	1	0.2586	1.0000	2.173	−1.864	1.241	1.550	1.793	1.595	4.937	3.318
2	5/2	−3/2			1	1	0.2320	−1.0000	1.749	−1.338	1.080	1.491	1.999	1.618	5.109	3.490
3	3/2	−1/2			1	1	0.2734	1.0000	2.429	−1.971	1.444	1.902	1.727	1.606	5.236	3.617
5	5/2	3/2			4	4	0.2450	−1.0000	1.951	−1.413	1.244	1.782	1.915	1.628	5.326	3.707
3	5/2	−3/2			1	1	0.2507	−1.0000	2.043	−1.446	1.320	1.917	1.883	1.632	5.432	3.813
0	3/2	−3/2			0	0	0.2139	1.0000	1.487	−0.617	1.179	2.049	2.257	1.703	6.010	4.391
2	3/2	−3/2			0	0	0.2339	1.0000	1.778	−0.675	1.441	2.545	2.088	1.712	6.345	4.726
2	5/2	−1/2			2	2	0.2076	−1.0000	1.401	−0.299	1.251	2.353	2.375	1.750	6.479	4.860
3	5/2	1/2			3	3	0.2114	−1.0000	1.452	−0.305	1.300	2.448	2.335	1.751	6.534	4.915
3	5/2	−1/2			2	2	0.2277	−1.0000	1.685	−0.328	1.521	2.878	2.185	1.755	6.818	5.199
4	5/2	1/2			3	3	0.2360	−1.0000	1.810	−0.340	1.640	3.109	2.121	1.756	6.986	5.367
4	3/2	−3/2			0	0	0.2666	1.0000	2.311	−0.769	1.926	3.468	1.891	1.723	7.081	5.462
								$A=200$, two extra nucleons, $j=9/2$								
0	9/2	−9/2			0	0	0.3180	−1.0000	3.287	−5.836	0.369	−2.179	1.228	1.342	0.390	0.000
2	9/2	−9/2			0	0	0.3338	−1.0000	3.621	−6.125	0.559	−1.945	1.216	1.367	0.638	0.248
4	9/2	−9/2			0	0	0.3610	−1.0000	4.236	−6.625	0.923	−1.465	1.200	1.404	1.139	0.749
6	9/2	−9/2			0	0	0.3911	−1.0000	4.972	−7.177	1.383	−0.822	1.187	1.438	1.804	1.414
2	9/2	−9/2			0	2	0.3606	−0.9267	4.227	−6.133	1.160	−0.745	1.242	1.366	1.863	1.473
3	9/2	−9/2			0	2	0.3749	−0.9334	4.568	−6.421	1.357	−0.497	1.229	1.407	2.139	1.749
8	9/2	−9/2			0	0	0.4210	−1.0000	5.759	−7.725	1.897	−0.068	1.178	1.467	2.576	2.186
8	9/2	7/2			8	8	0.2806	−1.0000	2.560	−3.433	0.843	−0.030	1.529	1.467	2.966	2.576
1	9/2	−7/2			1	1	0.2862	−1.0000	2.663	−3.501	0.912	0.073	1.514	1.474	3.060	2.670
0	1/2	−1/2			0	0	0.3008	1.0000	2.941	−3.680	1.101	0.362	1.477	1.491	3.330	2.940
0	9/2	−9/2			0	0	0.4496	−1.0000	6.568	−8.249	2.444	0.763	1.172	1.490	3.424	3.034
2	1/2	−1/2			0	0	0.3131	1.0000	3.185	−3.830	1.270	0.626	1.451	1.505	3.581	3.191
9	9/2	7/2			8	8	0.3203	−1.0000	3.334	−3.918	1.375	0.791	1.437	1.512	3.740	3.350
2	3/2	1/2			2	2	0.2924	1.0000	2.778	−3.130	1.214	0.862	1.549	1.525	3.937	3.547
4	1/2	−1/2			0	0	0.3361	1.0000	3.672	−4.112	1.616	1.176	1.410	1.527	4.113	3.723
3	3/2	1/2			2	2	0.3047	1.0000	3.017	−3.261	1.386	1.142	1.518	1.537	4.196	3.806
6	1/2	−1/2			0	0	0.3636	1.0000	4.296	−4.448	2.072	1.920	1.372	1.549	4.841	4.451
1	3/2	−1/2			1	1	0.3321	1.0000	3.584	−3.555	1.806	1.836	1.460	1.560	4.856	4.466
2	9/2	−5/2			2	2	0.2736	−1.0000	2.432	−2.092	1.386	1.727	1.713	1.594	5.034	4.644
7	9/2	5/2			7	7	0.2843	−1.0000	2.627	−2.174	1.540	1.993	1.673	1.602	5.268	4.878
0	3/2	−3/2			0	0	0.3187	1.0000	3.301	−2.924	1.839	2.216	1.530	1.587	5.333	4.943
0	7/2	−7/2			0	0	0.2652	−1.0000	2.286	−1.622	1.475	2.139	1.798	1.632	5.568	5.178
2	7/2	−7/2			0	0	0.2781	−1.0000	2.514	−1.701	1.663	2.476	1.743	1.640	5.859	5.469
1	5/2	−3/2			1	1	0.2976	1.0000	2.878	−1.820	1.968	3.026	1.673	1.651	6.350	5.960
4	7/2	−7/2			0	0	0.3022	−1.0000	2.969	−1.849	2.044	3.165	1.659	1.653	6.477	6.087
6	7/2	5/2			6	6	0.2337	−1.0000	1.775	−0.357	1.596	3.013	2.135	1.753	6.902	6.512
0	5/2	−5/2			0	0	0.2748	1.0000	2.454	−0.840	2.034	3.647	1.847	1.720	7.215	6.825
1	7/2	−5/2			1	1	0.2589	−1.0000	2.179	−0.396	1.981	3.764	1.971	1.758	7.493	7.103
								$A=100$, four extra nucleons, $j=9/2$								
0	9/2	7/2	−7/2	−9/2	0	0	0.4486	−1.0000	6.540	−10.975	1.052	−3.383	2.172	2.448	1.237	0.000
2	9/2	7/2	−7/2	−9/2	0	0	0.4653	−1.0000	7.037	−11.385	1.345	−3.003	2.158	2.479	1.634	0.397
4	9/2	7/2	−7/2	−9/2	0	0	0.4963	−1.0000	8.006	−12.143	1.934	−2.203	2.136	2.531	2.464	1.227
0	3/2	1/2	−1/2	−3/2	0	0	0.4340	1.0000	6.121	−9.291	1.476	−1.694	2.334	2.531	3.171	1.934
2	3/2	1/2	−1/2	−3/2	0	0	0.4500	1.0000	6.581	−9.634	1.764	−1.288	2.310	2.558	3.580	2.343
6	9/2	7/2	−7/2	−9/2	0	0	0.5327	−1.0000	9.221	−13.033	2.705	−1.106	2.117	2.584	3.595	2.358
2	9/2	7/2	−7/2	−9/2	0	2	0.5033	−0.9342	8.232	−11.504	2.480	−0.792	2.194	2.523	3.926	2.689
4	3/2	1/2	−1/2	−3/2	0	0	0.4802	1.0000	7.493	−10.279	2.353	−0.433	2.272	2.604	4.443	3.206
1	9/2	7/2	−5/2	−9/2	1	1	0.4494	−1.0000	6.564	−8.934	2.097	−0.273	2.376	2.610	4.712	3.475
8	9/2	7/2	−7/2	−9/2	0	0	0.5699	−1.0000	10.557	−13.945	3.585	0.198	2.103	2.629	4.930	3.693
2	9/2	7/2	−5/2	−9/2	1	1	0.4583	−1.0000	6.823	−9.109	2.269	−0.016	2.362	2.622	4.968	3.731
8	9/2	7/2	5/2	−5/2	8	8	0.3917	−1.0000	4.988	−3.594	3.191	4.584	2.975	2.909	10.468	9.231
1	9/2	5/2	−5/2	−7/2	1	1	0.3963	−1.0000	5.104	−3.636	3.286	4.754	2.956	2.913	10.623	9.386

mately the same value as used here has been used with success in the theory of fission.[13]

The parameter B [Eq. (3)] enters the coefficient of the rotational energy term W_3 [Eq. (18) and caption of Table III]. It is proportional to the moment of inertia of the nucleus and determines through ϑ_κ the mass involved in the collective nuclear rotation. B is also contained in nearly every off-diagonal term [Eqs. (19)], so that the validity of the approximation depends on the size of B. Since $B\sim A^{5/3}$, the strong coupling approximation improves rapidly with increasing A. The values of $\hbar^2/B$ used in the calculations at $A=100$ and 200 are given in Table IV.

[13] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).

The parameter $\bar{T}$ (same as Bohr's $k/2$) appears in the interaction term W_2 [Eq. (14)]. From the averaging process of the first-order Rainwater interaction term, $\bar{T}$ is defined by

$$\bar{T}=\tfrac{1}{2}V_0R_0^3|\rho_{nl}(R_0)|^2, \tag{25}$$

where $\rho_{nl}(R_0)$ is the value of the radial part of the particle wave function at the edge of the nuclear well, R_0 is the radius of the well, and V_0 the depth of the well. In the limit of a deep well, $\bar{T}$ approaches particle kinetic energy. For a nuclear well, Feenberg and Hammack[11] estimate that $\bar{T}$ is reduced by about 25 percent from its deep well value. It is at any rate a rather uncertain quantity. In the calculations described here, $\bar{T}$ was taken to be constant and equal to 20 Mev.

A simple method for a first check of the validity of the strong coupling approximation has been suggested by Davidson and Feenberg.[14] This is a comparison of the calculated diagonal matrix element of the Hamiltonian in first order (the $E-E_p$ of Table III) with the zero-order energy, $(E-E_p)_0=(5/2)\hbar\omega$, the zero-point energy of the lowest mode of surface vibration. The energy $\hbar\omega$ is given by Eq. (1). This comparison is shown in Table V, where it is seen that for one extra particle the ground-state energy (surface energy+interaction energy) is increased in first order, while for two or more extra particles the energy is lowered in first order. It can be concluded, therefore, that the strong coupling approximation is probably poor for only one extra nucleon outside closed shells and that it is perhaps good for two or more extra nucleons. The fact that the energy is raised in first order does not prove that the approximation is useless. Level order might be given correctly, for example, although level spacing would be greatly in error.*

The off-diagonal terms [Eqs. (19)] contain largely terms from the surface rotational energy. A criterion of validity of the approximation should therefore be that the rotational terms which are included, W_3, are small compared to the surface distortion energy, W_1, and the interaction energy, W_2; i.e., $W_3\ll W_1$ or $|W_2|$. For the most favorable one-particle case treated, $A=200$, $j=9/2$, W_3 is about $\frac{1}{3}$ of W_1 or $|W_2|$ in the ground state. W_3 becomes relatively larger for the excited states. For two extra particles at the same A and j, however, W_3 is about 1/9 of W_1 and $\frac{1}{16}$ of $|W_2|$. The approximation for two extra particles appears definitely better than for one by this criterion. A corollary to the rule that W_3 should be relatively small is the rule that the equilibrium deformation β should not be greatly altered by the inclusion of the rotation terms in W. Calling β_{static} the equilibrium β calculated from W_1+W_2 only, it is found that the inclusion of the rotation greatly increases this value for one extra nucleon, but increases it only slightly for two extra nucleons.

TABLE IV. Energy parameters of liquid droplet model. A = mass number; C and B defined by Eqs. (2) and (3) in text.

A	C (Mev)	$\hbar^2/B$ (Mev)
10	21.2	
30	41.5	
50	54.0	0.251
75	62.5	
100	65.6	0.0794
150	69.6	
200	62.9	0.0249
240	56.3	

TABLE V. Ground state energies in zero order, and in first-order strong coupling approximation. Zero-order energy, $E^{(0)}=(5/2)\hbar\omega$ = zero-point vibrational energy about spherical equilibrium position. First-order energy, $E^{(1)}$ = surface energy + interaction energy in strong coupling approximation. Energies are given in Mev.

No. of extra nucleons	$A=100$, $E^{(0)}=5.73$ Mev			$A=200$, $E^{(0)}=3.14$ Mev		
	j	$E^{(1)}$	$E^{(1)}-E^{(0)}$	j	$E^{(1)}$	$E^{(1)}-E^{(0)}$
1				3/2	4.92	+1.78
	5/2	8.58	+2.85	5/2	4.16	+1.02
				7/2	3.83	+0.69
	9/2	7.98	+2.25	9/2	3.68	+0.54
2	5/2	5.18	−0.55	5/2	1.62	−1.52
	9/2	3.71	−2.02	9/2	0.39	−2.75
4	9/2	1.24	−4.49			
6	9/2	1.24	−4.49			
8	9/2	3.71	−2.02			

Some data indicative of the validity of the approximation are collected in Table VI. It is clear that the approximation is considerably better for two extra nucleons than for one. For the particular two-particle case listed in Table VI, the effect of U_0 on the three lowest $I=0$ states, at 0, 1.18, and 4.39 Mev, was calculated in more detail. The ground state (5/2, −5/2) was shifted downward by 0.02 Mev, the state $[(\frac{1}{2},-\frac{1}{2})$ etc. $\cdots]$ was shifted down by 0.08 Mev, and the state $(\frac{3}{2},-\frac{3}{2})$ was shifted up to 0.10 Mev. The perturbation U_0 mixed into the ground state only 0.6 percent of the state $(\frac{1}{2},-\frac{1}{2})$ and less than 0.1 percent of the state $(\frac{3}{2},-\frac{3}{2})$, a negligible effect. The result would not be so favorable at $A=100$, where $\hbar^2/B$ is three times greater than at $A=200$, and the off-diagonal matrix elements are greater by about the same factor.

The situation with the distance to the first excited state is not so favorable, however. Since the level spacing depends principally on the rotation term W_3, it is also small compared to W_1 and $|W_2|$ and is likely to be strongly affected by the off-diagonal terms. This idea was tested by first-order perturbation calculations on the lowest two states of each of the two and four extra particle cases shown in Fig. 2. U_0 mixes the lowest two states with a higher pair of levels with spin 0 and 2 and about the same separation. Therefore, it has the effect to lower the ground and first excited states by about the same amount and not alter their spacing appreciably. U_2 and U_4 behave like U_0 in this

[14] J. Davidson and E. Feenberg, Phys. Rev. **89**, 856 (1953).

* A. Bohr and B. Mottelson (private communication) find that improved expressions for E_β and E_γ lead to a lowering of the energy also for one extra nucleon.

TABLE VI. Validity of the strong coupling approximation (all energies in Mev).

		$A=100$, one extra particle, $j=5/2$	$A=200$, one extra particle, $j=5/2$	$A=200$, two extra particles, $j=5/2$
Ground state	W_1	1.98	1.25	2.18
	W_2	−1.78	−1.41	−3.73
	W_3	1.09	0.54	0.31
Total zero-order energy		5.73	3.14	3.14
Total first-order energy		8.58	4.16	1.62
Approx. spacing of lowest levels connected by off-diagonal terms		1.0	0.7	1.7
Rough average values of off-diagonal terms between lowest connected states	$\bar{U}_0$	—	—	0.5
	$\bar{U}_1$	1.0	0.3	0.2
	$\bar{U}_2$	0.3	<0.3	≪2.0
	$\bar{U}_3$	≪0.3	<0.08	<0.05
	$\bar{U}_4$	—	—	<0.05
β_{static}		0.111	0.111	0.222
β_1		0.247	0.196	0.259

respect, and are smaller. U_1 and U_3, however, connect the first excited level with a higher excited level of spin 2, but have no nonvanishing matrix elements with the ground state. U_1 and U_3, therefore, act to lower the first excited energy and decrease the spacing of the lowest two states. The effect is very appreciable, as shown in Table VII. The energy spacing is diminished by 15 percent to 30 percent. The shift is small compared to the spacing of the levels mixed (so that the wave functions do not mix to a large extent), but it is an appreciable fraction of the lowest level spacing. In considering the validity of the approximation, therefore, it is important to refer to the particular nuclear properties of interest.

The computed distortions in the first-order strong coupling approximation are too large, because the coefficient of β in the interaction term W_2 is maximized by the ground-state quantum numbers Ω_i. For *any* admixture of other Ω_i, the coefficient will be decreased, and the computed equilibrium distortion, therefore, will be decreased.

It can be concluded that explicit calculation with the strong coupling approximation *should* be valid for several extra nucleons and for $A \gtrsim 100$. In addition, the ideas of the strong coupling model, e.g., that nuclear distortion plays an important role in even nuclei, should be valid after the strong coupling approximation in first order breaks down. A distortion $\beta \gtrsim 0.2$ appears to be necessary for the validity of the approximation in first order.

III. CORRELATION OF EXCITATION ENERGY WITH QUADRUPOLE MOMENTS

Both the first excited energies of even nuclei and the quadrupole moments (largely of odd-even nuclei) are known to exhibit a regular behavior as a function of N and Z, each with marked shell structure effects. The collective model discussed above predicts that the first excited energy of even nuclei depends principally on the equilibrium value of the nuclear distortion, provided the distortion is large enough. Likewise quadrupole moments, especially the large moments, are most easily interpreted in terms of a distorted nucleus. Because of the observed regularities in both of these quantities, it is natural to test the idea of the collective model and the idea that the nuclear distortion is a reasonably smooth function of N and Z for all nuclei by attempting to correlate the known quadrupole moments with the known first excited energies of even nuclei.

In order to carry out this correlation, we develop highly simplified formulas for these two quantities in terms of the nuclear distortion. On the basis of the detailed calculations summarized in Table III, the following assumptions and simplifications are made for the even nuclei. The distortion parameters β_1, γ_1 are taken to be the same in the ground and first excited state. In fact, β_1 is somewhat larger for the first excited level. The zero-point vibration energies E_β and E_γ are taken to be the same in ground and first excited state. The quantum numbers of the ground state are taken to be $K=\Omega=0$, $I=0$, and of the first excited state, $K=\Omega=0$, $I=2$. The extra nucleon states are assumed the same for ground and first excited state. As a result of these simplifications, only the rotation term W_3 differs between the ground and first excited states. The first excited energy is, therefore,

$$E_1=\frac{1}{16}\frac{\hbar^2}{B\beta_1^2}\left(\frac{1}{\sin^2(\gamma_1-2\pi/3)}+\frac{1}{\sin^2(\gamma_1-2\pi/3)}\right)$$
$$\times\{[I(I+1)-K^2$$
$$+f(\text{particle quantum numbers})]_1$$
$$-[I(I+1)-K^2$$
$$+f(\text{particle quantum numbers})]_0\}. \quad (26)$$

Inserting the values $\gamma_1=0$ or π, $I_1=2$, $I_0=0$, $K_1=K_0=0$, $\hbar^2/B=171/A^{5/3}$ Mev, and f_1 (particle quantum

TABLE VII. First-order corrections to detailed calculations.

Example	First excited energy in first order (Mev)	Shift due to U_1+U_3 (Mev)	Energy in second order	Percent decrease
$A=100$, two extra nucleons, $j=5/2$	0.581	−0.119	0.462	20
$A=100$, two extra nucleons, $j=9/2$	0.529	−0.161	0.368	30
$A=200$, two extra nucleons, $j=5/2$	0.315	−0.042	0.273	13
$A=200$, two extra nucleons, $j=9/2$	0.248	−0.039	0.209	16
$A=100$, four extra nucleons, $j=9/2$	0.397	−0.064	0.333	16

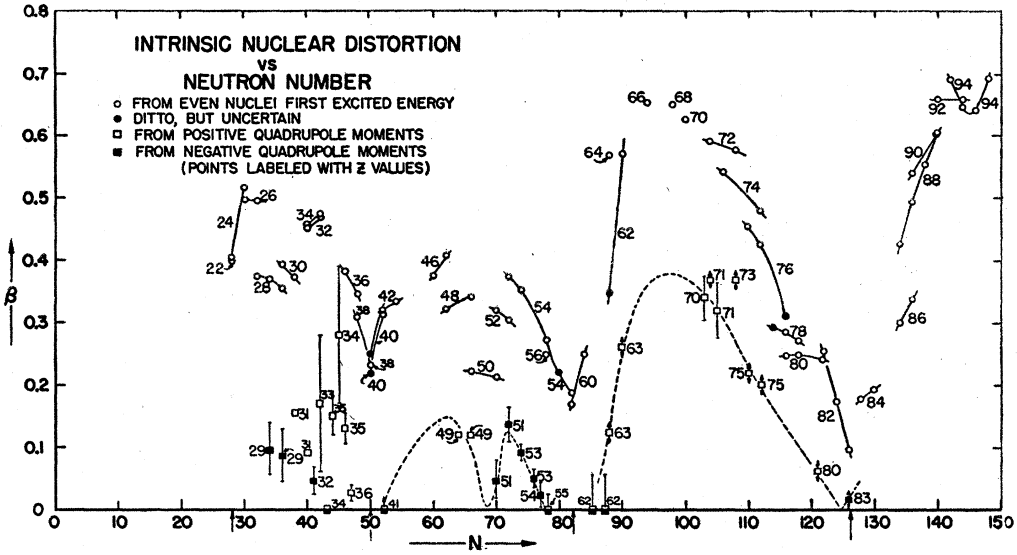

FIG. 4. Correlation of even nuclei first excited levels with quadrupole moments. The circled points are computed from Eq. (30a) in the text and represent upper limits to the nuclear distortion as computed from energy levels. The least accurate points are those near the maxima at $N=100$ and $N=145$, because here the energies are minimum and are least well known. The squared points are computed from quadrupole moments. Experimental errors are indicated on those points where they are known (to this author). Vertical arrows indicate unknown experimental errors. The distortions computed from quadrupole moments are seen to exhibit a regular behavior *vs* N only for $N>50$. If (a) the interpretation of the nature of the first excited states of even nuclei is correct, (b) quadrupole moments give a correct indication of the magnitude of the nuclear distortion, and (c) neighboring even-even and odd-even nuclei have comparable distortions, then Eq. (30a) overestimates the nuclear distortion by a factor of about 1.7 for large distortions and by very much more for small distortions. The factor of 1.7 is reasonable within the framework of the strong coupling approximation; factors much greater than 2 are not.

numbers) $=f_0$ (particle quantum numbers), one gets

$$E_1=\hbar^2/B\beta_1^2=171/A^{5/3}\beta_1^2 \text{ Mev}. \tag{27}$$

In order to get an equally simple formula for the quadrupole moment, it is assumed that the nuclear charge is uniformly distributed over a cylindrically symmetric ellipsoid, i.e., that the contribution to Q from the nonspherical distribution of extra nucleons is small compared to the contribution from the distorted core. For a stationary ellipsoid one obtains to first order in β,

$$Q_{\text{stat}}=(2/5)ZR_0^2(3/2)(5/\pi)^{\frac{1}{2}}\beta\cos\gamma. \tag{28a}$$

A distortion β corresponds to a fractional extension of the radius along the symmetry axis of the ellipsoid of $\frac{1}{2}(5/\pi)^{\frac{1}{2}}\beta\cos\gamma=0.631\beta\cos\gamma$ and a fractional extension perpendicular to the symmetry axis of

$$-(1/4)(5/\pi)^{\frac{1}{2}}\beta\cos\gamma=-0.315\beta\cos\gamma.$$

Positive $\cos\gamma$ corresponds to the cigar shape (prolate) and negative $\cos\gamma$ to the pancake shape (oblate). Putting the undistorted nuclear radius $R_0=1.40\times10^{-13}A^{\frac{1}{3}}$ cm, we have

$$Q_{\text{stat}}=0.0148ZA^{\frac{2}{3}}\beta\cos\gamma. \tag{28b}$$

Because of the quantum-mechanical fluctuations in the direction of **I**, the expectation value of Q is reduced[8] by the factor $I(2I-1)/[(I+1)(2I+3)]$. Therefore, one obtains finally

$$Q_{\text{obs}}=0.0148ZA^{\frac{2}{3}}\frac{I(2I-1)}{(I+1)(2I+3)}\beta\cos\gamma. \tag{29}$$

Formulas (27) and (29) are inverted to define nuclear distortions:

$$|\tilde{\beta}|=\frac{13.1}{A^{5/6}[E_1(\text{Mev})]^{\frac{1}{2}}}, \tag{30a}$$

$$\beta_Q=\frac{67.5}{ZA^{\frac{2}{3}}}\frac{(I+1)(2I+3)}{I(2I-1)}Q_{\text{obs}}(\text{barns}). \tag{30b}$$

If it is assumed that the total quadrupole moment is the sum of a moment due to the distorted core plus a moment due to the nonspherical distribution of extra protons outside closed shells, then β_Q may be too large or too small depending on whether these two moments have the same or opposite sign. (Note that β_Q has the sign of Q, whereas the β in Bohr's theory is intrinsically positive.)

The distortion $|\tilde{\beta}|$ computed from the energy levels is always too large, and in general very much too large: First, within the first-order approximation, Eq. (30a) is an oversimplification, and it can be shown that $|\tilde{\beta}|>\beta$ (first excited state) $>\beta$ (ground state). When the strong coupling approximation is very good, and β

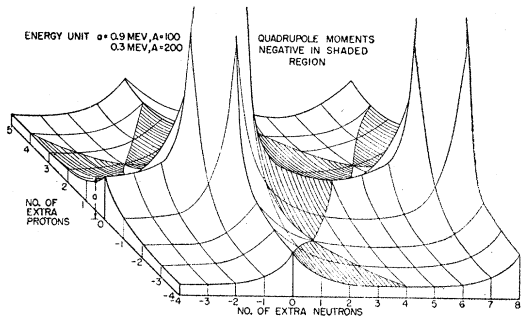

FIG. 5. Energy surface of first excited states of even nuclei for idealized nuclei. Shells for both neutrons and protons are assumed to consist of states of $j=7/2$ only. The nuclear distortion is computed in the strong coupling limit ignoring the rotational energy term in W. The excitation energy is computed in the same limit by means of Eq. (30a) in the text. The lines separating the shaded and unshaded regions are loci of equal energy of the prolate and oblate forms of nuclear distortion. The strong coupling limit used is not even approximately valid at the double magic nuclei, where it yields infinite spikes. The opposite limit of weak coupling yields a finite maximum at the double magic nuclei given by Eq. (1). This maximum varies approximately as $A^{-5/6}$, while the energy surface plotted varies as $A^{-5/3}$. The empirical energy surface (Scharff-Goldhaber) shows a similar behavior, with spikes at the double magic nuclei and ridges along the magic numbers.

is large, then $|\tilde{\beta}|$ is larger than β (ground state) by only a few percent. When the strong coupling approximation is barely valid, i.e., W_3 is comparable to W_1 and $|W_2|$, $|\tilde{\beta}|$ may be too large by up to a factor two. Next, the second-order corrections, as indicated in Table VII, will act to lower the numerator of Eq. (30a). For a 30 percent decrease in energy spacing due to second-order terms, Eq. (30a) will overestimate β by 20 percent. An approximate formula taking into account the effect of U_1 only is

$$|\tilde{\beta}|=\frac{13.1}{A^{5/6}[E_1(\text{Mev})]^{\frac{1}{2}}}[1-(j+\bar{\Omega})(j-\bar{\Omega}+1)/3\Gamma]^{\frac{1}{2}}. \quad (31)$$

U_1 connects states with *one* Ω_i differing by one, other quantum numbers unchanged. $\bar{\Omega}$ above is the larger of the two different Ω_i; j is associated with that particle whose Ω is changed. Γ is the ratio of the energy spacing of the connected states to the energy spacing of the ground and first excited states. For example, for $j=\bar{\Omega}=5/2$, $\Gamma=5$, Eq. (31) gives a value smaller than Eq. (30a) by 19 percent. Finally, the admixture of states of $K=1$ or 2 to the first excited state ($K=0$) will lower $|\tilde{\beta}|$, since the numerator of Eq. (30a) contains effectively $[I(I+1)-K^2]^{\frac{1}{2}}$.

Unfortunately, no simple or calculable expressions exist for applying these three corrections to Eq. (30a) for nuclei in general. We are forced, therefore, to use Eq. (30a) for the correlation, but with the anticipation that it will yield values of $|\tilde{\beta}|$ considerably too large by roughly 10 to 50 percent if the strong coupling picture is valid; by more if it is not. For any detailed calculation on specific nuclei, this rough formula would, of course, have to be improved.

Figure 4 gives the computed distortions $|\beta_Q|$ and $|\tilde{\beta}|$ *vs* neutron number N for known quadrupole moments and first excited states of even nuclei beyond $N=28$. The first evident feature is that $|\tilde{\beta}|$ is everywhere considerably greater than $|\beta_Q|$, as expected. It is larger by nearly a factor two at the maximum. The two sets of data represented in Fig. 4 show qualitative similarities, however. Just beyond neutron number 82, both curves rise precipitously, reaching a maximum at the same place, as nearly as can be guessed, and falling more slowly to minima at neutron number 126. The neutron shell 50 to 82 is more confused because the line of stable nuclei crosses proton number 50 in this shell. Curves of $|\tilde{\beta}|$ *vs* N join smoothly points for $Z=50$, for $Z=50\pm2$, and for $Z=50\pm4$, with a sharp decrease at proton number 50 as well as at neutron magic numbers. This behavior fits the simple ideas of the collective model (Fig. 5). In the same region $|\beta_Q|$ appears to have two maxima and to reach zero between. A possible explanation for this is offered schematically in Fig. 6. The collective model predicts a change in the sign of Q in the middle of a shell where the distortion is large. At this point the prolate and oblate forms are of

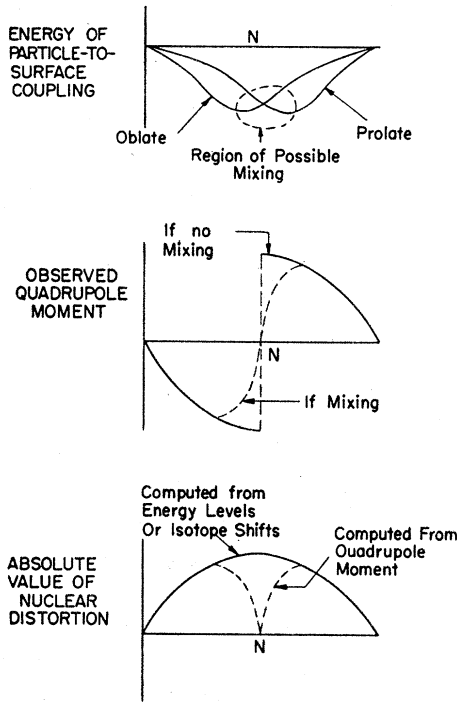

FIG. 6. Prolate-oblate crossover in nuclei. The diagrams show schematically how a mixing of the prolate and oblate forms of distortion in the region where these are of nearly equal energy could yield small quadrupole moments despite a large intrinsic distortion. Such a mixing requires an interaction between states of different Ω_i. This interaction is afforded in the strong coupling theory by the off-diagonal terms U_0, U_2 and U_4, Eqs. (19). Also a breakdown of the approximation that the particle motion can be treated relative to the ellipsoid axes would permit such mixing.

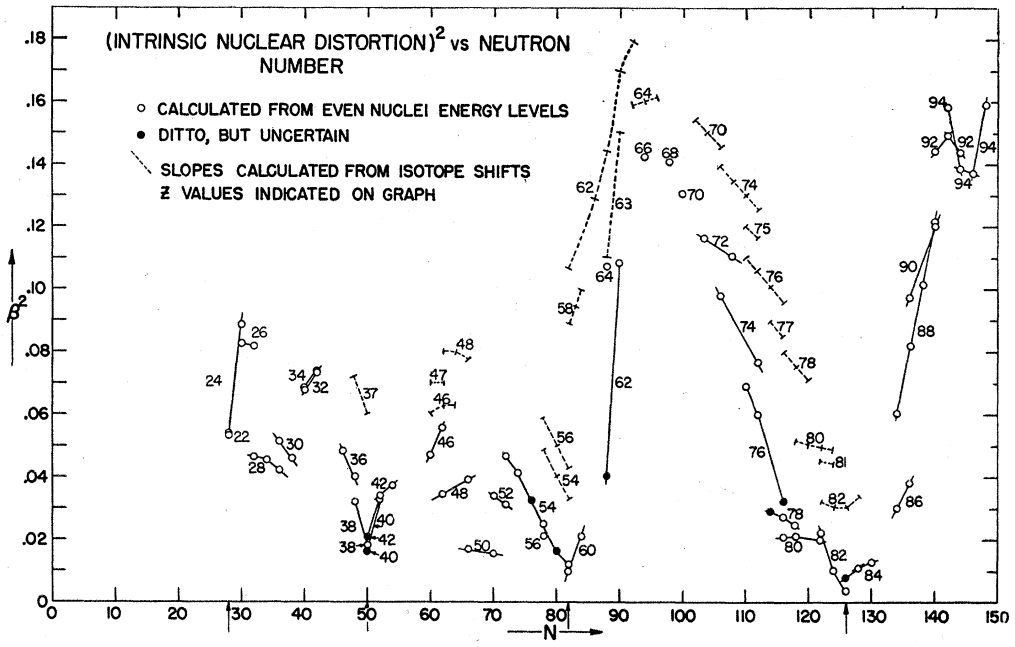

FIG. 7. Correlation of even nuclei energy levels with isotope shifts. β^2 is taken to be $\frac{1}{8}|\tilde{\beta}|^2$ in order to bring the maximum distortions calculated from energy levels and from quadrupole moments into agreement. The solid lines connect computed distortions for isotopes of the same element. The dashed lines represent the slope $[d(\beta^2)/dN]_Z$ taken from isotope shifts. The positions vertically of the dashed lines on the graph are meaningless —only the slopes are to be compared. A single adjustable parameter in the isotope shift data fitted at Pb^{208} brings all the slopes into agreement except in the region below $N=65$, where the isotope shift slopes are slightly too small, i.e., too negative. (The fluctuations of the energy level points at $Z=92$ and 94 are probably not significant. These correspond to energy levels around 40 kev which are not accurately known. The curves β^2 *vs* N appear to be reaching another maximum in this region.)

equal energy and the simple model predicts an abrupt change from one shape to the other. If the nuclear ground state in this region is a mixture of the nearly equal energy prolate and oblate forms, a very small *measured* quadrupole moment will result, while the intrinsic distortion, as measured, for example, by the energy levels, will remain large.

Wilets has suggested that nuclear distortion may account for the regularities[15] in the isotope shift anomalies. Isotope shifts could throw light on the prolate-oblate crossover question discussed above and on the question whether even nuclei distortions may be larger than odd-even distortions. An analysis which will be published elsewhere[16] shows that the isotope shift anomalies are best explained in terms of a distortion with a *shape vs* N and Z derived from energy levels of even nuclei but with a lower *magnitude* corresponding to that found from quadrupole moments.

The energy level—isotope shift correlation is shown in Fig. 7, in which the values of $|\tilde{\beta}|^2$ found from energy levels are arbitrarily reduced by a factor of three everywhere to bring them into agreement with the quadrupole moments at their maximum. The isotope shift data are also altered with a single adjustable parameter chosen to make the anomaly positive at $Pb^{210-208}$ and negative at $Pb^{208-206}$. The isotope shifts yield only the slope $[d(\beta^2)/dN]_Z$, and the magnitude of the line segments plotted are meaningless. The slopes, however, show a generally reasonable agreement with the trend of points calculated from energy levels. Especially to be noted are the large positive slopes just beyond $N=82$ and the smaller negative slopes below $N=126$, as were observed also in the quadrupole moment data.

On the basis of the results shown in Fig. 1, certain regularities in the low levels of odd-even nuclei can also be predicted. With each low-lying single particle level of spin $I=j$ should be associated a level of spin $I=j+1$, same parity, higher by the order of magnitude of neighboring even-even first excited energies. To the same approximation as Eq. (27), the energy difference of these level pairs is given by

$$\Delta E_{j+1,\,j}=\frac{\hbar^2}{B\beta_1^2}\left(\frac{2j+2}{6}\right). \tag{32}$$

IV. CONCLUSIONS

The successes of the collective model as applied to the low states of even nuclei are (a) the order of levels for the first few levels agrees with experiment; (b) the spacing of the 0+–2+ and 2+–4+ levels are comparable, as observed experimentally; (c) the first

[15] P. Brix and H. Kopfermann, Phys. Rev. **85**, 1050 (1952).
[16] Wilets, Hill, and Ford (to be published).

excited energy surface has the same qualitative features as the experimental energy surface (see Fig. 5); and (d) details of the shape of the curve of distortion *vs* N in the neutron shell 82 to 126 as calculated from energy levels agree with the curves found from quadrupole moments and from isotope shifts.

The serious defects of the model is that nuclear distortions calculated from first principles appear to be larger than is reasonable, implying that the particle-to-surface interaction may be weaker than is assumed. Because Eq. (30a) gives only an upper limit to β, the large distortions calculated from even nuclei energy levels do not necessarily speak against the validity of the strong coupling approximation. The required correction of nearly a factor two is larger than expected, however, for the case that the strong coupling approximation is valid (for wave functions) in first order. The very high first excited state of Pb is also a particular defect of the theory.

On the basis of Figs. 4 and 7, large quadrupole moments (5–8 barns) are predicted near uranium, and large isotope shifts (about twice the theoretical value) are predicted near radium.

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Cosmic-Ray Neutron Production in Elements as a Function of Latitude and Altitude*

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The relative rates of local neutron production in Al, Cu, Sn, and Pb were obtained at geomagnetic latitudes $\lambda=40°$ and 54° at atmospheric depth 312 g-cm^{-2} (30 000 ft pressure altitude). The latitude and altitude dependence of local neutron production in carbon and lead were measured in the latitude interval 0°–54° with pile geometries containing BF_3 proportional counters. From these observations several results were obtained:

1. The relative neutron multiplicities in elements were measured and found to be in good agreement with reported low altitude observations.
2. A neutron transition maximum in lead at ~20 g-cm^{-2} Pb was obtained at 33 000 ft pressure altitude.
3. The absorption mean free path for the neutron producing radiation in lead was 350 g-cm^{-2}.
4. An anomalous air absorption mean free path for the nucleonic component has been found for measurements derived from local neutron production in lead at $\lambda \gtrsim 40°$.
5. Aside from the absorption anomaly in elements of high atomic weight A, local neutron production in elements of high A as a function of λ is in fair agreement with the free air neutron latitude effect.
6. Local production in carbon as a function of λ and altitude is in agreement with corresponding free air neutron measurements.

I. INTRODUCTION

THE production and development of the intermediate and low energy portion of the nucleonic component has been studied by observing nuclear disintegrations and neutrons as a function of both altitude and latitude. The behavior of this low energy component may also be explored by determining the properties of nuclear disintegrations produced in a local mass absorber by incident nucleons. We shall define this process as local nuclear disintegration production. These local disintegrations principally yield neutrons, protons, and alpha-particles, i.e., disintegration products. Within a large local absorber mass the charged particles of low energy disappear by ionization energy loss and are not readily detected outside the local mass. The disintegration neutrons, however, escape from the local mass and may be detected. We define the observed neutron production as local neutron production.

In this paper we describe measurements of local neutron production in the elements C, Al, Cu, Sn, and Pb as a function of geomagnetic latitude and as a function of altitude in the range of atmospheric depths 200 to 600 g-cm^{-2}. From these measurements we determine the average neutron multiplicity from low energy nuclear disintegrations. The measurements were obtained in a series of aircraft flights from January, 1948 through November, 1949.

Local neutron production has been extensively investigated at sea level and mountain altitudes by the Cornell University and Yale groups for carbon and lead close to the latitudes $\lambda=50-52°$N. Neutron produc-

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