TABLE II. X-ray spectrum.

X-ray	Energy from previous determinations (kev)	Intensity from previous determinations (%)	Energy this paper (kev)	Intensity this paper (%)	References
$K\alpha$ x-ray	77	0.8±0.3	71±2	0.06	8
$K\beta$ x-ray	87	0.2	85±2	less than 0.02	8
L x-ray	9−16	20−35	9−16	22	2, 7, 16
M x-ray	···	···	1.9−3.4	···	···

TABLE III. Additional lines in proportional counter electromagnetic spectrum.

Line description	Energy this paper (kev)	Intensity this paper (%)	Remarks
W	57 ±1	0.02	Possible coincidence peak
X	34.5±1	0.1	Not associated with RaD
Y	18.6±1	0.3	Possible coincidence peak
Z	17.3±0.3	∼0.3	Peak reported by Cohen and Jaffe[a]
X-ray?	5.5±0.5	0.3	Fe x-ray or argon K x-ray escape peak
E	28.0±0.5	0.05	Possible argon K x-ray escape peak

[a] See reference 17.

lines in the K x-ray region is an additional illustration of the difficulty. A striking dearth of information on source materials exists in the published papers on both the β- and γ-spectra of RaD, and on the β-spectrum of RaE. The question of possible complexity in the latter must be left open until the complete decay scheme of RaD is elucidated, since the L x-ray intensity suggests that the 46.7-kev state is promptly traversed by less than 80 percent of the RaD transitions, and no high energy β-branch has been observed in RaD decay.[23]

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[23] Insch, Balfour, and Curran, Phys. Rev. 85, 805 (1952).

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A Nonperturbation Approach to Quantum Electrodynamics

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The equations governing the interaction of an electron with the electromagnetic fields are used in the form given by Schwinger to derive a linear integral equation for the function Γ, whose kernel is expressed as a power series in α. The first approximation to this kernel is used and the resulting integral equation solved without recourse to perturbation theory. With the aid of the solution first approximations can be found for self-energy effects, which are now finite, and some discussion of the analytic behavior of these and related quantities is given. An application of the method to meson theory illustrates the classification of the types of integral equation which arise. The possibility of extending the method is discussed.

1. INTRODUCTION

PERTURBATION theory has been extremely successful in electrodynamics in explaining experimental results since the renormalization program has been adopted. But almost all experimental results are associated with interactions in which the integrals concerned are quite convergent after renormalization, whereas for self-energy effects as well as the other renormalization coefficients there has been a complete failure to obtain finite results. At this juncture two alternative approaches suggest themselves: (1) the Lagrangian used is not adequate to handle self-energy effects, though accurate enough for interaction effects, and new ideas are required to attack these problems, (2) we may object that since only the simplest mode of solution has been used, that of straightforward expansion, some more powerful approach is required, which would also be superior to perturbation theory when the coupling constant was large. The self-energy of the electron, for example, is expressed in perturbation theory by

$$m=e^2(m_1+e^2m_2+e^4m_3+\cdots),$$

where all the m's are infinite. Such an expansion is only valid if me^{-2} is analytic in e^2 at the origin, and there is no *a priori* reason to believe this. A final point in favor of the second approach is that many theories, such as meson theory with gradient coupling, diverge still after renormalization, and it is not clear if this is not also due to the method employed.

It is this second approach which is considered here, and an attempt to solve the quantum electrodynamics of electrons without recourse to perturbation theory is presented below. The formulation of Schwinger,[1] in

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[1] J. S. Schwinger, Proc. Natl. Acad. Sci. 37, 452 (1951). We follow Schwinger's notation conventions.

which the propagation functions are shown to satisfy certain functional integro-differential equations, is used. These equations are renormalized, and the approach of this paper is to reduce the problem to the solution of a linear integral equation whose kernel is expressed as a power series in the coupling constant. The first approximation to this kernel is taken and the corresponding integral equation solved, and the solution discussed. The method thus provides only a partial departure from the expansion method, but it hoped that it will be of methodological interest since interesting results concerning the behavior, as functions of α, of the solutions and quantities calculated with their aid are obtained.

The method is applied to the various types of meson theory, and they are found to provide a classification of the various types of equation which occur.

2. DERIVATION OF THE EQUATION

With the Lagrangian

$$\mathcal{L}=-\tfrac{1}{4}[\bar{\psi},\gamma_\mu(-i\partial_\mu-eA_\mu)\psi+m\psi]+\tfrac{1}{4}F_{\mu\nu}{}^2+\tfrac{1}{4}\{F_{\mu\nu},\partial_\mu A_\nu-\partial_\nu A_\mu\}+\text{Hermitian conjugate,}\quad(2.1)$$

the equations for the Green's functions for the electron G, the photon D, and the current operator Γ, are given in momentum space by

$$[\gamma(p-eA)+M]G=1,\quad(2.2)$$

$$[k^2+P]D=1,\quad(2.3)$$

$$\Gamma=-(\delta/\delta eA)G^{-1},\quad(2.4)$$

$$M=m+ie^2\gamma G\Gamma D,\quad(2.5)$$

$$P=-ie^2\gamma G\Gamma G.\quad(2.6)$$

In these equations the indices and variables are supressed, a "matrix" multiplication being assumed. A and γ are considered as matrices in configuration space:

$$\gamma(\xi;xx')=\gamma\delta(\xi-x)\delta(x-x'),\quad(2.7)$$

$$(x|A|x')=\delta(x-x')A(x),\quad(2.8)$$

e.g.,

$$\gamma A\rightarrow\int(d\xi)\gamma_\mu(\xi)A_\mu(\xi),\quad(2.9)$$

where ξ is the "photon coordinate." G, M are functions of the electron coordinates $G(x,x')$, $M(x,x')$, and

$$MG\equiv\int M(x,x')G(x',x'')d^4x''.\quad(2.10)$$

In (2.2) this becomes $M(p)G(p)$. Γ is the generalization of γ which takes into account the radiative corrections and is a function of three variables,

$$\Gamma_\mu(\xi;x,x')=-[\delta/\delta eA_\mu(\xi)]G^{-1}(x,x').\quad(2.11)$$

D and P are functions of two photon coordinates,

$$P(\xi,\xi')=-ie^2\,\mathrm{Tr}\int\gamma(\xi;x,x')G(x',x'')\times\Gamma(\xi;x'',x''')G(x''',x)d^4x\,d^4x'\,d^4x''\,d^4x'''.\quad(2.12)$$

Details of the derivation of these expressions may be found in Schwinger's paper.

Our interest is to be concentrated on the function Γ. By iteration of the above equations it can be expressed as a power series in e^2:

$$\Gamma=-(\delta/\delta eA)[\gamma(p-eA)+M]\quad(2.13)$$

$$=\gamma-ie^2(\delta/\delta eA)(\gamma G\Gamma D).\quad(2.14)$$

Since $GG^{-1}=1$,

$$(\delta G/\delta eA)G^{-1}+(\delta G^{-1}/\delta eA)G=0,\quad(2.15)$$

$$\Gamma=\gamma+ie^2\gamma G\Gamma G\Gamma D-ie^2\gamma G(\delta/\delta eA)(\Gamma D),\quad(2.16)$$

and so on. This Γ can be used to find G and D as power series expansions, and further expansion can take place in terms of $G_1=[\gamma(p-eA)+m]^{-1}$ and $D_1=(k^2)^{-1}$. It is well known that this approach as it stands leads to a series of infinite terms, and renormalization must take place before these equations are in fact the equations which deal with physical quantities. The experimental mass is the value of M for a freely moving electron, i.e., one for which $G_1{}^{-1}=\gamma p+M=0$. The value of the physical Γ for the emission of a photon of vanishingly small momentum by a free electron is γ, and the forms of the Green's functions for free electron and photon of vanishingly small momentum are, respectively,

$$G_0{}^{-1}{}_{\text{exp}}=(\gamma p+m)_{\text{exp}},\quad(2.17)$$

$$D_0{}^{-1}{}_{\text{exp}}=k^2.\quad(2.18)$$

It has been shown by Dyson[2] that the equations (2.2)–(2.6), when expressed in terms of experimental quantities can be solved by perturbation theory to all orders in e^2. In order to obtain the renormalized perturbation theory, certain operations are performed upon the series. These operations will be performed on the closed equations (2.2)–(2.6), and it will be assumed that the resulting functions G', Γ', D' exist, and that if they could be obtained without recourse to perturbation theory, then would give the same series expansion as is obtained in perturbation theory. Since the complete nonperturbation approach is not achieved in this paper, this procedure cannot be rigorously justified, but appears very reasonable.

Renormalization effects a change in scale of G, Γ, D and e^2. To accomplish this the expansion (2.16) must be expressed in a symmetric form. Thus, if we write

$$\Gamma=\gamma+\Lambda,\quad(2.19)$$

[2] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

Λ may be expressed entirely in term of Γ:

$$\Gamma=\gamma-ie^2(\delta/\delta eA)(\Gamma-\Lambda)G\Gamma D \quad (2.20)$$

$$=\gamma-ie^2(\delta/\delta eA)\Gamma G\Gamma D$$
$$+ie^2(\delta/\delta eA)\{-ie^2(\delta/\delta eA)\Gamma G\Gamma D\}G\Gamma D+\cdots. \quad (2.21)$$

This amounts in perturbation theory to regrouping the terms in the expansion. If we now define Γ', Λ' by $Z\Gamma'=\Gamma$ and $Z\Lambda'=\Lambda$, then since eA and e^2D as a consequence of gauge (charge) invariance must remain unaltered under such a change,[3] from the definition of Γ,

$$G'=Z^{-1}G, \quad (2.22)$$

$$Z\Gamma'=\gamma-ie^2(\delta/\delta eA)Z(\Gamma'-\Lambda')G'\Gamma'D,$$
$$\Gamma'=Z^{-1}\gamma-ie^2(\delta/\delta eA)(\Gamma'-\Lambda')G'\Gamma'D$$
$$=Z^{-1}\gamma+\Lambda' \quad (2.23)$$

$$=Z^{-1}\gamma-ie^2\Gamma'G'\Gamma'G'\Gamma'D'+e^4\text{ terms}+\cdots. \quad (2.24)$$

This result is unaltered by charge renormalization, in which

$$e'=Z^{*\frac{1}{2}}e,\quad A'=(Z^*)^{-\frac{1}{2}}A,\quad D'=Z^{*-1}D,$$
$$D'^{-1}=Z^{*-1}k^2-ie'^2(\Gamma'-\Lambda')G'\Gamma'G'. \quad (2.25)$$

The Z's are chosen so that

$$\Gamma_0'=\gamma,\quad D_0'=k^2,\quad G_0'=\gamma p+m, \quad (2.26)$$

where the subscript 0 denotes the value of these functions for free electron and free photon, e.g.,

$$\Gamma'=\gamma+\Lambda'-(\Lambda')_0.$$

The equation for Γ' is a nonlinear integral equation of increasing complexity as the expansion proceeds, and cannot be directly solved in this form.

Our approach will be to expand all the functions on the right by perturbation theory with the exception of one of the Γ''s, which leaves a linear integral equation with an expanded kernel. The simplest approach is to retain that Γ' which arises in each term from the application of the $(\delta/\delta eA)$ upon M. In this case the integral equation involves only one variable, and it can be written (conventionally since commutation relations must be maintained in bringing the Γ' to the right)

$$\Gamma'(k,j)=Z^{-1}\gamma+\int\Gamma'(k,l)[\sum_0^\infty e^{2n}\mathrm{K}_n(l,j;k)d^4l, \quad (2.27)$$

where $\Gamma'(k,j)$ is the Fourier transform of $\Gamma(\xi;x,x')$ based on $\xi-x$, $x-x'$, and the k's are independent of e^2. Written out explicitly,

$$\Gamma'=Z^{-1}\gamma-ie^2\gamma G_1'\Gamma'G_1'\gamma D_1'+e^4\text{ terms}. \quad (2.28)$$

Z^{-1} is to be fixed by the boundary condition (2.28).

3. SOLUTION OF THE EQUATION

The existence and behavior of a solution depend on the behavior of the kernel, and to have it expressed as a power series is not very satisfactory, since very few conclusions can be drawn concerning the behavior of a function from the first few terms of its series expansion. The simplest approximation, that of solving with only the K_0 present, will be adopted, and the possibility of a better approximation will be discussed later. This procedure may be viewed in the following way. Of the infinite series of terms in the *full* perturbation expansion of (2.27), an infinite set is picked out and the remainder, of order e^4 and higher, are neglected.[4] By approaching from the viewpoint of an integral equation, the sum of this series is expressed as a closed function which if expanded would again give the perturbation series, but which may be used outside the radius of convergence of the power series.

The equation to be solved is then, dropping primes,

$$\Gamma_\mu(k,j)=Z'^{-1}\gamma_\mu-ie^2\int\gamma_\nu G_1\left(\frac{k}{2}+l\right)\Gamma_\mu(k,l)$$
$$\times G_1\left(l-\frac{k}{2}\right)\gamma_\nu D_1\left(j-l-\frac{k}{2}\right)d^4l. \quad (3.1)$$

Z'^{-1} is the renormalization constant for this equation, and it is not, in general, equal to Z^{-1}. The solution is the operator which governs the emission of a photon momentum k by an electron of momentum j, $Z^{-1}\gamma$ being the contribution of the "bare" electron, and the remaining term the effect of the photon field, which in this approximation is treated by perturbation theory, retaining only those effects due to the non-overlapping emission and absorption of virtual photons. The integral equation (3.1) is singular since the range of integration is infinite. Moreover, since $\int\gamma_\nu G_1\gamma_\mu G_1\gamma_\nu D_1$ diverges, the usual methods of Neumann, and more particularly of Fredholm, cannot be applied, there being a pole in the kernel at infinity.

The method adopted is to find eigenfunctions of the homogeneous part of (3.1) and investigate how the complete solution can be built up from them, to satisfy the boundary condition. It might be noticed at this point that although this equation contains only part of the total solution, it contains all the difficulties as far as the divergences associated with straightforward expansion are concerned. The only reason that (3.1), indeed (2.27), can be solved by perturbation theory is that the divergence associated with the kernel is so weak that the alternative form $\Gamma'=\gamma+\Lambda'-(\Lambda)_0'$ can be treated.

Equation (3.1) still presents considerable difficulty

[3] J. C. Ward, Phys. Rev. 78, 182 (1950).

[4] It will be seen that this has similarity to the "ladder" approximation which has been suggested in the two-body equation by Bethe and Salpeter. In terms of Feynman diagrams we retain only those in which photon lines do not cross, and closed loops and self-energy parts of electron lines are omitted. The ladder approximation, in the two-body problem, leads in certain cases to integral equations of the type displayed here, and the solution of the two-body equation in this approximation from the integral equation point of view has been investigated by Dr. J. Goldstein, to whom the author is indebted for some interesting discussions.

and so is simplified in the following way. Since the integral equation operates on j only, it is possible to consider the case $k=0$, which introduces special simplifications. Also, we can look for a solution in the form $\gamma_\mu f(j^2)$, and neglecting the terms in $\sigma_{\mu\nu}p_\nu$, etc., and the finite part, which arise from the commutation properties of γ. We then get

$$f(j^2)=Z'^{-1}-\frac{ie^2}{(2\pi)^4}\int\frac{f(l^2)d^4l}{(l^2+m^2)(j-l)^2}. \quad (3.2)$$

The terms neglected can always be brought in later on the basis of the solutions of (3.2). In meson theory with $\gamma_5 f(j^2)$, Eq. (3.2) would in fact be complete. In the case of (3.2), solutions in a closed form can be obtained to the homogeneous equation,

$$f(j^2)=-i\lambda\left(\frac{2}{\pi^3}\right)\int\frac{f(l^2)d^4l}{(l^2+m^2)(j-l)^2}, \quad (3.3)$$

details of which are given in the Appendix. Since the method there does not work for $k\neq 0$, or when a meson mass is present, a wider approach is made to (3.3), which can be generalized to these cases. In order to make its structure clearer a transform is used, which reduces this to a one-dimensional integral equation. Write f in the form of a "Stieltjes transform":[4]

$$f(l^2)=\int\frac{F(a)da}{a+l^2}, \quad (3.4)$$

where a small imaginary part is added to the denominator, thus giving the form of a superposition of propagation functions. Integrating in the usual manner,

$$\begin{aligned}&\int\frac{f(l^2)d^4l}{(l^2+m^2)(j-l)^2}\\ &=\int da\int\frac{F(a)d^4l}{(l^2+m^2)(l^2+a)(l^2-2lj+j^2)}\\ &=\int da\int_0^1 dx\int_0^1 dy\\ &\quad\times\frac{xF(a)d^4l}{[l^2+j^2xy(1-xy)+m^2x(1-y)+a(1-x)]^3}\\ &=\frac{i}{8}(2\pi)^2\int da\int_0^1 dx\int_0^1 dyF(a)x\\ &\quad\times[j^2xy(1-xy)+a(1-x)+m^2x(1-y)]^{-1}.\end{aligned}$$

Now invert the transformation (3.4),

$$\frac{i\pi^2}{2}\int_0^1 dx\int_0^1 dyx(1-x)^{-1}\times f[\{i^2xy(1-xy)+m^2x(1-y)\}/(1-x)], \quad (3.5)$$

which can be transformed into

$$f(j^2+m^2)=\lambda\int_0^1 dy\int_0^\infty \zeta d\zeta(1+\zeta)^{-2}\times f[(j^2y+m^2)\zeta(1-\zeta[\zeta+1]^{-1}y)]. \quad (3.6)$$

The boundary condition is now $f(0)=1$, and since only positive arguments appear in f under the integral, we can consider this a one-dimensional equation in a positive variable $\xi=j^2$. To extend the solution to negative ξ, when f appears in an apparently non-singled valued form, the form (3.4) is implied, the small imaginary addition to the denominator ensuring that the functions exist and are single valued. Since the usual method of obtaining eigenvalues and eigenfunctions are not available, we consider the equation when ξ is very large.

Then (3.6) becomes

$$f(\xi)\sim\lambda\int_0^1 dy\int_0^\infty \zeta d\zeta f(\zeta)[\zeta+\xi(1-y)]^{-2}. \quad (3.7)$$

This belongs to a general class of equations for which, if $f(\xi)$ is a solution, then so also is $f(\mu\xi)$, and the solutions are powers $\xi^{-\beta}$.[5] Substituting this form into (3.6), where for reason of convergence $0<R\beta<1$, and $\xi\ll m^2$,

$$\xi^{-\beta}\sim\lambda\beta^{-1}(\beta-1)^{-1}\xi^{-\beta};$$

i.e., $\beta=\frac{1}{2}[1\pm(1-4\lambda)^{\frac{1}{2}}]$, which if λ is small gives $\beta=\lambda$, $1-\lambda$, and is complex when $\lambda>\frac{1}{4}$. There are no other solutions,[6] so that the general solution is $C\xi^{-\beta_1}+B\xi^{-\beta_2}$, β_1, β_2 being the two roots for a given λ. The spectrum of (3.6) is continuous, therefore, and the eigenfunctions are not square integrable; moreover none of the available methods can be employed to construct an orthonormal set. Returning to (3.2), we are forced to the conclusion that $Z'^{-1}=0$, and there exists a solution to (3.2) with $Z'^{-1}=0$, for all values of $e^2/4\pi=\alpha$, this being $8\pi\lambda$ above. The solution for (3.2) is obtained in closed form as a hypergeometric function in the appendix, and the various results above are confirmed there. For the solution to (3.2), and the more general equation we proceed from the form at infinity, $\xi^{-\lambda}$, λ the smaller root, by writing it in a form equivalent to expansion about a point A; e.g., $A^\lambda(j^2+m^2+A)^{-\lambda}$ is a form be-

[5] This transform assumes that f has only poles and branch points as singularities. Before the integration can be carried out, the position of the singularities of f must be known, and the transformation is a device for accomplishing this; with its aid many complicated functions can be integrated in the Feynman sense, with very little difficulty.

[6] This has been proved rigorously for the kernel $(x+y)^{-1}$ by G. H. Hardy and E. T. Titchmarsh; see, for instance D. V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, 1941). The approach in the appendix proves this rigorously for (3.3), and it is assumed that the result holds also for the more general equations. Equations of this type have a considerable though rather unsystematic literature. T. Carleman, *Sur les equations integrales singulieres* (Almqvist and Viksells, Uppsala, 1923). A general discussion is found in E. C. Titchmarsh, *The Fourier Integral* (Oxford University Press, Oxford, 1931).

having correctly at the origin and at infinity. But more generally an expansion along a line is to be expected, e.g.,

$$\int_1^\infty f(x)(j^2x+m^2)^{-\lambda}dx,$$

i.e.,

$$\int_1^\infty \phi(x)\left(\frac{m^2(1-x)}{j^2x+m^2}\right)^{-\lambda}dx, \tag{3.8}$$

where $\phi(x)\sim 1$, and $\int_0^1\phi(x)dx=1$. The work in the appendix shows that for (3.2), $\phi(x)=x^{-\lambda}(1+\lambda)$, and this yields

$$\Gamma_\mu^{(0)}(0,j)\sim\gamma_\mu\int_0^1\frac{x^{-\alpha/8\pi}}{1-\alpha/8\pi}\left(\frac{m^2(1-x)}{j^2x+m^2}\right)^{\alpha/8\pi}dx, \tag{3.9}$$

which agrees with the perturbation theory expansion of (2.2):

$$f(j^2)=1-\frac{\alpha}{8\pi}\int_0^1\log\left(\frac{m^2(1-x)}{j^2x+m^2}\right)+O(\alpha^2)\cdots.$$

When j^2 is negative the fractional power is to be understood as having been put in the form (3.4).

Though in (3.2) we have dealt with a very simple case, the arguments are quite general. The case of $k\neq 0$ may be considered, as also can the nonsymmetric terms such as $\sigma_{\mu\nu}p_\nu$, without any difference in approach, though with considerable mathematical complication.

4. A DISCUSSION OF $\Gamma^{(0)}$

$\Gamma^{(0)}$ expresses the nonlocalization of the electron induced by the field in a closed form, which is clearly expressed if we consider $\Gamma^{(0)}$ in configuration space for the static case $j^2=k^2-m^2$, k a three-vector, when we find the distribution $C\alpha^{-1}r^{-3+\alpha/8\pi}\chi(r)$, where $\chi(r)$ is finite at the origin and decays exponentially.

The self-energy of the electron and polarization of the vacuum can be calculated with $\Gamma^{(0)}$. The convention must be adopted that, as in derivation of $\Gamma^{(0)}$, only one $\Gamma^{(0)}$ is retained in the integrals concerned, and in all the rest of the integral the first approximations are used. Then both these quantities are found to be of order unity, whereas in perturbation theory they are assumed to be of order α, so that in perturbation theory an attempt is made to expand a pole as a power series, which leads to a series of divergent terms.

When α is negative, there is no solution for $\Gamma^{(0)}$, which must therefore be considered as defined for $\alpha>0$ only, and similarly everything calculated with it. Perturbation theory appears to give solutions equally valid for either sign of α, but from the present point of view this is invalid. This kind of behavior has been conjectured by Dyson[7] from general physical arguments, and through $\Gamma^{(0)}$ is only a partial solution it may be regarded as an illustration of Dyson's remarks.

[7] F. J. Dyson, Phys. Rev. 85, 631 (1952).

The result $Z'^{-1}=0$ is at first sight surprising since it implies the absence of the "bare" electron. In perturbation theory $Z^{-1}=1+\Sigma\alpha^nZ_n$, where all the Z_n's are infinite, a result which would follow if Z'^{-1} were forced into this form. The two results arise from different viewpoints, depending on whether we ascribe unity to the bare electron and attempt to express the effect of photons present in a series in α, or concentrate our interest upon the effect of the photons, whereupon Z'^{-1} is zero, as in the integral equation approach.

The Green's function can be found with Γ, though the case with $k=0$ is not adequate. The result is that when p^2 is very large, $G(p)\sim(\gamma p+m)(p^2)^{-\alpha/8\pi}$, and it is seen that the convention mentioned above must be employed to obtain definite answers, since the subsequent use of this function can alter the convergence behavior of M or Γ.

5. APPLICATION TO MESON THEORY

In meson theory interest centers upon cases in which the coupling is large. This weakens the applicability of the methods of this paper, but in spite of this we shall apply them to the various types of meson theory, since they are still of considerable interest, and moreover illustrate the various types of equation arising from different kinds of coupling.

(A) A Nonlinear Scalar Field

As an example of a theory which offers fewer difficulties than electrodynamics, consider a scalar field, self-coupled by a term ϕ^3. Its Lagrangian, including a source function $J(x)$ is

$$\mathfrak{L}=\tfrac{1}{2}(i\partial_\mu\phi)^2+\tfrac{1}{2}\kappa^2\phi^2-\tfrac{2}{3}\lambda\phi^3+J\phi. \tag{5.1}$$

This theory is especially simple since only one kind of particle is involved. Its Green's function Δ is defined in Schwinger's notation by

$$\begin{aligned}\Delta(x,x')&=\delta\phi(x)/\delta J(x')\\&=i\langle\phi(x)\phi(x')\rangle_+-i\langle\phi(x)\rangle\langle\phi(x')\rangle,\end{aligned} \tag{5.2}$$

and there is one auxiliary function Ω, the analog of Γ:

$$\Omega(xx'x'')=-\delta\Delta(x',x'')/\delta\lambda\phi(x). \tag{5.3}$$

The equation of motion is

$$(-\Box+\kappa^2)\phi-\lambda\phi^2=J, \tag{5.4}$$

which gives

$$\Delta^{-1}=k^2-\lambda\langle\phi\rangle+\kappa^2-i\lambda^2\Delta\Omega\Delta \tag{5.5}$$

$$\Omega=1+i\lambda^2\frac{\delta}{\delta\lambda\phi}\Delta\Omega\Delta \tag{5.6}$$

$$=1+i\lambda^2(2\Delta\Omega\Delta\Omega\Delta)+i\lambda^2\Delta\frac{\delta\Omega}{\delta\lambda\phi}\Delta. \tag{5.7}$$

The latter equation can now be iterated. The integrals appearing in (5.7), when iterated, are all finite and the renormalization of this equation may be postponed until the equation is solved. Apart from a mass renormalization which is infinite in perturbation theory, these equations may be considered as they stand. The linear integral equation is

$$\Omega=1+i\lambda^2 2\Omega(\Delta_0\Delta_0\Delta_0+\lambda^4\cdots) \\ =1+i2\lambda^2\Omega\sum_0 \lambda^{2n}\mathrm{K}_n, \quad (5.8)$$

where all the kernels K_n have $\int \mathrm{K}_n(jj)d^4j$ bounded so that Fredholm's method can be used upon it to any order. Providing $\Sigma\lambda^{2n}\mathrm{K}_n$ converges, the Fredholm method is guaranteed to converge.[8]

(B) Pseudoscalar Meson Theory with Pseudoscalar Coupling

This theory is of considerable interest, and falls into the same class as electrodynamics. Its general equations can be derived conveniently using the methods of Schwinger's paper, with the addition that for the full theory with photons present, third variations of the Lagrangian are required:

$$\delta''\delta'\langle\delta\mathfrak{L}(x)\rangle \\ =i\int_{\sigma_2}^{\sigma_1}(dx')(dx'')[\langle\delta\mathfrak{L}(x)\delta'\mathfrak{L}(x')\delta''\mathfrak{L}(x'')\rangle_+ \\ -\langle\delta\mathfrak{L}(x)\delta''\mathfrak{L}(x'')\rangle_+\langle\delta'\mathfrak{L}(x)\rangle \\ -\langle\delta\mathfrak{L}(x)\delta''\mathfrak{L}(x'')\rangle_+\langle\delta'\mathfrak{L}(x')\rangle \\ -\langle\delta'\mathfrak{L}(x')\delta''\mathfrak{L}(x'')\rangle_+\langle\delta\mathfrak{L}(x)\rangle \\ +2\langle\delta\mathfrak{L}(x)\rangle\langle\delta'\mathfrak{L}(x)\rangle\langle\delta''\mathfrak{L}(x'')\rangle]. \quad (5.9)$$

We give the equations in full for the Lagrangian,

$$\mathfrak{L}=\tfrac{1}{2}\sum_i\{(\partial_\mu-ie\mathcal{T}A_\mu)\phi_i(\partial_\mu-ie\mathcal{T}A_\mu)\phi_i\}+\tfrac{1}{2}\sum_i\kappa_i^2\phi_i^2 \\ -\tfrac{1}{4}[\bar\psi,(-i\partial_\mu-eTA_\mu)\psi+m\psi]+\sum_i\tfrac{1}{4}g_i^2[\bar\psi,\gamma_5\tau\psi]\phi_i \\ +\tfrac{1}{4}F_{\mu\nu}^2-\tfrac{1}{4}\{F_{\mu\nu},\partial_\mu A_\nu-\partial_\nu A_\mu\}+J_\mu A_\mu \\ +\sum\mathrm{K}_i\phi_i+\tfrac{1}{2}[\bar\psi,\eta]+g^2\lambda\sum_i(\phi 1\phi)_i(\phi\mathrm{U}\phi)_i \\ +\text{Herm. conj. of all term in }\psi,\bar\psi.$$

Here it is understood that $g\tau\phi$, etc., are abbreviations for $\sum g_i\tau_i\phi_i$, ϕ_i having three components. Usually the index i on g, K will be dropped. The ψ has two sets of components in isotopic space. $\mathcal{T}$, T, and U are matrices that conserve change:

$$T=\frac{1+\tau_3}{2},\quad \mathcal{T}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\quad (\phi\mathrm{U}\phi)=\begin{pmatrix}3\phi_3^2+\phi_1\phi_2\\3\phi_3^2+\phi_1\phi_2\\3\phi_1\phi_2+\phi_3^2\end{pmatrix}.$$

η, J_μ, K_i are sources of the Dirac, Maxwell, and meson fields, respectively. This gives rise to three types of Green's function G, Δ, and D, where we shall use these letters as they stand letting their components be implicit. G has four components, two of which have nonzero expectation values, Δ has three components, and D has one. With these a series of auxiliary function can be defined:[9]

$$\Gamma_\mu=-\frac{\delta}{\delta eA_\mu}G^{-1}, \quad (5.10)$$

$$\Gamma_5=-\frac{\delta}{\delta g\phi}G^{-1}, \quad (5.11)$$

$$V_\mu=-\frac{\delta}{\delta eA_\mu}\Delta^{-1}, \quad (5.12)$$

$$C_{\mu\nu}=\frac{\delta}{\delta eA_\mu}\frac{\delta}{\delta eA_\nu}\Delta^{-1}, \quad (5.13)$$

$$N=\frac{\delta}{\delta g\phi}\frac{\delta}{\delta g\phi}\Delta^{-1}. \quad (5.14)$$

Here again it is understood that Γ_μ has two components, Γ_5 six (two of them zero) and so on. For example, in full,

$$N_{jk}{}^i=\frac{\delta}{\delta g_j\phi_j(\xi_1)}\frac{\delta}{\delta g_k\phi_k(\xi_2)}\Delta_i^{-1}(\xi_3,\xi_4).$$

The final equations for the Green's functions are

$$[\gamma p+M]G=1, \quad (5.15)$$

$$[k^2+P]D=1, \quad (5.16)$$

$$[k^2+\kappa^2+\Pi]\Delta=1, \quad (5.17)$$

where

$$M=m+ig^2\tau\gamma_5G\Gamma_5\Delta+ie^2T\gamma_\mu G\Gamma_\lambda D_{\lambda\mu}, \quad (5.18)$$

$$P_{\mu\lambda}=e^4\mathcal{T}^2\delta_{\mu\rho}D_{\rho\nu}C_{\lambda\nu}\Delta\Delta+2e^2\mathcal{T}\partial_\mu\Delta V_\lambda\Delta \\ -ie^2\mathcal{T}^2\Delta\delta_{\mu\lambda}+ie^2T\,\mathrm{tr}\gamma_\mu G\Gamma_\lambda G \quad (5.19)$$

$$\pi=2e^2\mathcal{T}\partial_\mu\Delta V_\lambda D_{\lambda\mu}+e^4\mathcal{T}^2\delta_{\lambda\mu}D_{\mu\rho}C_{\lambda\rho}D-ig^2\lambda n\Delta \\ -ie^2\mathcal{T}^2D_{\mu\lambda}\delta_{\mu\lambda}+ig^2\,\mathrm{tr}\gamma_5G\Gamma_5G+g^4\lambda^2n\Delta\Delta\Delta N. \quad (5.20)$$

n is a matrix conserving charge.

These equations may now be treated as was electrodynamics earlier with a larger number of renormalizations required by the larger number of fields and the lack of gauge invariance, and with the ϕ^4 term present the convergence in perturbation theory has been demonstrated.[10] Integral equations for Γ_μ, Γ_5, $C_{\mu\nu}$, V_μ, and N can be set up, all falling into the same type. We confine

[8] The series $\Sigma\lambda^{2n}\mathrm{K}_n$, since perturbation theory has been used in it, will not converge, according to C. A. Hurst, Proc. Cambridge Phil. Soc. 48, 625 (1952). If, however, we follow Hurst's interpretation that the expansion is asymptotic and should be broken off at a certain point to give best results, the arguments above can be considered adequate.

[9] These definitions are the most useful, but alternative ways can be used and are sometimes needed, e.g., $C_{\mu\nu}=g^2e^{-2}(\delta/\delta g\phi)\times(\delta/\delta g\phi)D_{\mu\nu}^{-1}$ and $(\delta/\delta g\phi)(\Gamma_\mu D)=DV_\mu\Gamma_5\Delta+D\delta\Gamma_5/\delta eA_\mu$.

[10] P. T. Matthews and A. Salam, Revs. Modern Phys. 23, 311 (1951).

our attention to Γ_5 which satisfies

$$\Gamma_5=\gamma_5 Z'^{-1}+ig^2\int\gamma_5\tau G\Gamma_5 G\gamma_5\tau\Delta. \tag{5.21}$$

The solution $\Gamma_5=\gamma_5 f$ contains no approximation as far as γ_5 is concerned, but the closed form cannot be obtained, and the approximate method must be used. The τ's give $+1$, 0, -1 according to whether symmetric charged, or neutral theory is employed, and only the former can be used in this approximation. $f(0, j)$ for large $(j)^2$ behaves like $(j^2)^{-\beta}$, where

$$\beta=\tfrac{1}{2}[1\pm(1-g^2/8\pi^2)^{\frac{1}{2}}],$$

with logarithmic solution when $\beta=\frac{1}{2}$. For large g^2, i.e., $g^2/4\pi>2\pi$, $\beta=\frac{1}{2}\pm i\epsilon$.[11] This means that for large j^2 the solution behaves like $\cos\log(j^2)$, and one can expect the Green's function also to have a series of roots. Since this behavior would apparently be ruled out on physical grounds, this may be taken as an upper limit in coupling constant to the applicability of the method.

When $\beta=\frac{1}{2}$, the expansion of $(j^2)^{-\frac{1}{2}}$ in powers of g^2 has lost all significance. The approximate solution for $\beta<\frac{1}{2}$ is

$$\gamma_5\tau\int_0^1(1-\beta)\{m^2x+\kappa^2(x^{-1}-1)\}^\beta \times\{j^2x(1-x)+m^2x+\kappa^2(1-x)\}^{-\beta}dx, \tag{5.22}$$

differing from the electrodynamic case by the κ^2 term, which may be interpreted by saying that the particle presents a dense core "radius" (the Compton wavelength of the nucleon) surrounded by a diffuse region "radius" (the Compton wavelength of the meson).

The interaction of the electromagnetic field with a nucleon is described by Γ_μ, and this suggests that some attempt might be made to estimate the proton neutron mass difference. Since the mass operator is dominated by its meson part, the approximate integral equation is

$$\Gamma_\mu=Z^{-1}\gamma_\mu(1+\tau_3)/2-ig^2\gamma_5\tau G\Gamma_\mu G\gamma_5\tau\Delta -ig^2\gamma_5\tau G\gamma_5\tau\Delta V_\mu\Delta. \tag{5.23}$$

Consider firstly only the first and second terms on the right. They give an integral equation which is essentially the same as (5.21) and has $Z^{-1}=0$ a condition for solution. This implies that the isotopic matrix in Γ_μ satisfies $t=C\tau_i t\tau_i$, i.e., $t=1$ not $\frac{1}{2}(1+\tau_3)$. This just states that if mesons are always present, then both proton and neutron interact in the same way with the electromagnetic field, the difference being in the charge on the meson cloud. Thus, the self-energy due to the term $ie^2\Gamma_\mu G\Gamma_\mu D$ is the same for both proton and neutron. A similar argument can be applied to the self-energy due to mesons interacting with one another, an argument which holds in perturbation theory. Thus, the difference comes from the interaction of the mesons with the nucleon, which has the same magnitude but opposite sign in the two cases. The types of approximation used so far are clearly not adequate to tackle this problem, but we may make the observation that since the effect of Γ, V is to introduce a cutoff which makes the result finite, and since the result will come mostly from the terms which are infinite in perturbation theory, the sign of our answer can be expected to be the sign of the infinite contribution of perturbation theory. By considering the first terms to be involved, e.g., $g^2e^2\Gamma_5 G\Gamma_\mu G\Gamma_5\Delta V_\mu\Delta$, the sign is found to be incorrect. This result, however, can clearly not be taken in any way as conclusive.

(C) Pseudoscalar Mesons, Gradient Coupling

The only difference to the Lagrangian of (B) is the coupling term $\frac{1}{4}ig[\bar{\psi}, \gamma_5\gamma_\mu\psi]\partial_\mu\phi$. The first approximate integral equation is

$$\Omega=Z'^{-1}\gamma_5\gamma_\mu\partial_\mu-ig^2\gamma_5\gamma_\mu\partial_\mu G_1\Omega G_1\gamma_5\gamma_\nu\partial_\nu\Delta_1, \tag{5.24}$$

where Ω is the analog of Γ in this theory.

The kernel of this equation has a linear, rather than logarithmic singularity. If the homogeneous equation is substituted into itself, an equation is obtained which must contain all of the solutions to the homogeneous equation itself. This is

$$\Omega=-g^4\gamma_5\gamma_\mu\partial_\mu G_1\gamma_5\gamma_\nu\partial_\nu G_1\Omega G_1\gamma_5\gamma_\lambda\partial_\lambda G_1\gamma_5\gamma_\pi\partial_\pi\Delta\Delta.$$

But if the subsidiary integration in this equation is carried out, it leads to a divergence in the kernel; the only solution is the trivial one, and (5.24) has no solution. This argument does not entirely rule out the possibility of a solution to the complete equation, since the approximation used to derive (5.24) depends on its having a solution. But clearly, if a solution exists, some much more powerful method is required to obtain it.

CONCLUSIONS AND OUTLOOK

The meson theories indicate a classification of types of equation, being soluble in the Fredholm sense, soluble though unbounded, and insoluble, according to whether the kernel behaves for large k^2 like $(k^2)^{-2-\delta}$, $(k^2)^{-2}$, $(k^2)^{-2+\delta}$, $\delta>0$. This categorization is similar to that obtained in perturbation theory of trivially renormalizable, nontrivially renormalizable, and nonrenormalizable, and will extend to the more complex types of theory whose general classification has recently been discussed by Sakata *et al.*[12] But it must be pointed out that such classifications are not conclusive, since the expanded kernel in (2.27) may, if expressed in closed form, fall into either of these classes,[13] and the final answer to the questions raised here can only be given when this function is obtained. Electrodynamics, for example, might in this

[11] This change is similar to the change from Legendre functions to conal functions, which are real.

[12] Sakata, Umezawa, and Kamefuchi, Prog. Theoret. Phys. 7, 337 (1952); H. Umezawa, Prog. Theoret. Phys. 7, 551 (1952).

[13] This remark may be considered valid even if the series expansion does not converge; for, providing the series is in the asymptotic category, there is still a generating function.

form be soluble in the Fredholm sense and have $Z^{-1}\neq 0$, while it also might be quite insoluble. The main difficulty lies in the expansion associated with the renormalization, since the approximation of putting $\Gamma G=\gamma G_1$ is valid at high momenta, and also the functional derivative can be replaced in certain cases by a partial derivative,[3] which, though not a systematic approach, agrees with the solution found above. We have chosen the first term in the expression in (2.27); however, any other term could have been chosen and would lead to a solution behaving like momentum squared raised to a power equal to a multiple of α when α is small, for high momentum, in spite of the extra powers of α appearing in front of the kernel, these being recompensed by the greater degree of unboundedness of the kernels. Similarly, the contribution to, say, the self-energy of the electron from each of these kernels is of order unity, and if this type of approach is to converge, it cannot rely on the smallness of α; then we have no criterion of convergence.

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APPENDIX

We give here a complete treatment of the approximate integral equation. The equation under consideration is

$$f(j^2)=Z^{-1}-\frac{i\alpha}{4\pi^3}\int\frac{f(k^2)d^4k}{(k^2+m^2)(k-j)^2}. \qquad (A.1)$$

Consider the homogeneous equation, and put $(k^2+m^2)^{-1}f=\phi$:

$$(j^2+m^2)\phi=i\lambda\int\frac{\phi(j^2)d^4k}{(k-j)^2}. \qquad (A.2)$$

Apply a transform of the type (3.4):

$$\phi(k^2)=\int\frac{\Phi(A)dA}{(A+k^2)^2}, \qquad (A.3)$$

$$(j^2+m^2)\phi(j^2)=\lambda\int\frac{\Phi(A)}{j^2}\ln\frac{A+j^2}{A}. \qquad (A.4)$$

Multiply by j^2. Putting $j^2=x$, and differentiating twice with respect to x, we return to the original form,

$$[x(x+1)\phi(x)]''=-\lambda\phi(x). \qquad (A.5)$$

Now return to f. We get

$$(xf)''=-\lambda f(x+1)^{-1}, \qquad (A.6)$$

or

$$x(x+1)f''+2(x+1)f'+\lambda f=0. \qquad (A.7)$$

It will be noticed that in the process of differentiation, arbitrary constants have been introduced, so that the solutions of (A.7) contain also the solutions to

$$f(j^2)=A+\frac{B}{j^2}-\frac{i\alpha}{4\pi^3}\int\frac{f(h^2)d^4k}{(h^2+m^2)(k-j)^2}, \qquad (A.8)$$

where A and B are arbitrary constants.

Equation (A.7) has the indicial equation at infinity,

$$\beta(\beta-1)+\lambda=0, \qquad (A.9)$$

which has been obtained earlier. Such functions cannot be used as the basis of a complete set in any interval containing the point at infinity, and so $A=0$, i.e., $Z^{-1}=0$. At the origin the indices are 0 and -1. The latter is to be expected in the light of the coefficient B in (A.8), and the solution containing it must be avoided. At $x+1=0$, where we wish to apply one boundary condition, the solutions are of the form

$$\sum_1^\infty a_n x^n \quad\text{and}\quad \sum_1^\infty a_n(\log x)x^n+\sum_0^\infty b_n x^n,$$

and the latter (corresponding to index 0) is the one appropriate to one problem. It is given in closed form in Whittaker and Watson,[14] the form suitable for one purpose being

$$\int_0^1 z^{-\beta}(1-z)^{\beta}(1+xz)^{-\beta}dz,$$

β being a root of (A.9), i.e., $\beta\sim\alpha/8\pi$, or in the notation of (3.9), normalized,

$$f(k^2)=\int_0^1\frac{z^{-\alpha/8\pi}}{[1-(\alpha/8\pi)]}\left\{\frac{(m^2)(1-z)}{(k^2z+m^2)}\right\}^{+\alpha/8\pi}dz,$$

which is the solution obtained earlier by less direct arguments.

[14] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (The MacMillan Company, New York, 1946), p. 297, Ex. (6).